POTENTIAL ALGEBRAS WITH FEW GENERATORS

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Abstract. We give a complete description of quadratic twisted potential algebras on three generators as well as cubic twisted potential algebras on two generators up to graded algebra isomorphisms under the assumption that the ground field is algebraically closed and has characteristic other than 2 or 3.

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1. Introduction. Throughout this article, $\mathbb{K}$ is an algebraically closed field of characteristic different from 2 or 3. The only dimension we use (dim) is the dimension of $\mathbb{K}$-vector spaces. If B is a $\mathbb{Z}_+$-graded vector space, B_m always stands for the mth component of B. If each B_m is finite dimensional, the *Hilbert series* of B is

$$H_B(t) = \sum_{j=0}^{\infty} \dim B_m\, t^m.$$

Classical potential algebras are defined as $\mathbb{K}[x_1, \ldots, x_n]/I_L$, where I_L is the ideal generated by all first-order partial derivatives $\frac{\partial L}{\partial x_j}$ of the potential $L \in \mathbb{K}[x_1, \ldots, x_n]$. Potential algebras have been defined in the noncommutative setting by Kontsevich [14], see also [4] (an alternative equivalent definition was suggested by Ginzburg [9]). An element $F \in \mathbb{K}\langle x_1, \ldots, x_n \rangle$ is called *cyclically invariant* if it is invariant for the linear map $C : \mathbb{K}\langle x_1, \ldots, x_n \rangle \to \mathbb{K}\langle x_1, \ldots, x_n \rangle$ defined on monomials by $C(1) = 1$ and $C(x_j u) = u x_j$ for all j and all monomials u. For example, $x^2 y + xyx + yx^2$ and x^3 are cyclically invariant, while $xy - yx$ and $x^2 y$ are not. The symbol $\mathbb{K}^{\text{cyc}}\langle x_1, \ldots, x_n \rangle$ stands for the vector space of all cyclically invariant elements of $\mathbb{K}\langle x_1, \ldots, x_n \rangle$. We define linear maps $\delta_{x_j} : \mathbb{K}\langle x_1, \ldots, x_n \rangle \to \mathbb{K}\langle x_1, \ldots, x_n \rangle$ and $\delta_{x_j}^R : \mathbb{K}\langle x_1, \ldots, x_n \rangle \to \mathbb{K}\langle x_1, \ldots, x_n \rangle$ by their action on monomials as follows:

$$\delta_{x_j} u = \begin{cases} v & \text{if } u = x_j v; \\ 0 & \text{otherwise,} \end{cases} \qquad \delta_{x_j}^R u = \begin{cases} v & \text{if } u = v x_j, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $F \in \mathbb{K}\langle x_1, \ldots, x_n \rangle$ is cyclically invariant if and only if $\delta_{x_j} F = \delta_{x_j}^R F$ for $1 \leqslant j \leqslant n$.

For a *potential* $F \in \mathbb{K}^{\text{cyc}}\langle x_1, \ldots, x_n \rangle$, the *potential algebra* A_F is defined as $\mathbb{K}\langle x_1, \ldots, x_n \rangle/I$, where I is the ideal generated by $\delta_{x_j} F$ for $1 \leqslant j \leqslant n$.

Note that if the characteristic of $\mathbb{K}$ is either 0 or is greater than the top degree of non-zero homogeneous components of $F \in \mathbb{K}^{\text{cyc}}\langle x_1, \ldots, x_n \rangle$, then $F = G^{\circlearrowleft}$ for some (non-unique) $G \in \mathbb{K}\langle x_1, \ldots, x_n \rangle$, where the linear map $G \mapsto G^{\circlearrowleft}$ from $\mathbb{K}\langle x_1, \ldots, x_n \rangle$ to $\mathbb{K}^{\text{cyc}}\langle x_1, \ldots, x_n \rangle$ is defined by its action on homogeneous elements by

$$u^{\circlearrowleft} = u + Cu + \ldots + C^{d-1}u, \text{ where } d \text{ is the degree of } u.$$

For example, $x^{4\circlearrowleft} = 4x^4$ and $x^2y^{\circlearrowleft} = x^2y + xyx + yx^2$. The usual partial derivatives $\frac{\partial\, G^{\mathrm{ab}}}{\partial\, x_j}$ of the abelianization G^{ab} of $G \in \mathbb{K}\langle x_1, \ldots, x_n\rangle$ (G^{ab} is the image of G under the canonical map from $\mathbb{K}\langle x_1, \ldots, x_n\rangle$ onto $\mathbb{K}[x_1, \ldots, x_n]$) are the abelianizations of $\delta_{x_j}(G^{\circlearrowleft})$. Thus, commutative potential algebras are the abelianizations of the noncommutative ones. The following lemma is elementary.

LEMMA 1.1. *For every $F \in \mathbb{K}\langle x_1, \ldots, x_n\rangle$ with trivial zero-degree component $(F_0 = 0)$,*

$$F = \sum_{j=1}^{n} x_j(\delta_{x_j}F) = \sum_{j=1}^{n}(\delta_{x_j}^R F)x_j.$$

Thus, F is cyclicly invariant if and only if $F = \sum_{j=1}^{n}(\delta_{x_j}F)x_j$. In particular,

$$F = \sum_{j=1}^{n} x_j(\delta_{x_j}F) = \sum_{j=1}^{n}(\delta_{x_j}F)x_j \ \text{ for every } F \in \mathbb{K}^{\mathrm{cyc}}\langle x_1, \ldots, x_n\rangle \text{ with } F_0 = 0,$$

$$\sum_{j=1}^{n}\big[x_j, \delta_{x_j}F\big] = 0 \ \text{ for every } F \in \mathbb{K}^{\mathrm{cyc}}\langle x_1, \ldots, x_n\rangle. \tag{1.1}$$

We consider a larger class of algebras. We call $F \in \mathbb{K}\langle x_1, \ldots, x_n\rangle$ a *twisted potential* if the linear spans of $\delta_{x_1}F, \ldots, \delta_{x_n}F$ and of $\delta_{x_1}^R F, \ldots, \delta_{x_n}^R F$ coincide. If F is a twisted potential, the corresponding *twisted potential algebra* A_F is given by generators $x_1, \ldots, x_n$ and relations $\delta_{x_1}F, \ldots, \delta_{x_n}F$ (the same algebra is presented by the relations $\delta_{x_1}^R F, \ldots, \delta_{x_n}^R F$). Note that there is a number of other generalizations of the concept of a potential algebra. For instance, one can replace the free algebra in the above definition by a (directed) graph algebra [4]. Our definition corresponds to the case of the *n*-petal rose (one vertex with *n* loops). Note also that *superpotential* algebras [4] are particular cases of twisted potential algebras: for them $\delta_{x_j}^R F = \pm\delta_{x_j}F$.

There is a complex of right A-modules associated with a twisted potential algebra $A = A_F$ with $F_0 = F_1 = 0$:

$$0 \to A \xrightarrow{d_3} A^n \xrightarrow{d_2} A^n \xrightarrow{d_1} A \xrightarrow{d_0} \mathbb{K} \to 0, \qquad \text{where } d_2(u_1, \ldots, u_n)_j = \sum_{k=1}^{n}(\delta_{x_j}\delta_{x_k}^R F)u_k,$$

d_0 is the augmentation map, $d_1(u_1, \ldots, u_n) = x_1u_1 + \ldots + x_nu_n$ and $d_3(u) = (x_1u, \ldots, x_nu)$. \tag{1.2}

It is straightforward to verify that (1.2) is indeed a complex and that it is exact at its three rightmost terms. We say that the twisted potential algebra $A = A$ is *exact* if (1.2) is an exact complex. Obviously, exactness of (1.2) is preserved under linear substitutions and therefore is an isomorphism invariant as long as degree-graded twisted potential algebras are concerned.

DEFINITION 1.2. We say that a twisted potential $F \in \mathbb{K}\langle x_1, \ldots, x_n\rangle$ is *non-degenerate* if $\delta_{x_1}F, \ldots, \delta_{x_n}F$ are linearly independent.

Note that if $F \in \mathbb{K}\langle x_1, \ldots, x_n\rangle$ is a non-degenerate twisted potential, then

$$\begin{pmatrix} \delta_{x_1}^R F \\ \vdots \\ \delta_{x_n}^R F \end{pmatrix} = M \begin{pmatrix} \delta_{x_1}F \\ \vdots \\ \delta_{x_n}F \end{pmatrix}, \tag{1.3}$$

for a unique $M \in GL_n(\mathbb{K})$. We say that *M provides the twist* or *is the twist*.

REMARK 1.3. It is elementary to verify that if $F \in \mathbb{K}\langle x_1, \ldots, x_n \rangle$ is a twisted potential for which the dimension of the linear span of $\delta_{x_1}F, \ldots, \delta_{x_n}F$ is $m < n$, then there is an m-dimensional subspace M in $V = \mathrm{span}\{x_1, \ldots, x_n\}$ such that F belongs to the tensor algebra of M. That is, there is a linear basis $y_1, \ldots, y_n$ in V for which only $y_1, \ldots, y_m$ feature in F written in terms of $y_1, \ldots, y_n$. Thus, F and generators $y_1, \ldots, y_m$ define a twisted potential algebra B such that the original A_F is the free product of B and the free $\mathbb{K}$-algebra on $n - m$ generators. Thus, in order to describe all twisted potential algebras with n generators, it is enough to describe non-degenerate twisted potential algebras with $\leqslant n$ generators.

By Lemma 1.1, cyclic invariance happens precisely when (1.3) is satisfied with M being the identity matrix. That is, every non-degenerate potential is a twisted potential with *trivial twist*. Non-degenerate twisted potential algebras are very similar to algebras defined by multilinear forms of Dubois-Violette [7, 8] (our definition generalizes the latter for non-homogeneous case).

REMARK 1.4. Assume that F is a non-degenerate twisted potential with the twist $M \in GL_n(\mathbb{K})$. If we perform a non-degenerate linear substitution $x_j = \sum_k c_{j,k} y_k$, then in the new variables y_j, F remains a non-degenerate twisted potential, with the new twist being the conjugate of M by the transpose of the substitution matrix C. We leave this calculation for the reader to verify. One useful consequence of this observation is that by means of a linear substitution, M can be replaced by a convenient conjugate matrix. For instance, M can be transformed into its Jordan normal form.

We say that a twisted potential $F \in \mathbb{K}\langle x_1, \ldots, x_n \rangle$ is *proper* if the equality

$$\sum_{j=1}^n x_j(\delta_{x_j}F) = \sum_{j=1}^n (\delta_{x_j}^R F)x_j, \tag{1.4}$$

from Lemma 1.1 provides the only linear dependence of the $2n^2$ elements $x_k(\delta_{x_j}F)$ and $(\delta_{x_j}^R F)x_k$ with $1 \leqslant j, k \leqslant n$ of $\mathbb{K}\langle x_1, \ldots, x_n \rangle$ up to a scalar multiple. Note that in this case, $\delta_{x_j}F$ are automatically linearly independent and therefore F is non-degenerate.

LEMMA 1.5. *Let $F \in \mathbb{K}\langle x_1, \ldots, x_n \rangle$ be a homogeneous twisted potential of degree $k \geqslant 3$ and $A = A_F$. Then $\dim A_k \geqslant n^k - 2n^2 + 1$. Moreover, F is non-degenerate if and only if $\dim A_{k-1} = n^{k-1} - n$ and F is proper if and only if $\dim A_k = n^k - 2n^2 + 1$. Furthermore, if F is proper, then F is uniquely determined by A_F up to a scalar multiple and therefore a linear substitution providing a graded algebra isomorphism between A_F and a twisted potential algebra A_G must transform F to G up to a scalar multiple.*

Proof. Let V be the linear span of x_j for $1 \leqslant j \leqslant n$, R_F be the linear span of $\delta_{x_j}F$ for $1 \leqslant j \leqslant n$ and I be the ideal of relations for A: I is the ideal in $\mathbb{K}\langle x_1, \ldots, x_n \rangle$ generated by R_F. Obviously, F is non-degenerate if and only if $\dim R_F = n$ if and only if $\dim A_{k-1} = n^{k-1} - n$. Clearly, I_k is spanned by $2n^2$ elements $x_j \delta_{x_m}F$ and $\delta_{x_m}Fx_j$ for $1 \leqslant j, m \leqslant n$. The equation (1.4) provides a non-trivial linear dependence of these elements. Hence, $\dim I_k \leqslant 2n^2 - 1$ and therefore $\dim A_k \geqslant n^k - 2n^2 + 1$. Clearly, the equality $\dim A_k = n^k - 2n^2 + 1$ holds if and only if there is no linear dependence of $x_j \delta_{x_m}F$ and $\delta_{x_m}Fx_j$ other than (1.4) (up to a scalar multiple). That is, F is proper if and only if $\dim A_k = n^k - 2n^2 + 1$.

Now let F be proper. Then $\dim I_k = \dim(VR_F + R_F V) = 2n^2 - 1$. By Lemma 1.1, $F \in VR_F \cap R_F V$. Since $\delta_{x_j}F$ are linearly independent, $\dim VR_F = \dim R_F V = n^2$. Hence, $\dim(VR_F \cap R_F V) = 1$. Thus, $VR_F \cap R_F V$ is the one-dimensional space spanned by F. It

follows that F is uniquely determined by A up to a scalar multiple. If a linear substitution C provides an isomorphism between A and a twisted potential algebra A_G, then C must transform $VR_F \cap R_F V$ to $VR_G \cap R_G V$. Since the first of these spaces is the one-dimensional space spanned by F and the second contains G, it must also be one-dimensional and must be spanned by G. Hence, A_G is proper and C transforms F into G up to a scalar multiple. $\quad\square$

REMARK 1.6. Since any homogeneous twisted potential, when proper, is uniquely (up to a scalar multiple) determined by the corresponding twisted potential algebra, we can call such algebras *proper twisted potential algebras*. Similarly, we say that a twisted potential algebra is *degenerate* if it is given by a degenerate twisted potential. As already mentioned, a proper degree-graded twisted potential algebra is always non-degenerate. Later, we shall see that every exact degree-graded twisted potential algebra is proper.

We introduce the following notation. For integers n, k satisfying $n \geqslant 2$ and $k \geqslant 3$ and $M \in GL_n(\mathbb{K})$,

$$\mathcal{P}_{n,k}(M) \text{ is the set of homogeneous degree } k \text{ elements,}$$
$$F \in \mathbb{K}\langle x_1, \ldots, x_n \rangle \text{ for which (1.3) is satisfied.} \tag{1.5}$$

Obviously, $\mathcal{P}_{n,k}(M)$ is a vector space. We denote

$$\mathcal{P}_{n,k} = \mathcal{P}_{n,k}(\text{Id}). \tag{1.6}$$

In other words, $\mathcal{P}_{n,k}$ consists of homogeneous degree k elements of $\mathbb{K}^{\text{cyc}}\langle x_1, \ldots, x_n \rangle$. Finally,

$$\mathcal{P}_{n,k}^* \text{ is the set of all homogeneous degree } k \text{ twisted potentials in } \mathbb{K}\langle x_1, \ldots, x_n \rangle. \tag{1.7}$$

If V is an n-dimensional vector space over $\mathbb{K}$ and R is a subspace of the n^2-dimensional space $V^2 = V \otimes V$, then the quotient of the tensor algebra $T(V)$ (in most cases, we use juxtaposition to denote multiplication in $T(V)$ instead of the symbol $\otimes$) by the ideal I generated by R is called a *quadratic algebra* and denoted $A(V, R)$. A quadratic algebra $A = A(V, R)$ is a *PBW algebra* (as defined in [16]) if there are linear bases $x_1, \ldots, x_n$ and $g_1, \ldots, g_m$ in V and R, respectively, such that with respect to some compatible with multiplication well-ordering (i.e., for any monomilas u, v, w $u \leqslant v$ implies $uw \leqslant vw$ and $wu \leqslant wv$) on the monomials in $x_1, \ldots, x_n$, $g_1, \ldots, g_m$ is a Gröbner basis of the ideal I of relations of A. A quadratic algebra A is called *Koszul* if $\mathbb{K}$ as a graded right A-module has a free resolution $\cdots \to M_m \to \cdots \to M_1 \to A \to \mathbb{K} \to 0$, where the second last arrow is the augmentation map and the matrices of the maps $M_m \to M_{m-1}$ with respect to some free bases consist of homogeneous elements of degree 1. If we pick a basis $x_1, \ldots, x_n$ in V, we get a bilinear form b on the free algebra $\mathbb{K}\langle x_1, \ldots, x_n \rangle$ (naturally identified with the tensor algebra $T(V)$) defined by $b(u, v) = \delta_{u,v}$ for every monomials u and v in the variables $x_1, \ldots, x_n$. The algebra $A^! = A(V, R^\perp)$, where $R^\perp = \{u \in V^2 : b(r, u) = 0 \text{ for each } r \in R\}$, is known as the *dual algebra* of $A = A(V, R)$. Recall that there is a specific complex of free right A-modules, called the Koszul complex, whose exactness is equivalent to Koszulity of A:

$$\cdots \xrightarrow{d_{k+1}} (A_k^!)^* \otimes A \xrightarrow{d_k} (A_{k-1}^!)^* \otimes A \xrightarrow{d_{k-1}} \cdots \xrightarrow{d_1} (A_0^!)^* \otimes A = A \longrightarrow \mathbb{K} \longrightarrow 0, \tag{1.8}$$

where the tensor products are over $\mathbb{K}$, the second last arrow is the augmentation map and d_k are given by $d_k(\varphi \otimes u) = \sum\limits_{j=1}^{n} \varphi_j \otimes x_j u$, where $\varphi_j \in (A_{k-1}^!)^*$, $\varphi_j(v) = \varphi(x_j v)$. Although $A^!$

and the Koszul complex seem to depend on the choice of a basis in V, it is not the case up to the natural equivalence [16]. Recall that

$$\text{every PBW algebra is Koszul; A is Koszul} \iff A^! \text{ is Koszul;}$$

$$\text{if A is Koszul, then } H_A(-t)H_{A^!}(t) = 1.$$

Note that if $F \in \mathcal{P}^*_{n,3}$, the twisted potential algebra A_F is quadratic. One can easily verify that the complex (1.2) is always a subcomplex of the Koszul complex for A_F, see [4]. The two complexes coincide precisely when A_F is a proper twisted potential algebra. Thus, we have the following curious fact:

$$\text{if } F \in \mathcal{P}^*_{n,3} \text{ is proper, then } A_F \text{ is Koszul} \iff A_F \text{ is exact.} \tag{1.9}$$

Throughout the paper, we perform linear substitutions. When describing a substitution, we keep the same letters for both old and new variables. We introduce a substitution by showing by which linear combination of (new) variables must the (old) variables be replaced. For example, if we write $x \to x + y + z$, $y \to z - y$ and $z \to 7z$, this means that all occurrences of x (in the relations, potential, etc.) are replaced by $x + y + z$, all occurrences of y are replaced by $z - y$, while z is swapped for $7z$. A **scaling** is a linear substitution with a diagonal matrix (swaps each variable with it own scalar multiple).

Our main objective is to provide a complete classification up to graded algebra isomorphisms of twisted potential algebras A_F for $F \in \mathcal{P}^*_{3,3} \cup \mathcal{P}^*_{2,4}$. This resonates with the Artin–Schelter classification [1]: many algebras we deal with are indeed Artin–Schelter regular. However, the classes are not exactly the same. The main difference though is that Artin and Schelter never provided a classification up to an isomorphism. Results are presented in s. The first column provides a label for further references (P in the label indicates a potential algebra, while T indicates that the algebra is twisted potential and non-potential). The exceptions column says which values of parameters are excluded. The isomorphism column provides generators of a group action on the space of parameters such that corresponding algebras are isomorphic precisely when the parameters are in the same orbit. We also need the following notation. Let

ξ_8 and ξ_9 be fixed elements of $\mathbb{K}^*$ of multiplicative orders 8 and 9, respectively,

(they exist since $\mathbb{K}$ is algebraically closed and has characteristic different from 2 or 3). We denote

$$\theta = \xi_9^3 \quad \text{and} \quad i = \xi_8^2. \quad \text{Obviously, } \theta^3 = 1 \neq \theta \text{ and } i^2 = -1.$$

THEOREM 1.7. *A is a potential algebra on three generators given by a homogeneous degree 3 potential if and only if A is isomorphic (as a graded algebra) to an algebra from the following table. The algebras from different rows are non-isomorphic. Algebras from (P1–P9) are proper, algebras from (P10–P14) are non-proper and non-degenerate, while algebras from (P15–P18) are degenerate.*

THEOREM 1.8. *A is a non-potential twisted potential algebra on three generators given by a homogeneous degree 3 twisted potential if and only if A is isomorphic (as a graded algebra) to an algebra from the following table. The algebras from different rows of the table are non-isomorphic. Algebras T1–T18 are proper, while T19–T23 are degenerate.*

	The potential F	Defining relations of A_F	Exceptions	Isomorphisms	Hilbert series	Koszul/ PBW/ Exact
P1	$x^3 + y^3 + z^3 + axyz^{\circlearrowleft} + bxzy^{\circlearrowleft}$	$x^2 + ayz + bzy;\ \ y^2 + azx + bxz;$ $z^2 + axy + byx$	$(a, b) \neq (0, 0);\ (a+b)^3 \neq -1$ $(a^3, b^3) \neq (1, 1)$	$(a, b) \mapsto (\theta a, \theta b)$ $(a, b) \mapsto \left(\frac{\theta a + \theta^2 b + 1}{a+b+1}, \frac{\theta^2 a + \theta b + 1}{a+b+1} \right)$	$(1-t)^{-3}$	Y/N/Y
P2	$xyz^{\circlearrowleft} + axzy^{\circlearrowleft}$	$yz + azy;\ \ zx + axz;\ \ xy + ayx$	$a \neq 0$	$a \mapsto a^{-1}$	$(1-t)^{-3}$	Y/Y/Y
P3	$(y+z)^3 + xyz^{\circlearrowleft} + axzy^{\circlearrowleft}$	$yz + azy;\ \ axz + zx + (y+z)^2;$ $xy + ayx + (y+z)^2$	$a \neq 0;\ a \neq -1$	$a \mapsto a^{-1}$	$(1-t)^{-3}$	Y/Y/Y
P4	$z^3 + xyz^{\circlearrowleft} + axzy^{\circlearrowleft}$	$yz + azy;\ \ axz + zx;\ \ xy + ayx + z^2$	$a \neq 0$	$a \mapsto a^{-1}$	$(1-t)^{-3}$	Y/Y/Y
P5	$y^3 + xz^{2\circlearrowleft} + xyz^{\circlearrowleft} - xzy^{\circlearrowleft}$	$yz - zy + z^2;\ \ xz - zx - y^2;$ $xy - yx + xz + zx$	none	trivial	$(1-t)^{-3}$	Y/Y/Y
P6	$xz^{2\circlearrowleft} + y^2z^{\circlearrowleft} + xyz^{\circlearrowleft} - xzy^{\circlearrowleft}$	$yz - zy + z^2;\ \ xz - zx - yz - zy;$ $xy - yx + xz + zx + y^2$	none	trivial	$(1-t)^{-3}$	Y/Y/Y
P7	$y^3 + z^3 + xyz^{\circlearrowleft} - xzy^{\circlearrowleft}$	$yz - zy;\ \ xz - zx - y^2;\ \ xy - yx + z^2$	none	trivial	$(1-t)^{-3}$	Y/Y/Y
P8	$yz^{2\circlearrowleft} + xyz^{\circlearrowleft} - xzy^{\circlearrowleft}$	$yz - zy;\ \ xz - zx - z^2;\ \ xy - yx + yz + zy$	none	trivial	$(1-t)^{-3}$	Y/Y/Y
P9	$(y+z)^3 + xyz^{\circlearrowleft}$	$yz;\ \ zx + (y+z)^2;\ \ xy + (y+z)^2$	none	trivial	$\frac{(1+t)(1+t^2)(1+t+t^2)}{1-t-t^3-2t^4}$	N/N/N
P10	$xz^{2\circlearrowleft} + y^3$	$z^2;\ \ y^2;\ \ xz + zx$	none	trivial	$\frac{1+t}{1-2t}$	Y/Y/N
P11	$x^3 + y^3 + z^3$	$x^2;\ \ y^2;\ \ z^2$	none	trivial	$\frac{1+t}{1-2t}$	Y/Y/N
P12	$xyz^{\circlearrowleft}$	$yz;\ \ zx;\ \ xy$	none	trivial	$\frac{1+t}{1-2t}$	Y/Y/N
P13	$xz^{2\circlearrowleft} + y^2z^{\circlearrowleft}$	$z^2;\ \ yz + zy;\ \ xz + zx + y^2$	none	trivial	$\frac{1+t}{1-2t}$	Y/Y/N
P14	$z^3 + xyz^{\circlearrowleft}$	$yz;\ \ zx;\ \ xy + z^2$	none	trivial	$\frac{1+t+t^2+t^3+t^4}{1-2t+t^2-t^3-t^4}$	N/N/N
P15	$y^3 + z^3$	$y^2;\ \ z^2$	none	trivial	$\frac{1+t}{1-2t-t^2}$	Y/Y/N
P16	$yz^{2\circlearrowleft}$	$z^2;\ \ yz + zy$	none	trivial	$\frac{1+t}{1-2t-t^2}$	Y/Y/N
P17	z^3	z^2	none	trivial	$\frac{1+t}{1-2t-2t^2}$	Y/Y/N
P18	0	none	none	trivial	$(1-3t)^{-1}$	Y/Y/N

	Twisted potential F	Defining relations of A_F	Exceptions	Isomorphisms	Hilbert series	Koszul/ PBW/ Exact
T1	$bxyz + ayzx + czxy - abyxz - bcxzy - aczyx$	$xy - ayx;\ zx - bxz;\ yz - czy$	$abc \neq 0;$ $(a-b, a-c) \neq (0,0)$	$(a,b,c) \mapsto (b,c,a)$ $(a,b,c) \mapsto (a^{-1}, c^{-1}, b^{-1})$	$(1-t)^{-3}$	Y/Y/Y
T2	$axyz + byzx + azxy - abyxz - a^2xzy - abzyx - az^3$	$xy - byx - z^2;\ zx - axz;\ yz - azy$	$ab \neq 0;\ a \neq b$	$(a,b) \mapsto (a^{-1}, b^{-1})$	$(1-t)^{-3}$	Y/Y/Y
T3	$xzy^{\circlearrowright} - xyz^{\circlearrowright} + a(xz^2 + z^2x + z^2y)$ $+ \frac{1-a}{2}(y^2z + zy^2 - 2zxz - zyz) - \frac{1+a}{2}yzy$	$yz - zy - az^2;\ xz - zx - azy + \frac{a(1-a)}{2}z^2;$ $xy - yx + (1-2a)zx + \frac{a-1}{2}y^2 + \frac{(1+a)(1-2a)}{4}zy + \frac{a^2(1-a)}{2}z^2$	$a \neq \frac{1}{3}$	trivial	$(1-t)^{-3}$	Y/Y/Y
T4	$xzy^{\circlearrowright} - xyz^{\circlearrowright} + \frac{1}{3}xz^2 + \frac{1}{3}z^2x - \frac{2}{3}zxz + \frac{1}{3}y^2z$ $+\frac{1}{3}zy^2 - \frac{2}{3}yzy + \frac{1}{2}z^2z - \frac{1}{3}zyz + \frac{a}{27}z^3$	$yz - zy - \frac{1}{3}z^2;\ xz - zx - \frac{1}{3}zy - \frac{1}{9}z^2;$ $xy - yx - \frac{1}{3}y^2 + \frac{1}{3}zx + \frac{2}{9}zy + \frac{1-a}{27}z^2$	none	trivial	$(1-t)^{-3}$	Y/Y/Y
T5	$zyx + byxz + b^2xzy - bzxy - yzx$ $-b^2xyz + (ab-1)zxz + az^2x + ab^2xz^2$	$bxy + (1-ab)xz - yx - azx;\ bxz - zx;\ yz - zy - az^2$	$b \neq 0$	trivial	$(1-t)^{-3}$	Y/Y/Y
T6	$yxz - xzy + zyx + yzx - xyz - zxy$ $+(a-1)yzy + ay^2z + azy^2 + z^3$	$xy - yx - ay^2 - z^2;\ xz + zx + (a-1)zy + ayz;\ yz + zy$	none	trivial	$(1-t)^{-3}$	Y/Y/Y
T7	$xzy^{\circlearrowright} - xyz^{\circlearrowright} - yzy + ay^2z + by^3 + z^3$	$xy - yx - ay^2 - z^2;\ xz + by^2 + ayz - zx + (a-1)zy;\ yz - zy$	none	$(a,b) \mapsto (a, -b)$	$(1-t)^{-3}$	Y/Y/Y
T8	$xzy^{\circlearrowright} - xyz^{\circlearrowright} - yzy + yz^{2\ \circlearrowright} + ay^3$	$xy - yx - yz - zy;\ xz + ay^2 - zx - zy + z^2;\ yz - zy$	$a \neq 0$	trivial	$(1-t)^{-3}$	Y/Y/Y
T9	$a^2xyz + yzx + azxy - a^2xzy$ $-zyx - ayxz + a^2xz^2 + zyz + azxz$	$axy - yx + 2zx;\ axz - zx;\ yz - zy + z^2$	$a \neq 0$	trivial	$(1-t)^{-3}$	Y/Y/Y
T10	$xyz - yzx + zxy - yxz + xzy - zyx$ $+y^2z - yzy + zy^2 + z^3$	$xy - yx + y^2 + z^2;\ xz + zx + 2zy;\ yz + zy$	none	trivial	$(1-t)^{-3}$	Y/Y/Y
T11	$x^2z + axzx + a^2zx^2 + y^2z - ayzy + a^2zy^2$	$xz + azx;\ yz - azy;\ x^2 + y^2$	$a \neq 0$	$a \mapsto -a$	$(1-t)^{-3}$	Y/Y/Y
T12	$z^2y + izyz - yz^2 + y^2x - yxy + xy^2 + x^3$	$x^2 + y^2;\ xy - yx + z^2;\ zy + iyz$	none	trivial	$(1-t)^{-3}$	Y/N/Y
T13	$z^2y - izyz - yz^2 + y^2x - yxy + xy^2 + x^3$	$x^2 + y^2;\ xy - yx + z^2;\ zy - iyz$	none	trivial	$(1-t)^{-3}$	Y/N/Y

T14	$xyx + yxy + zyx + yzy + zyz$ $+ \theta xzy + \theta zxz + \theta^2 zxx + \theta^2 yxz$	$yx + \theta zy + \theta^2 zx;\;\; xy + zy + \theta^2 xz;\;\; yx + yz + \theta xz$	none	trivial	$(1-t)^{-3}$	Y/N/Y
T15	$xyx + yxy + zyx + yzy + zyz$ $+ \theta^2 xzy + \theta^2 zxz + \theta xzx + \theta yxz$	$yx + \theta^2 zy + \theta zx;\;\; xy + zy + \theta xz;\;\; yx + yz + \theta^2 xz$	none	trivial	$(1-t)^{-3}$	Y/N/Y
T16	$y^2 z^{\circlearrowleft} + z^3 + x^2 z - xzx + zx^2$	$x^2 + y^2 + z^2;\;\; xz - zx;\;\; yz + zy$	none	trivial	$(1-t)^{-3}$	Y/Y/Y
T17	$xy^{2\circlearrowleft} + y^3 + xz^2 - zxz + z^2 x$	$xz - zx;\;\; xy + yx + y^2;\;\; y^2 + z^2$	none	trivial	$(1-t)^{-3}$	Y/Y/Y
T18	$y^3 + yz^{2\circlearrowleft} + az^3 + x^2 z - xzx + zx^2$	$xz - zx;\;\; yz + zy + x^2 + az^2;\;\; y^2 + z^2$	$a^2 + 4 \neq 0$	$a \mapsto -a$	$(1-t)^{-3}$	Y/N/Y
T19	$x^2 y + axyx + a^2 yx^2 + z^3$	$x^2;\;\; xy + ayx;\;\; z^2$	$a \neq 0;\;\; a \neq 1$	trivial	$\frac{1+t}{1-2t}$	Y/Y/N
T20	$xy^2 + ayxy + a^2 y^2 x + x^2 z + a^2 xzx + a^4 zx^2$	$x^2;\;\; xy + ayx;\;\; xz + a^2 zx + y^2$	$a \neq 0;\;\; a \neq 1$	trivial	$\frac{1+t}{1-2t}$	Y/Y/N
T21	$y^3 + z^3 + x^2 z - xzx + zx^2$	$y^2;\;\; x^2 + z^2;\;\; xz - zx$	$a \neq 0;\;\; a \neq 1$	trivial	$\frac{1+t}{1-2t}$	Y/Y/N
T22	$x^2 y + axyx + a^2 yx^2$	$xy + ayx;\;\; x^2$	$a \neq 0;\;\; a \neq 1$	trivial	$\frac{1+t}{1-2t-t^2}$	Y/Y/N
T23	$x^2 y - xyx + yx^2 + y^3$	$xy - yx;\;\; x^2 + y^2$	none	trivial	$\frac{1+t}{1-2t-t^2}$	Y/Y/N

	Potential F	Defining relations of A_F	Exceptions	Isomorphisms	Hilbert series	Exact
P19	$x^4 + ax^2y^2{}^\circlearrowleft + bxyxy^\circlearrowleft + y^4$	$x^3 + axy^2 + ay^2x + 2byxy;$ $ax^2y + ayx^2 + 2bxyx + y^3$	$4(a+b)^2 \neq 1;\ (a,b) \neq (0,0)$ $(a,b) \neq \pm(1, 1/2)$	$(a,b) \mapsto (-a,-b)$ $(a,b) \mapsto \left(\frac{1-2b}{1+2a+2b}, \frac{1-2a+2b}{2(1+2a+2b)}\right)$	$(1+t)^{-1}(1-t)^{-3}$	Y
P20	$x^2y^2{}^\circlearrowleft + \frac{a}{2}xyxy^\circlearrowleft$	$xy^2 + y^2x + axyx;\ x^2y + yx^2 + ayxy$	none	trivial	$(1+t)^{-1}(1-t)^{-3}$	Y
P21	$x^4 + x^2y^2{}^\circlearrowleft + \frac{a}{2}xyxy^\circlearrowleft$	$x^3 + xy^2 + y^2x + ayxy;\ x^2y + yx^2 + ayxy$	none	trivial	$(1+t)^{-1}(1-t)^{-3}$	Y
P22	$x^3y^\circlearrowleft + x^2y^2{}^\circlearrowleft - xyxy^\circlearrowleft$	$x^2y^\circlearrowleft + xy^2{}^\circlearrowleft - 3yxy;\ x^3 + x^2y + yx^2 - 2xyx$	none	trivial	$(1+t)^{-1}(1-t)^{-3}$	Y
P23	$x^4 + \frac{1}{2}xyxy$	$x^3 + yxy;\ xyx$	none	trivial	$\frac{(1+t^2)(1-t^5)}{(1-t-t^4-t^5)(1-t)}$	N
P24	$x^4 + y^4$	$x^3;\ y^3$	none	trivial	$\frac{1+t+t^2}{1-t-t^2}$	N
P25	$x^3y^\circlearrowleft$	$x^2y + yx^2 + xyx;\ x^3$	none	trivial	$\frac{1+t+t^2}{1-t-t^2}$	N
P26	$xyxy^\circlearrowleft$	$yxy;\ xyx$	none	trivial	$\frac{1+t+t^2}{1-t-t^2}$	N
P27	x^4	x^3	none	trivial	$\frac{1+t+t^3}{1-t-t^2-t^3}$	N
P28	0	none	none	trivial	$(1-2t)^{-1}$	N

	Twisted potential F	Defining relations of A_F	Exceptions	Hilbert series	Exact
T24	$x^2y^2 + a^2y^2x^2 + axy^2x + ayx^2y + bxyxy + abyxyx$	$a^2yx^2 + ax^2y + abxyx; \quad xy^2 + ay^2x + byxy$	$a \neq 0; \ a \neq 1$	$(1+t)^{-1}(1-t)^{-3}$	Y
T25	$x^2y^2 + y^2x^2 - xy^2x - yx^2y + (a-1)x^2yx$ $+(1-a)xyx^2 + ayx^3 - ax^3y + \frac{a}{2}x^4$	$xy^2 - y^2x + (a-1)xyx + (1-a)yx^2 - ax^2y + \frac{a}{2}x^3;$ $yx^2 - x^2y + ax^3$	none	$(1+t)^{-1}(1-t)^{-3}$	Y
T26	$x^2y^2\circlearrowleft - xyxy\circlearrowleft + ayx^3 + ax^3y + (a-1)xyx^2$ $+(a+1)x^2yx$	$xy^2 + y^2x - 2yxy + ax^2y + (a-1)yx^2 + (a+1)xyx;$ $ax^3 + x^2y + yx^2 - 2xyx$	none	$(1+t)^{-1}(1-t)^{-3}$	Y
T27	$x^2y^2\circlearrowleft - xyxy\circlearrowleft - xyx^2 + x^2yx + ax^4$	$x^2y + yx^2 - 2xyx; \quad xy^2 + y^2x - 2yxy - yx^2 + xyx + ax^3$	none	$(1+t)^{-1}(1-t)^{-3}$	Y
T28	$x^2y^2 + a^2y^2x^2 + axy^2x - ayx^2y$	$a^2yx^2 - ax^2y; \quad xy^2 + ay^2x$	$a \neq 0$	$(1+t)^{-1}(1-t)^{-3}$	Y
T29	$x^3y + yx^3 + \theta xyx^2 + \theta^2 x^2yx + y^4$	$x^2y + \theta yx^2 + \theta^2 xyx; \quad x^3 + y^3$	none	$(1+t)^{-1}(1-t)^{-3}$	Y
T30	$x^3y + yx^3 + \theta^2 xyx^2 + \theta x^2yx + y^4$	$x^2y + \theta^2 yx^2 + \theta xyx; \quad x^3 + y^3$	none	$(1+t)^{-1}(1-t)^{-3}$	Y
T31	$x^4 - iyx^3 - y^2x^2 + iy^3x + y^4 + xy^3 + x^2y^2 + x^3y$	$x^3 + x^2y + xy^2 + y^3; \quad -ix^3 - yx^2 + iy^2x + y^3$	none	$(1+t)^{-1}(1-t)^{-3}$	Y
T32	$x^4 + iyx^3 - y^2x^2 - iy^3x + y^4 + xy^3 + x^2y^2 + x^3y$	$x^3 + x^2y + xy^2 + y^3; \quad ix^3 - yx^2 - iy^2x + y^3$	none	$(1+t)^{-1}(1-t)^{-3}$	Y
T33	$x^2y^2 - yx^2y + y^2x^2 - xy^2x + y^3x - xy^3 + yxy^2 - y^2xy$	$-x^2y + yx^2 + y^2x + xy^2 - yxy; \quad xy^2 - y^2x - y^3$	none	$(1+t)^{-1}(1-t)^{-3}$	Y
T34	$x^3y + ax^2yx + a^2xyx^2 + a^3yx^3$	$x^2y + axyx + a^2yx^2; \quad x^3$	$a \neq 0; \ a \neq 1$	$\frac{1+t+t^2}{1-t-t^2}$	N

THEOREM 1.9. *A is a potential algebra on two generators given by a homogeneous degree* 4 *potential if and only if A is isomorphic* (*as a graded algebra*) *to an algebra from the following table. The algebras from different rows of the table are non-isomorphic. Algebras from* (P19–P23) *are proper, algebras from* (P24–P26) *are non-proper and non-degenerate, while algebras from* (P27–P28) *are degenerate.*

THEOREM 1.10. *A is a non-potential twisted potential algebra on two generators given by a homogeneous degree* 4 *potential if and only if A is isomorphic* (*as a graded algebra*) *to an algebra from the following table. Distinct algebras anywhere in the table are non-isomorphic. Algebras from* (T24–T33) *are proper, while the algebras in* (T34) *are non-proper and non-degenerate.*

REMARK 1.11. Recall [**9**] that a $\mathbb{K}$-algebra A is called n-Calabi–Yau if A admits a projective A-bimodule resolution $0 \to P_0 \to \ldots \to P_n \to A \to 0$ such that the sequence $0 \to \text{Hom}(P_n, A) \to \ldots \to \text{Hom}(P_0, A) \to 0$ is quasi-isomorphic to $0 \to P_0 \to \ldots \to P_n \to 0$ (Poincaré duality). Algebras from (P1–P8) and (P19–P22) are 3-Calabi–Yau with the required resolution provided by tensoring the complex (1.2) by A (over $\mathbb{K}$) on the left and interpreting the result as a bimodule complex. Actually, this captures (up to an isomorphism) all 3-Calabi–Yau algebras which are potential with the potential from $\mathcal{P}_{3,3}$ or $\mathcal{P}_{2,4}$. This augments the coarse description of Bocklandt [**3**] of graded 3-Calabi–Yau algebras. What he provides is a description of directed graphs and degrees such that there exists a homogeneous potential F of given degree with the quotient of the graph path algebra by the relations $\delta_{x_j} F$ (F is written in terms of generators of the path algebra) being 3-Calabi–Yau and proves that (in the category of degree-graded algebras) every 3-Calabi–Yau algebra emerges this way. On the other hand, we take two specific situations: 3-petal rose and degree 3 and 2 petal rose and degree 4 and describe the corresponding 3-Calabi–Yau algebras up to an isomorphism.

One can also consider the twisted version of this remark and observe that the list of twisted 3-CY algebras [**4**] comprises P1–P18, P19–P22, T1–T8, and T24–T33.

Section 2 is devoted to recalling relevant general information as well as to proving few auxiliary results of general nature. In Section 3, we prove a number of general results on potential and twisted potential algebras and provide examples. In Sections 3–6, we prove Theorems 1.9, 1.7, 1.10, and 1.8, respectively. We make extra comments and discuss some open questions in the final Section 7.

2. Preliminary results. Note that if $A = A(V, R)$ is a quadratic algebra, $x_1, \ldots, x_n$ is a fixed basis in V and the monomials in x_j are equipped with an order compatible with multiplication, then we can choose a basis $g_1, \ldots, g_m$ in R such that the leading monomials $\overline{g}_j$ of g_j are pairwise distinct. We call $S = \{\overline{g}_1, \ldots, \overline{g}_m\}$ the *set of leading monomials of R*. Although there are multiple bases in R with pairwise distinct leading monomials, S is uniquely determined by R (provided x_j and the order are fixed). The following result is an improved version of a lemma from [**11**].

LEMMA 2.1. *Let $A = A(V, R)$ be a quadratic algebra. Then, the following statements are equivalent:*

(A1) *A is PBW, $\dim A_1 = 3$, $\dim A_2 = 6$ and $\dim A_3 = 10$;*
(A2) *A is PBW and $H_A = (1 - t)^{-3}$;*
(A3) *$\dim A_3 = 10$ and there is a basis x, y, z in V and a well-ordering on x, y, z monomials compatible with multiplication, with respect to which the set of leading monomials of R is $\{xy, xz, yz\}$.*

Proof. The implication (A2)$\Longrightarrow$(A1) is obvious. Next, assume that (A1) is satisfied. Then, dim $V =$ dim $R = 3$ and dim $A_3 = 10$. Let a, b, c be a PBW basis for A, while f, g, h be corresponding PBW generators. Since f, g, and h form a Gröbner basis of the ideal of relations of A, it is easy to see that dim A_3 is 9 plus the number of overlaps of the leading monomials $\overline{f}$, $\overline{g}$, and $\overline{h}$ of f, g, and h. Since dim $A_3 = 10$, the monomials $\overline{f}$, $\overline{g}$, and $\overline{h}$ must have exactly one overlap. It is a routine check that if at least one of three degree 2 monomials in three variables is a square, these monomials overlap at least twice. The same happens, if the three monomials contain uv and vu for some distinct $u, v \in \{a, b, c\}$. Finally, the triples (ab, bc, ca) and (ba, cb, ac) produce three overlaps apiece. The only option left is for $(\overline{f}, \overline{g}, \overline{h})$ to be $\{xy, xz, yz\}$, where (x, y, z) is a permutation of (a, b, c). This completes the proof of implication (A1)$\Longrightarrow$(A3). Finally, assume that (A3) is satisfied. Then, the leading monomials of defining relations have exactly one overlap. If this overlap produces a non-trivial degree 3 element of the Gröbner basis of the ideal of relations of A, then dim $A_3 = 9$, which contradicts the assumptions. Hence, the overlap resolves. That is, a linear basis in R is actually a Gröbner basis of the ideal of relations of A and A is PBW. Furthermore, the leading monomials of the defining relations are the same as for $\mathbb{K}[x, y, z]$ with respect to the left-to-right lexicographical ordering with $x > y > z$. Hence, A and $\mathbb{K}[x, y, z]$ have the same Hilbert series: $H_A = (1 - t)^{-3}$. This completes the proof of (A3)$\Longrightarrow$(A2). $\square$

2.1. Canonical forms. The following lemma is a well-known fact. We provide a proof for the sake of completeness.

LEMMA 2.2. *Let $\mathbb{K}$ be an arbitrary algebraically closed field (characteristics 2 and 3 are allowed here), M be a two-dimensional vector space over $\mathbb{K}$ and S be a one-dimensional subspace of $M^2 = M \otimes M$. Then, S satisfies exactly one of the following conditions:*
(I1) $S = \mathrm{span}\{y^2\}$; (I2) $S = \mathrm{span}\{yx\}$; (I3) $S = \mathrm{span}\{xy - \alpha yx\}$ *with $\alpha \in \mathbb{K}^*$*; (I4) $S = \mathrm{span}\{xy - yx - y^2\}$, *for some linear basis x, y in M. Furthermore, if $S = \mathrm{span}\{xy - \alpha yx\} = \mathrm{span}\{x'y' - \beta y'x'\}$ with $\alpha\beta \neq 0$ for two different bases x, y and x', y' in M, then either $\alpha = \beta$ or $\alpha\beta = 1$.*

Proof. If M is spanned by a rank 1 element, then $S = \mathrm{span}\{uv\}$, where u, v are non-zero elements of M uniquely determined by S up to non-zero scalar multiples. If u and v are linearly independent, we set $y = u$ and $x = v$ to see that (I2) is satisfied. If u and v are linearly dependent, we set $y = u$ and pick an arbitrary $x \in M$ such that y and x are linearly independent. In this case, (I1) is satisfied. Obviously, (I1) and (I2) cannot happen simultaneously. Since S in (I3) and (I4) are spanned by rank 2 elements, neither of them can happen together with either (I1) or (I2). Now let u, v be an arbitrary basis in M and S be spanned by a rank 2 element $f = auu + buv + cvu + dvv$ with $a, b, c, d \in \mathbb{K}$. A linear substitution $u \to u$, $v \to v + su$ with an appropriate $s \in \mathbb{K}$ turns a into 0 (one must use the fact that f has rank 2 and that $\mathbb{K}$ is algebraically closed: s is a solution of a quadratic equation). Thus, we can assume that $a = 0$. Since f has rank 2, it follows that $bc \neq 0$. If $b + c \neq 0$, we set $x = u + \frac{dv}{b+c}$ and $y = bv$ to see that (I3) is satisfied with $\alpha = -\frac{c}{b} \neq 1$. Note also that the only linear substitutions which send $xy - \alpha yx$ to $xy - \beta yx$ (up to a scalar multiple) with $\alpha\beta \in \mathbb{K}^*$, $\alpha \neq 1$ are scalings and scalings composed with swapping x and y. In the first case, $\alpha = \beta$. In the second case, $\alpha\beta = 1$. Finally, assume that $b + c = 0$. If, additionally, $d = 0$, S is spanned by $xy - yx$ with $x = u$ and $y = v$, which falls into (I3) with $\alpha = 1$. Note that any linear substitution keeps the shape of $xy - yx$ up to a scalar multiple. If $d \neq 0$, we set $x = u$ and $y = \frac{dv}{b}$ to see that S is spanned by $xy - yx - y^2$ yielding (I4). The

remarks on linear substitutions made in each of considered above cases show that S satisfies only one of I1–I4, and that α in I3 is defined uniquely up to taking the reciprocal. $\qquad\square$

We need the following canonical form result, which goes all the way back to Weierstrass. Note that if $\mathbb{K}$ is not algebraically closed or if the characteristic of $\mathbb{K}$ is 2 or 3, the result does not hold. We sketch a proof for the sake of convenience. With a little bit of extra work, it is possible to show that polynomials F_b and $F_{b'}$ of (W1) are equivalent if and only if b and b' are in the same orbit under the group action generated by two maps $b \mapsto \theta b$ and $b \mapsto \frac{1-b}{1+b}$ (this group has 12 elements).

LEMMA 2.3. *Assume that $\mathbb{K}$ is algebraically closed and has characteristic different from 2 or 3. Then every homogeneous degree 3 polynomial $F \in \mathbb{K}[x, y, z]$ by means of a non-degenerate linear substitution can be transformed into exactly one of the following forms*:

(W1) $F = F_b = x^3 + y^3 + z^3 + 3bxyz$ with $b \in \mathbb{K}$, $b^3 \neq -1$;   (W2) $F = xz^2 + y^3$;

(W3) $F = xyz + (y+z)^3$;   (W4) $F = xz^2 + y^2z$;   (W5) $F = xyz + z^3$;

(W6) $F = xyz$; (W7) $F = y^3 + z^3$;   (W8) $F = yz^2$;   (W9) $F = z^3$;   (W10) $F = 0$.
$$\tag{2.1}$$

Sketch of the proof of Lemma 2.3. Since 0 stays 0 under linear substitutions, we can deal with non-zero polynomials only. Let $F \in \mathbb{K}[x, y, z]$ be a non-zero homogeneous degree 3 polynomial. First, assume that F is non-degenerate in the sense that the gradient $\nabla F = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right)$ does not vanish at every $(x, y, z) \in \mathbb{K}^3 \setminus \{(0, 0, 0)\}$. We encourage the reader to do calculations themselves (alternatively, one can look up details in [10, 15]). The equation $F(x, y, z) = 0$ defines a smooth curve C on the projective plane $\mathbb{K}P^2$ (an elliptic curve) and the linear span of the union of the lines through the origin representing the inflection points of C coincides with $\mathbb{K}^3$. Since the vectors $(1, -1, 0)$, $(1, -\theta, 0)$, and $(1, 0, -1)$ form a linear basis in $\mathbb{R}^3$, there exists a non-degenerate linear substitution, which makes the lines spanned by these three vectors into inflection points of C. Finally, one easily checks that if these three lines (treated as points of $\mathbb{K}P^2$) are inflection points of C, then F is a scalar multiple of an F_b of (W1). By scaling, we can get rid of the scalar multiple, turning F into one of the members of (W1). Of course, a non-degenerate F cannot be equivalent to a degenerate one.

For the rest of the proof, we can assume that F is degenerate. This includes the possibility for F to be reducible as a polynomial. If F can be written as a product of three (automatically homogeneous) polynomials of degree 1, $F = u_1u_2u_3$, we have options based on linear dependencies between u_j. This leads to F being equivalent to one of (W6–W9), which are pairwise non-equivalent. Next, if $F = uf$, where u and f are homogeneous irreducible polynomials of degrees 1 and 2, respectively, we start by transforming u into z by a linear substitution. Thus, we can assume that $F = zf$. Irreducibility of the quadratic form f is equivalent to invertibility of its matrix. If the x, y part of f is a non-degenerate quadratic form on two variables, it is easy to see that a linear substitution sending z to a scalar multiple of z transforms F to the form (W5). Otherwise, a substitution of the same type transforms F to the form (W4). Since a non-degenerate x, y quadratic form is non-equivalent to a degenerate one, (W4) and (W5) are non-equivalent. This covers all options for F to be reducible.

It remains to consider the case when F is degenerate and irreducible. By means of a linear substitution, we can ensure that ∇F vanishes at $(1, 0, 0)$, which means that

$F = xf(y,z) + g(y,z)$, where f and g are homogeneous polynomials of degrees 2 and 3, respectively. Since F is irreducible, f is non-zero. If f is of rank 1, a linear substitution (leaving x unchanged) transforms f into z^2, while if f has rank 2, a linear substitution transforms f into yz. Thus, it suffices to look at F of the forms $F = xyz + g(y,z)$ and $F = xz^2 + g(y,z)$. In the case $F = xyz + g(y,z) = xyz + ay^3 + by^2z + cyz^2 + dz^3$, we have $ad \neq 0$ (otherwise F is reducible) and a scaling turns F into $xyz + y^3 + by^2z + cyz^2 + z^3$. Then, the sub $x \to x + (3-b)y + (3-c)z$, $y \to y$, $z \to z$ transforms F into $xyz + (y+z)^3$ of (W3). Finally, if $F = xz^2 + g(y,z) = xz^2 + ay^3 + by^2z + cyz^2 + dz^3$, we must have $a \neq 0$ (otherwise F is reducible). A substitution of the form $x \to x$, $y \to py + qz$, $z \to z$ with $p \in \mathbb{K}^*$ and $q \in \mathbb{K}$ turns a into 1 and kills b: $F = xz^2 + y^3 + cyz^2 + dz^3$ (with new c and d). Now the substitution $x \to x - cy - dz$, $y \to y$, $z \to z$ transforms F into $xz^2 + y^3$ of (W2). Since we have ploughed through all possibilities, every F is equivalent to one of (W1–W10). It remains to show that F from different (Wj) are non-equivalent. From the above considerations, this is clear for all cases except for (W2) and (W3). The Hessian (the determinant of the matrix of second derivatives) of F from (W2) is $-24yz^2$, while the Hessian of F from (W3) is $2xyz - 6(y+z)^3$. Since the first is reducible, while the second is not (equivalence of ternary cubics yields equivalence of their Hessians), (W2) and (W3) are non-equivalent.   □

LEMMA 2.4. *Let* $G \in \mathbb{K}[x,y]$ *be a homogeneous degree 4 polynomial. Then by a linear substitution (=natural action of* $GL_2(\mathbb{K})$*), G can be turned into one of the following forms*:

(C1) $G = 0$; (C2) $G = x^4$; (C3) $G = x^3y$; (C4) $G = x^2y^2$;

(C5) $G = x^4 + x^2y^2$; (C6) $G = x^4 + ax^2y^2 + y^4$ *with* $a^2 \neq 4$.

Moreover, to which of the above six forms G can be transformed is uniquely determined by G. Furthermore, for $a \in \mathbb{K}$, $a^2 \neq 4$, the set of $S \in GL_2(\mathbb{K})$, the substitution by which turns $x^4 + ax^2y^2 + y^4$ into $\lambda(x^4 + bx^2y^2 + y^4)$ for some $\lambda \in \mathbb{K}^*$ and $b \in \mathbb{K}$ with $b^2 \neq 4$ does not depend on a, forms a subgroup H of $GL_2(\mathbb{K})$ and consists of non-zero scalar multiples of the matrices of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}, \begin{pmatrix} 1 & p \\ -1 & p \end{pmatrix}, \begin{pmatrix} 1 & p \\ 1 & -p \end{pmatrix}, \begin{pmatrix} qp & 1 \\ p & q \end{pmatrix},$$

where $p, q \in \mathbb{K}$, $p^4 = 1$, and $q^2 = -1$.   (2.2)

Proof. If $G = 0$ to begin with, it stays this way after any linear sub. Thus, we can assume that $G \neq 0$. Since G is homogeneous of degree 4 and $\mathbb{K}$ is algebraically closed, G is the product of four non-zero homogeneous degree 1 polynomials $G = u_1u_2u_3u_4$. Analyzing possible linear dependencies of u_j, we see that unless u_j are pairwise linearly independent, a linear substitution turns G into a unique form from (C2–C5). Indeed, all u_j being proportional leads to (C2), three being proportional with one outside their one-dimensional linear span gives (C3), two pairs of proportional u_j generating distinct one-dimensional spaces corresponds to (C4), while only one pair of proportional u_j leads to (C5). This leaves the case of u_j being pairwise linearly independent. Note that for $a \in \mathbb{K}$ satisfying $a^2 \neq 4$, this is the case with $G = x^4 + ax^2y^2 + y^4$. Now, we shall see that an arbitrary G with this property can be turned into a G from (C6) by means of a linear substitution. We achieve this in three steps. First, by the substitution which turns u_1 into x and u_2 into y, we make G divisible by xy: $G = xy(px^2 + qxy + ry^2)$ with $p, q, r \in \mathbb{K}$. Note that $pr \neq 0$ (degree 1 divisors of G are

linearly independent). A scaling turns G into $G = xy(x^2 + qxy + y^2)$ with $q \in \mathbb{K}$. The substitution $x \to x + y$, $y \to x - y$ together with a scaling transforms G into $x^4 + ax^2y^2 + y^4$. Finally, pairwise linear independence of degree 1 factors translates into $a^2 \neq 4$ and we are done with the first part of the lemma.

It is easy to see that H consisting of non-zero scalar multiples of the matrices in (2.2) is a subgroup of $GL_2(\mathbb{K})$ and that substitutions provided by matrices from H preserve the class (C6) up to scalar multiples. Assume now that $a, b \in \mathbb{K}$, $a^2, b^2 \neq 4$ and the linear substitution provided by

$$S = \begin{pmatrix} \alpha & \beta \\ p & q \end{pmatrix} \in GL_2(\mathbb{K}),$$

turns $x^4 + ax^2y^2 + y^4$ into $G = \lambda(x^4 + bx^2y^2 + y^4)$ for some $\lambda \in \mathbb{K}^*$. The proof will be complete if we show that $S \in H$. The condition that $x^4 + ax^2y^2 + y^4$ is mapped to G yields the following system:

$$\alpha^4 + p^4 + a\alpha^2p^2 = \beta^4 + q^4 + a\beta^2q^2 \neq 0,$$

$$2\alpha^3\beta + a\alpha^2pq + a\alpha\beta p^2 + 2p^3q = 0; \quad 2\alpha\beta^3 + a\alpha\beta q^2 + a\beta^2pq + 2pq^3 = 0. \tag{2.3}$$

Indeed, the first equation ensures that after the sub the x^4 and y^4 coefficients of G are equal and non-zero, while the remaining two equations are responsible for the absence of x^3y and xy^3 in G.

If $q\alpha = 0$, the above system immediately gives $\alpha = q = 0$ and $\beta^4 = p^4 \neq 0$ and therefore $S \in H$. If $p\beta = 0$, we similarly have $\beta = p = 0$ and $\alpha^4 = q^4 \neq 0$ ensuring the membership of S in H. Thus, it remains to consider the case $pq\alpha\beta \neq 0$. Set $s = \alpha/\beta$ and $t = q/p$. The last two equations in (2.3) now read

$$2s^3 + as^2t + as + 2t = 0, \qquad 2s + ast^2 + at + 2t^3 = 0.$$

Multiplying the first equation by t, the second by s and subtracting yields $(s^2 - t^2)(st - 1) = 0$. Since S is non-degenerate, $st \neq 1$. Hence, $t = s$ or $t = -s$. If $t = s$, the first equation of (2.3) implies $s^2 = -1$. If $t = -s$, one similarly gets $s^2 = 1$. Thus, we have the following options for (s, t): (i, i), $(-i, -i)$, $(1, -1)$, and $(-1, 1)$. The inclusion $S \in H$ becomes straightforward. $\square$

LEMMA 2.5. *For $a, b \in \mathbb{K}$ satisfying $4(a + b)^2 \neq 1$, let*

$$F_{a,b} = x^4 + ax^2y^{2\circlearrowleft} + bxyxy^{\circlearrowleft} + y^4 \in \mathbb{K}^{\mathrm{cyc}}\langle x, y \rangle.$$

Then, $F_{a,b}$ and $F_{a',b'}$ are equivalent (=can be obtained from one another by a linear substitution) if and only if (a, b) and (a', b') belong to the same orbit of the group action generated by two involutions $(a, b) \mapsto (-a, -b)$ and $(a, b) \mapsto \left(\frac{1-2b}{1+2a+2b}, \frac{1-2a+2b}{2(1+2a+2b)} \right)$. This group has six elements and is isomorphic to S_3.

Proof. Note that the abelianization $F_{a,b}^{\mathrm{ab}} \in \mathbb{K}[x, y]$ of $F_{a,b}$ is given by $F_{a,b}^{\mathrm{ab}} = x^4 + 4(a + b)x^2y^2 + y^4$. By the assumption $4(a + b)^2 \neq 1$, each $F_{a,b}^{\mathrm{ab}}$ is of the form (C6) of Lemma 2.4. Since every linear substitution transforming $F_{a,b}$ into $F_{a',b'}$ must also transform $F_{a,b}^{\mathrm{ab}}$ into $F_{a',b'}^{\mathrm{ab}}$, the relevant substitutions can only be provided by matrices from the group H of Lemma 2.4. Factoring out scalar matrices, we get the group $H_0 = H/\mathbb{K}^*I$ (H_0 is isomorphic to S_4 since it has 24 elements and trivial center). Note also that the substitutions $x \to -x$,

$y \to y$ and $x \to y$, $y \to x$ transform each $F_{a,b}$ to itself. After factoring these out from H_0, we are left with a group of order 6, which is easily seen to be isomorphic to S_3 and to act essentially freely on $F_{a,b}$. Two involutions generating S_3 correspond to substitutions $x \to x$, $y \to iy$ and $x \to x + iy$, $y \to x - iy$, which act by $(a, b) \mapsto (-a, -b)$ and $(a, b) \mapsto \left(\frac{1-2b}{1+2a+2b}, \frac{1-2a+2b}{2(1+2a+2b)} \right)$ on the parameters (a, b). $\square$

2.2. A few general results on twisted potential algebras.   For an algebra A generated by $x_1, \ldots, x_n$, we say that $u \in A$ is a *right annihilator* if $x_j u = 0$ for $1 \leqslant j \leqslant n$. A right annihilator u is *non-trivial* if $u \neq 0$.

LEMMA 2.6. *Let $F \in \mathcal{P}_{n,k}^*$ with $n \geqslant 2$, $k \geqslant 3$ and let $A = A_F$. Then the following statements are equivalent*:

(1)   *A is an exact twisted potential algebra*;
(2)   *A has no non-trivial right annihilators and $H_A = (1 - nt + nt^{k-1} - t^k)^{-1}$.*

Moreover, if A is exact, then A is proper. Finally,

$$
\text{if } k = 3, \text{ then} \quad
\begin{cases}
A \text{ is exact} \Longrightarrow A \text{ is Koszul,} \\
A \text{ is proper and Koszul} \Longrightarrow A \text{ is exact.}
\end{cases}
\tag{2.4}
$$

Proof. Assume that A is exact. Denote $a_j = \dim A_j$ and set $a_{-1} = 0$. Since defining relations of A are of degree $k - 1$, $a_j = n^j$ for $0 \leqslant j < k - 1$, exactness of (1.2) yields the recurrent equality $a_{m+k} - na_{m+k-1} + na_{m+1} - a_m = 0$ for $m \geqslant -1$. Together with the initial data $a_j = n^j$ for $0 \leqslant j < k - 1$ and $a_{-1} = 0$, this determines a_n for $n \geqslant 0$ uniquely, yielding $H_A = (1 - nt + nt^{k-1} - t^k)^{-1}$. Since (1.2) is exact, d_3 is injective and therefore A has no non-trivial right annihilators. Now assume that $H_A = (1 - nt + nt^{k-1} - t^k)^{-1}$ and that A has no non-trivial right annihilators. Then, d_3 in (1.2) is injective and therefore the complex is exact at the leftmost A. Since, as we have already mentioned, (1.2) is exact at its three rightmost terms, the only place where the exactness may fail is the left A^n. Considering the graded slices of the complex and using the exactness of the complex everywhere after the left A^n, we can compute the dimension of the intersection of the kernel of d_2 with A_{m+1}^n, which is $a_{m+k} - na_{m+k-1} + na_{m+1}$. On the other hand, $\dim d_3(A_m) = a_m$. Since a_m are the Taylor coefficients of $(1 - nt + nt^{k-1} - t^k)^{-1}$, they satisfy $a_{m+k} - na_{m+k-1} + na_{m+1} - a_m = 0$, which proves that the above image and kernel have the same dimension and therefore coincide. Thus, exactness extends to the missing term. The fact that A is proper when exact now follows from Lemma 1.5 (just look at $\dim A_k$), or directly from definitions. The Koszulity statement is a consequence of (1.9). $\square$

REMARK 2.7. It is easy to construct examples of exact potential algebras with n generators given by a homogeneous degree k potential for all n, k satisfying $n \geqslant 2$, $k \geqslant 3$, and $(n, k) \neq (2, 3)$. The case $(n, k) = (2, 3)$ yields no exact algebras, as will be demonstrated below. The following lemma together with Lemma 2.6 provides easy means to construct exact twisted potential algebras.

LEMMA 2.8. *Let $F \in \mathcal{P}_{n,k}^*$ with $n \geqslant 2$, $k \geqslant 3$ and let monomials in x_j be equipped with a well-ordering compatible with multiplication. Assume also that the leading monomials of defining relations $r_j = \delta_{x_j} F$ of $A = A_F$ exhibit exactly one overlap. Then the defining relations form a Gröbner basis in the ideal of relations of A.*

Proof. The equation (1.4) provides a non-trivial linear dependence of $x_j r_m$ and $r_m x_j$ for $1 \leqslant j, m \leqslant n$. Since the degree of each r_m is $k - 1 > 1$, this dependence produces a

non-trivial noncommutative syzygy of degree k. If there are no overlaps of degree k or if the only overlap does not resolve (=produces a new element of the reduced Gröbner basis), then we arrive to a contradiction with the existence of a non-trivial degree k syzygy. Hence, the only overlap resolves and therefore the defining relations indeed form a Gröbner basis.   □

We shall use the following observation on a number of occasions.

LEMMA 2.9. *Let $F \in \mathcal{P}_{2,4}^*$ be such that x^2y^2 and yx^2y are in F with non-zero coefficients, while the monomials x^4, x^3y, x^2yx, and yx^3 do not occur in F. Then, $A = A_F$ is exact and $H_A = (1 + t)^{-1}(1 - t)^{-3}$. The same holds if $F \in \mathcal{P}_{2,4}^*$ contains y^2x^2 and xy^2x with non-zero coefficients, while y^4, y^3x, y^2xy, and xy^3 do not feature in F.*

Proof. The two statements are clearly equivalent (just swap x and y). Thus, we may assume that x^2y^2 and yx^2y are in F with non-zero coefficients and F, while the monomials x^4, x^3y, x^2yx, and yx^3 do not occur in F. Equip the x, y monomials with the left-to-right degree-lexicographical order assuming $x > y$. Then, xy^2 is the leading monomial of $\delta_x F$, while x^2y is the leading monomial of $\delta_y F$. Since these monomials exhibit just one overlap $x^2y^2 = (x^2y)y = x(xy^2)$, Lemma 2.8 implies that $\delta_x F$ and $\delta_y F$ form a Gröbner basis in the ideal of relations of A. It immediately follows that $H_A = (1 + t)^{-1}(1 - t)^{-3}$ (the corresponding normal words are $y^n(xy)^m x^k$ with $n, m, k \in \mathbb{Z}_+$) and that A has no non-trivial right annihilators (the map $u \mapsto yu$ from A to A is injective since the set of normal words is closed under multiplication by y on the left). By Lemma 2.6, A is exact.   □

LEMMA 2.10. *Let $F \in \mathcal{P}_{3,3}^*$ be such that xyz, yxz, and zxy are present in F with non-zero coefficients, while x^3, x^2y, x^2z, xyx, xy^2, yx^2, yxy, and zx^2 do not feature in F. Then $A = A_F$ is PBW, Koszul, exact and satisfies $H_A = (1 - t)^{-3}$.*

Proof. Equip the x, y, z monomials with the left-to-right degree-lexicographical order assuming $x > y > z$. By assumptions, the leading monomials of $\delta_z F$, $\delta_y F$, and $\delta_x F$ are xy, xz, and yz, respectively. Since the said monomials exhibit just one overlap, the defining relations form a quadratic Gröbner basis by Lemma 2.8. Since $\{xy, xz, yz\}$ is the set of leading monomials of members of the said basis, the equality $H_A = (1 - t)^{-3}$ immediately follows. Since A admits a quadratic Gröbner basis in the ideal of relations, A is PBW and therefore Koszul. As in the proof of Lemma 2.9, A has no non-trivial right annihilators (the map $u \mapsto zu$ from A to A is injective). By Lemma 2.6, A is exact.   □

For power series $F = \sum_{n=0}^{\infty} a_n t^n$ and $G = \sum_{n=0}^{\infty} b_n t^n$ with real coefficients, we write $F \leqslant G$ if $a_n \leqslant b_n$ for all n.

LEMMA 2.11. *Let $F \in \mathcal{P}_{3,3}^*$ and $A = A_F$. Then, $H_A \geqslant (1 - t)^{-3}$ if $F \in \mathcal{P}_{3,3}$. Furthermore, the following statements are equivalent:*

(K1) *A is exact;*
(K2) *$H_A = (1 - t)^{-3}$ and A has no non-trivial right annihilators;*
(K3) *$H_A = (1 - t)^{-3}$ and A is Koszul.*

Proof. Without loss of generality, we can assume that $\mathbb{K}$ is uncountable (just replace $\mathbb{K}$ with an uncountable field extension, if necessary). A standard argument shows that the set of $F \in \mathcal{P}_{n,k}$ for which the complex (1.2) with $A = A_F$ is non-exact is the union of countably many Zarissky closed sets. By Lemma 2.10, the set of $F \in \mathcal{P}_{3,3}$ for which A_F is exact is non-empty. Indeed, just take $F = xyz^{\circlearrowleft} - xzy^{\circlearrowleft}$. Hence, A_F is exact for generic $F \in \mathcal{P}_{3,3}$ (=for

all F except for F from a countable union of proper algebraic varieties). By Lemma 2.6, $H_{A_F} = (1-t)^{-3}$ for generic $F \in \mathcal{P}_{3,3}$. According to Ufnarovski [20], the Hilbert series H_{A_F} is componentwise minimal (among such series) for generic F. Hence, $H_{A_F} \geqslant (1-t)^{-3}$ for all $F \in \mathcal{P}_{3,3}$.

The equivalence of (K1) and (K3) follows from the already mentioned fact that the complex (1.2) coincides with the Koszul complex if the potential F is proper, while the latter happens if and only if $\dim A_3 = 10$ (see Lemma 1.5). The equivalence of (K1) and (K2) follows from Lemmas 2.6. $\qquad \square$

The same argument with an example of exact A_F given by $F = x^2 y^{2^\circlearrowleft}$ yields the following lemma.

LEMMA 2.12. *Let $F \in \mathcal{P}_{2,4}^*$ and $A = A_F$. Then, $H_A \geqslant (1+t)^{-1}(1-t)^{-3}$ if $F \in \mathcal{P}_{2,4}$. Furthermore, the following statements are equivalent:*

(L1) *A is exact;*
(L2) *$H_A = (1+t)^{-1}(1-t)^{-3}$ and A has no non-trivial right annihilators.*

2.3. Algebras A_F with $F \in \mathcal{P}_{2,3}^*$. The case $(n, k) = (2, 3)$ is an odd one out. In this case, there are no exact twisted potential algebras and the formula for the minimal series fails to follow the pattern. For esthetic reasons, we use x and y instead of x_1 and x_2. We equip x, y monomials with the left-to-right degree-lexicographical order assuming $x > y$.

PROPOSITION 2.13. *There are just four pairwise non-isomorphic algebras in the variety $W = \{A_F : F \in \mathcal{P}_{2,3}\}$. These are the algebras corresponding to the potentials $F = 0$, $F = x^3$, $F = xy^{2^\circlearrowleft}$, and $F = x^3 + y^3$. Their Hilbert series are $\frac{1}{1-2t}$, $\frac{1+t}{1-t-t^2}$ and $\frac{1+t}{1-t}$ for the last two algebras. All algebras in W are PBW, Koszul, infinite dimensional, and non-exact.*

Proof. An $F \in \mathcal{P}_{2,3}$ has the form $F = ax^3 + bxy^{2^\circlearrowleft} + cx^2 y^{\circlearrowleft} + dy^3$ with $a, b, c, d \in \mathbb{K}$. Then, the abelianization of F is $F^{ab} = ax^3 + 3bxy^2 + 3cx^2 y + dy^3 \in \mathbb{K}[x, y]$. Since $\mathbb{K}$ is not of characteristic 3, F recovers uniquely from its abelianization. Since a degree 3 cubic form on two variables is a product of three linear forms, we see that by a linear substitution F^{ab} can be turned into one of the following forms x^3, xy^2, or $x^3 + y^3$ unless it was zero to begin with. This corresponds to F turning into one of the four potentials listed in the statement of the lemma by means of a linear substitution. If $F = 0$, A_F is the free algebra on two generators and $H_A = \frac{1}{1-2t}$. If $F = x^3$, A_F is defined by one relation x^2, which forms a one-element Gröbner basis, yielding $H_{A_F} = \frac{1+t}{1-t-t^2}$. If $F = xy^{2^\circlearrowleft}$ or $F = x^3 + y^3$, the defining relations of A_F are $xy + yx$, y^2 or x^2, y^2, respectively. Again, they form a Gröbner basis, yielding $H_{A_F} = \frac{1+t}{1-t}$. Since the last two algebras are easily seen to be non-isomorphic and the Hilbert series of the first three are pairwise distinct, the four algebras are pairwise non-isomorphic. As all four algebras have quadratic Gröbner basis, they are PBW and therefore Koszul. Obviously, they are infinite dimensional. If any of these algebras were exact, Lemma 2.6 would imply that its Hilbert series is $\frac{1}{1-2t+2t^2-t^3}$, which does not match any of the above series. Thus, all algebras in W are non-exact. $\qquad \square$

PROPOSITION 2.14. *Any non-potential twisted potential algebra A on two generators given by a homogeneous degree 3 twisted potential is isomorphic to either A_G or A_{G_α} with $\alpha \in \mathbb{K} \setminus \{0, 1\}$, where $G = x^2 y - xyx + yx^2 + y^3$ and $G_\alpha = x^2 y + \alpha xyx + \alpha^2 yx^2$. Furthermore, these algebras are pairwise non-isomorphic, non-degenerate, infinite dimensional, non-exact, PBW, Koszul and have the Hilbert series $\frac{1+t}{1-t}$.*

Proof. We know that $A = A_F$. If $\delta_x F$ and $\delta_y F$ are linearly dependent, one easily sees that either $F = 0$ or F is the cube of a degree one homogeneous element. In either case A is potential, which contradicts the assumptions. Thus, F is non-degenerate. Let $M \in GL_2(\mathbb{K})$ be the matrix providing the twist. By Remark 1.4, we can assume that M is in Jordan normal form. Since A is non-potential, M is not the identity matrix. Let α and β be the eigenvalues of M. One easily sees that $\mathcal{P}_{2,3}(M)$ is trivial if $1 \notin \{\alpha^2\beta, \alpha\beta^2\}$. Without loss of generality, we can assume that $\alpha^2\beta = 1$. If $\alpha = \beta = 1$ and M is a single Jordan block, it is easy to see that every $F \in \mathcal{P}_{2,3}(M)$ is degenerate. Thus, it remains to consider the following options for M:

$$M = N_\alpha = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix} \text{ with } \alpha^3 = 1 \neq \alpha \text{ and } M = M_\alpha = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-2} \end{pmatrix} \text{ with } \alpha \neq 1.$$

In the case $M = M_\alpha$, $\mathcal{P}_{2,3}(M)$ contains $x^2y + \alpha xyx + \alpha^2 yx^2$ and consists only of its scalar multiples unless $\alpha^3 = 1$ or $\alpha = -1$. In the case $\alpha^3 = 1 \neq \alpha$, $\mathcal{P}_{2,3}(M)$ sits in the two-dimensional space spanned by $x^2y + \alpha xyx + \alpha^2 yx^2$ and $xy^2 + \alpha^2 yxy + \alpha y^2x$. If $\alpha = -1$, $\mathcal{P}_{2,3}(M)$ is contained in the two-dimensional space spanned by $x^2y - xyx + yx^2$ and y^3. In the case $M = N_\alpha$ with $\alpha^3 = 1 \neq \alpha$, $\mathcal{P}_{2,3}(M)$ consists of scalar multiples of $\frac{x^3}{1-\alpha} + yx^2 + \alpha x^2y + \alpha^2 xyx$. All this is obtained by translating (1.3) into a system of linear equations on coefficients of F and solving it.

If F is $\frac{x^3}{1-\alpha} + yx^2 + \alpha x^2y + \alpha^2 xyx$ with $\alpha^3 = 1 \neq \alpha$, then $A_F = A_{G_\alpha}$. When $F = x^2y + \alpha xyx + \alpha^2 yx^2 = G_\alpha$, the defining relations of $A = A_F$ are x^2 and $xy + \alpha yx$ with $\alpha \in \mathbb{K}^*$, $\alpha \neq 1$. It is easy to see that different α correspond to non-isomorphic algebras. Indeed, a linear substitution providing an isomorphism must map x to its scalar multiple (x^2 is the only square among quadratic relations) and taking this into account, it is easy to see that y also must be mapped to its own scalar multiple. Such a substitution preserves the space of defining relations and therefore α is an isomorphism invariant. If $F = s(x^2y + \alpha xyx + \alpha^2 yx^2) + t(xy^2 + \alpha^2 yxy + \alpha y^2x)$ with $s, t \in \mathbb{K}$, $(s, t) \neq (0, 0)$ and $\alpha^3 = 1 \neq \alpha$, we have options. If $st = 0$, then by means of a scaling combined with the swap of x and y in the case $s = 0$, we can transform F into G_α. If $st \neq 0$, a scaling reduces considerations to the case $t = s = 1$. Then $F = x^2y + \alpha xyx + \alpha^2 yx^2 + xy^2 + \alpha^2 yxy + \alpha y^2x$. The substitution $x \rightarrow x + \alpha y$, $y \rightarrow y$, provides an isomorphism of $A = A_F$ and A_{G_α}. Now let $F = s(x^2y - xyx + yx^2) + ty^3$ with $s, t \in \mathbb{K}$. If $s = 0$, A is potential. Thus, $s \neq 0$. If $t = 0$ and $s \neq 0$, then up to a scalar multiple, $F = G_{-1}$. Thus, we can assume that $st \neq 0$. By scaling, we can turn both s and t into 1, which transforms F into G. The defining relations of $A = A_F$ are now $xy - yx$ and $x^2 + y^2$. This time the space of quadratic relations fails to contain a square of a degree one element and therefore the corresponding algebra is not isomorphic to any of A_{G_α}. It is easy to see that for $F \in \{G\} \cup \{G_\alpha : \alpha \in \mathbb{K} \setminus \{0, 1\}\}$, the defining relations of A_F form a Gröbner basis in the ideal of relations. Hence, A is PBW and Koszul and $H_A = \frac{1+t}{1-t}$. This series fails to coincide with $\frac{1}{1-2t+2t^2-t^3}$ and therefore none of these algebras is exact according to Lemma 2.6. $\qquad\square$

Proposition 2.14 and Remark 1.3 provide a complete description of degenerate twisted potential non-potential algebras on three generators with homogeneous degree 3 twisted potentials. Indeed, the latter are free products of algebras from Proposition 2.14 with the algebra of polynomials on one variable. This observation is recorded as follows.

LEMMA 2.15. *A is a non-potential twisted potential algebra on three generators given by a homogeneous degree 3 degenerate twisted potential if and only if A is isomorphic*

to an algebra from (T22) *or* (T23) *of Theorem* 1.8. *The algebras with different labels are non-isomorphic and the information in the table from Theorem* 1.8 *concerning* (T22) *and* (T23) *holds true.*

3. Potential algebras A_F for $F \in \mathcal{P}_{2,4}$. Throughout this section, we equip the monomials in x, y with the left-to-right degree-lexicographical ordering assuming $x > y$. The following statement is elementary.

LEMMA 3.1. *The kernel of the canonical homomorphism from $\mathbb{K}\langle x, y \rangle$ onto $\mathbb{K}[x, y]$ (=abelianization) intersects $\mathcal{P}_{2,4}$ by the one-dimensional space spanned by $x^2 y^{2^\circlearrowleft} - xyxy^\circlearrowleft$.*

LEMMA 3.2. *Let $F \in \mathcal{P}_{2,4}$. Then by a linear substitution F can be turned into one of the following forms*:

(H1) $F = 0$; (H2) $F = x^4$; (H3) $F = x^4 + \frac{1}{2}xyxy^\circlearrowleft$; (H4) $F = \frac{1}{2}xyxy^\circlearrowleft$; (H5) $F = x^3 y^\circlearrowleft$;

(H6) $F = x^4 + y^4$; (H7) $F = x^3 y^\circlearrowleft + x^2 y^{2^\circlearrowleft} - xyxy^\circlearrowleft$;

(H8) $F_a = x^4 + x^2 y^{2^\circlearrowleft} + \frac{a}{2}xyxy^\circlearrowleft$; (H9) $F_a = x^2 y^{2^\circlearrowleft} + \frac{a}{2}xyxy^\circlearrowleft$;

(H10) $F_{a,b} = x^4 + ax^2 y^{2^\circlearrowleft} + bxyxy^\circlearrowleft + y^4$

with $4(a + b)^2 \neq 1$, $(a, b) \notin \{(0, 0), (1, \frac{1}{2}), (-1, -\frac{1}{2})\}$,

where $a, b \in \mathbb{K}$. Moreover, to which of the above 10 *forms F can be turned into is uniquely determined by F and the parameter a in* (H8) *and* (H9) *is uniquely determined by F. As for* (H10), $F_{a,b}$ *and $F_{a',b'}$ can be obtained from one another by a linear substitution if and only if they belong to the same orbit of the group action generated by two involutions $(a, b) \mapsto (-a, -b)$ and $(a, b) \mapsto \left(\frac{1-2b}{1+2a+2b}, \frac{1-2a+2b}{2(1+2a+2b)} \right)$. This group has six elements and is isomorphic to S_3.*

Proof. Let $F \in \mathcal{P}_{2,4}$. First, we show that F can be turned into exactly one of (H1–H10) by a linear sub. Let $G \in \mathbb{K}[x, y]$ be the abelianization of F. By Lemma 2.4, a linear substitution turns G into exactly one of the forms (C1–C6). Thus, we can assume from the start that G is in one of the forms (C1–C6).

If $G = 0$, then by Lemma 3.1, $F = s(x^2 y^{2^\circlearrowleft} - xyxy^\circlearrowleft)$ with $s \in \mathbb{K}$. If $s = 0$, F is given by (H1). If $s \neq 0$, a scaling brings F to the form (H9) with $a = -2$. If $G = x^4$, then by Lemma 3.1, $F = x^4 + s(x^2 y^{2^\circlearrowleft} - xyxy^\circlearrowleft)$ with $s \in \mathbb{K}$. If $s = 0$, F is given by (H2). If $s \neq 0$, a scaling brings F to the form (H8) with $a = -2$. If $G = x^3 y$, then by Lemma 3.1, $F = \frac{1}{4}x^3 y^\circlearrowleft + s(x^2 y^{2^\circlearrowleft} - xyxy^\circlearrowleft)$ with $s \in \mathbb{K}$. If $s = 0$, F acquires the form (H5) after scaling. If $s \neq 0$, a scaling brings F to the form (H7). If $G = x^2 y^2$, then by Lemma 3.1, $F = \frac{1}{4}x^2 y^{2^\circlearrowleft} + s(x^2 y^{2^\circlearrowleft} - xyxy^\circlearrowleft)$ with $s \in \mathbb{K}$. If $1 + 4s = 0$, F acquires the form (H4) after scaling. Otherwise, a scaling brings F to the form (H9) with $a \neq -2$. If $G = x^4 + x^2 y^2$, then by Lemma 3.1, $F = x^4 + \frac{1}{4}x^2 y^{2^\circlearrowleft} + s(x^2 y^{2^\circlearrowleft} - xyxy^\circlearrowleft)$ with $s \in \mathbb{K}$. If $4s + 1 = 0$, F acquires the form (H3) after scaling. Otherwise, a scaling brings F to the form (H8) with $a \neq -2$. Finally, assume that $G = x^4 + cx^2 y^2 + y^4$ with $c^2 \neq 4$. By Lemma 3.1, $F = x^4 + \frac{c}{4}x^2 y^{2^\circlearrowleft} + y^4 + s(x^2 y^{2^\circlearrowleft} - xyxy^\circlearrowleft)$ with $s \in \mathbb{K}$. That is, $F = F_{a,b} = x^4 + ax^2 y^{2^\circlearrowleft} + bxyxy^\circlearrowleft + y^4$ with $a = s + \frac{c}{4}$ and $b = -s$. The condition $c^2 \neq 4$ translates into $4(a + b)^2 \neq 1$. By Lemma 2.5, $F_{0,0}$,

$F_{1,1/2}$, and $F_{-1,-1/2}$ are all equivalent and are non-equivalent to any other $F_{a,b}$. Thus, if $(a, b) \in \left\{ (0, 0), \left(1, \frac{1}{2}\right), \left(-1, -\frac{1}{2}\right) \right\}$, F is equivalent to the potential from (H5). Otherwise, F is in (H10).

The fact that F with different labels from the list (H1–H10) are non-equivalent (cannot be obtained from one another by a linear sub) follows from the above observations, the non-equivalence of polynomials with the different labels from the list (C1–C6) as well as the trivial observation that a symmetric (not just cyclically) element of $\mathcal{P}_{2,4}$ cannot be equivalent to a non-symmetric one. It remains to prove the statements about equivalence within each of (H8), (H9), and (H10). The latter follows directly from Lemma 2.5. It remains to deal with (H7) and (H8). We may remove the exceptional cases $a = -2$ from consideration. One easily sees that the only subs that transform an F_a with $a \neq -2$ from (H7) (respectively, (H8)) to another $F_{a'}$ from (H7) (respectively, (H8)) up to scalar multiples are scalings combined with a possible swap of x and y in the (H8) case. Since none of the latter has any effect on the parameter, we have $a = a'$. It follows that distinct F_a from (H7) or (H8) are non-equivalent. $\qquad \square$

LEMMA 3.3. *Let $F \in \mathcal{P}_{2,4}$ be a potential from* (Hj) *of Lemma* 3.2 *for* $1 \leqslant j \leqslant 6$. *Then $A = A_F$ is non-exact. Its Hilbert series is $H_A = \frac{1+t+t^2}{1-t-t^2}$ for $j \in \{4, 5, 6\}$, $H_A = \frac{(1+t^2)(1-t^5)}{(1-t)(1-t-t^4-t^5)}$ for $j = 3$, $H_A = \frac{1+t+t^2}{1-t-t^2-t^3}$ for $j = 2$, and $H_A = \frac{1}{1-2t}$ for $j = 1$. If $j = 3$, A is proper, while A is non-proper in all other cases.*

Proof. It is straightforward to verify that the defining relations themselves form a Gröbner basis in the ideal of relations for all F under consideration except for F from (H3). In the case $j = 3$, we swap x and y to begin with. After this, the reduced Gröbner basis in the ideal of relations comprises yxy, $xyx + y^3$, and y^4. In each case, knowing a finite Gröbner basis (more specifically, knowing the leading monomials of its members), it is a routine calculation to find H_A in the form of a rational function to confirm the required formulae. Since none of the resulting series coincides with $(1 + t)^{-1}(1 - t)^{-3}$, Lemma 2.12 implies that A is non-exact. By Lemma 1.5, A is proper if and only if $\dim A_4 = 9$. Knowing the Hilbert series, we see that this happens precisely when F given by (H3). $\qquad \square$

LEMMA 3.4. *Let $F \in \mathcal{P}_{2,4}$ be either from* (H7–H9) *or from* (H10) *of Lemma* 3.2 *with (a, b) such that $ab(a^2 - 1)(4b^2 - a^2)(4b^2 - 1)(4b^2 - a^4)(4b^2 - 2a^2 + 1) = 0$. Then $A = A_F$ is exact and satisfies $H_A = (1 + t)^{-1}(1 - t)^{-3}$.*

Proof. If F is from (H7–H9), the result follows directly from Lemma 2.9. It remains to deal with the case of F given by (H10).

Case 1. $a = 0$. Since $(a, b) \neq (0, 0)$, we have $b \neq 0$. Then $F = x^4 + y^4 + bxyxy^{\circlearrowleft}$. Scaling x and y, we can turn F into $F = x^4 - \frac{1}{2}xyxy^{\circlearrowleft} + qy^4$ with $q \in \mathbb{K}^*$. The defining relations of A now are $x^3 = yxy$ and $yxy = qy^3$. It is now easy to compute the reduced Gröbner basis of the ideal of relations of A, which comprises $x^3 - yxy$, $xyx - qy^3$, $xy^4 - y^4x$, $x^2y^3 - \frac{1}{q}yxy^2x$, and $xy^2xy - qy^3x^2$. Knowing the basis, it is routine to verify that $H_A = (1 + t)^{-1}(1 - t)^{-3}$. Since none of the leading monomials of the members of the Gröbner basis starts with y, there are no non-trivial right annihilators in A. By Lemma 2.12, A is exact.

In the remaining cases, the proof goes along the following lines. We determine a non-zero homogeneous degree 4 central element g in A and consider the algebra $B = A/I$, where I is the ideal generated by g. The algebra B can be presented by the generators x, y and the relations $\delta_x F$, $\delta_y F$, and g. Then, we compute the reduced Gröbner basis of the ideal of relations of B, which turns out to be finite in all cases. Knowing the leading monomials

of members of the basis, it is routine to verify that $\dim B_n = 2n$ for all $n \in \mathbb{N}$. Observing that none of the leading monomials starts with y, we deduce that $yu \neq 0$ for every non-zero $u \in B$. Since g is central, $\dim A_n = \dim B_n + \dim gA_{n-4}$ for all $n \geqslant 4$. In particular, $\dim A_n \leqslant \dim A_{n-4} + 2n$ for $n \geqslant 5$ and all these inequalities turn into equalities precisely when g is not a zero divisor. The inequalities $\dim A_n \leqslant \dim A_{n-4} + 2n$ together with easily verifiable $\dim A_4 = 12$ imply that $H_A \leqslant (1+t)^{-1}(1-t)^{-3}$ and the equality is only possible if $\dim A_n \leqslant \dim A_{n-4} + 2n$ for all $n \geqslant 5$. By Lemma 2.12, $H_A \geqslant (1+t)^{-1}(1-t)^{-3}$. Hence, $H_A = (1+t)^{-1}(1-t)^{-3}$ and g is not a zero divisor. Now we check that $yu \neq 0$ for every non-zero $u \in A$. Assume the contrary and pick a non-zero homogeneous $u \in A$ of smallest possible degree such that $yu = 0$ in A. Since B is a quotient of A, $yu = 0$ in B. Hence, $u = 0$ in B. Then, $u = gv$ in A for some $v \in A$. The equality $yu = 0$ yields $gyv = 0$ and therefore $yv = 0$ in A. Since the degree of v is smaller (by 4) than the degree of u, we have arrived to a contradiction. Thus, $yu \neq 0$ for every non-zero $u \in A$ and therefore A has no non-trivial right annihilators. By Lemma 2.12, A is exact.

It remains to provide the central element g in A and a Gröbner basis in the ideal of relations of B.

Case 2. $b = 0$. Since $(a, b) \neq (0, 0)$, we have $a \neq 0$. Then, $F = x^4 + y^4 + ax^2y^{2\circlearrowleft}$. Scaling x and y, we can turn F into $F = x^4 + x^2y^{2\circlearrowleft} + qy^4$ with $q \in \mathbb{K}^*$. The defining relations of A now are $x^3 + xy^2 + y^2x$ and $x^2y + yx^2 + qy^3$. A routine verification shows that $g = xy^2x + (1-q)y^2x^2 - qy^4$ is central in A. Next, $x^3 + xy^2 + y^2x$, $x^2y + yx^2 + qy^3$, $xy^2x + (1-q)y^2x^2 - qy^4$, $xyx^2 - (1-q)xy^3 - y^2xy$, $xy^4 + y^4x + (2-q)y^2xy^2$, $xyxy^2 + (2-q)xy^3x + y^2xyx$, $xy^3x^2 + (q^2 - 3q + 1)y^2xy^3 + (1-q)y^4xy$, and $xy^3xy^2 - (q^2 - 4q + 3)y^2xy^3x - (2-q)y^4xyx$ form the reduced Gröbner basis in the ideal of relations of B and the above scheme yields the desired result.

Case 3. $a = 1$. Since $(a, b) \neq (1, 1/2)$, we have $b \neq 1/2$. Since $4(a + b)^2 \neq 1$, we have $b \neq -1/2$. Denoting $b = \frac{q}{2}$, we have $F = x^4 + x^2y^{2\circlearrowleft} + \frac{q}{2}xyxy^{\circlearrowleft} + y^4$ with $q^2 \neq 1$. The defining relations of A now are $x^3 + xy^2 + qyxy + y^2x$ and $x^2y + qxyx + yx^2 + y^3$. The element $g = xy^2x - y^4$ is central in A and $x^3 + xy^2 + qyxy + y^2x$, $x^2y + qxyx + yx^2 + y^3$, $xy^2x - y^4$, $xyx^2 - y^2xy$, $xyxyx + \frac{1}{q}xy^4 + \frac{1}{q}y^2xy^2 + \frac{1}{q}y^4x$, $xyxy^2 + xy^3x + y^2xyx + qy^5$, $xy^3x^2 + xy^5 + y^4xy + qy^5x$, $xy^3xy + \frac{1}{q}xy^4x + \frac{1}{q}y^4x^2 + \frac{1}{q}y^6$, and $xy^6 - y^6x$ form the reduced Gröbner basis in the ideal of relations of the corresponding B.

Case 4. $a = -1$ or $a = -2b$ or $b = \pm\frac{1}{2}$. These cases follow from the already considered ones due to the isomorphism conditions in (H10). Indeed, one easily sees that our algebras in the case $b = \pm\frac{1}{2}$ are isomorphic to those with $a = 0$. The cases $a = 1$, $a = -1$, and $a = -2b$ are linked in a similar way.

Case 5. $2b = a$. Since $(a, b) \neq (0, 0)$, we have $a \neq 0$ and $b \neq 0$. Scaling x and y, we can turn F into $F = x^4 + x^2y^{2\circlearrowleft} + \frac{1}{2}xyxy^{\circlearrowleft} + qy^4$ with $q = a^{-2} \in \mathbb{K}^*$. Since $(a, b) \neq \pm(1, 1/2)$, we have $q \neq 1$. The defining relations of A now are $x^3 + xy^2 - yxy + y^2x$ and $x^2y - xyx + yx^2 + qy^3$. The element $g = xyxy - y^2x^2$ is central in A and $x^3 + xy^2 - yxy + y^2x$, $x^2y - xyx + yx^2 + qy^3$, $xyxy - y^2x^2$, and $xy^3 - y^3x$ form the reduced Gröbner basis in the ideal of relations of the corresponding B.

Case 6. $2b = a^2$. Since $(a, b) \neq (0, 0)$, we have $a \neq 0$ and $b \neq 0$. Scaling x and y, we can turn F into $F = x^4 + x^2y^{2\circlearrowleft} + \frac{a}{2}xyxy^{\circlearrowleft} + \frac{1}{a^2}y^4$. Since the case $b = \pm\frac{1}{2}$ is already considered, we can assume that $a^2 \neq 1$. The defining relations of A now are $x^3 + xy^2 + ayxy + y^2x$ and $x^2y + axyx + yx^2 + \frac{1}{a^2}y^3$. The element $g = xyxy + axy^2x - \frac{1}{a}y^2x^2 - \frac{1}{a}y^4$ is central in A and $x^3 + xy^2 + ayxy + y^2x$, $x^2y + axyx + yx^2 + \frac{1}{a^2}y^3$, $xyxy + axy^2x - \frac{1}{a}y^2x^2 - \frac{1}{a}y^4$, $xyx^2 + \frac{1}{a^2}xy^3 - y^2xy - \frac{1}{a}y^3x$, $xy^2x^2 + \frac{1}{a}y^3xy$, $xy^3x + \frac{1}{a}y^5$, $xy^2xyx + \frac{1}{a^3}xy^5 -$

$\frac{1}{a^2}y^3xy^2 - \frac{1}{a^2}y^5x$, $xy^2xy^2 + xy^4x - \frac{1}{a}y^3xyx - y^6$, $xy^4xy + \frac{1}{a}xy^5x - \frac{1}{a}y^5x^2 - \frac{1}{a^2}y^7$, $xy^4x^2 + \frac{1}{a^2}xy^6 - \frac{1}{a}y^5xy - y^6x$, $xy^5xy + \frac{1}{a}xy^6x + \frac{1}{a}y^6x^2 + \frac{1}{a^3}y^8$, $xy^5x^2 + xy^7 + y^6xy + ay^7x$, $xy^6xy - y^7x$, and $xy^8 - y^8x$ comprise the reduced Gröbner basis in the ideal of relations of the corresponding B.

Case 7. $4b^2 - 2a^2 + 1 = 0$ or $2b = -a^2$. As in Case 4, these cases follow from the already considered ones due to the isomorphism conditions in (H10). Indeed, one easily sees that our algebras in the case $2b = -a^2$ are isomorphic to those with $2b = a^2$ as well as to those with $4b^2 - 2a^2 + 1 = 0$. It remains to notice that Cases 1–7 exhaust all possibilities. □

LEMMA 3.5. *Let $F \in \mathcal{P}_{2,4}$ be given by* (H10) *of Lemma* 3.2 *with parameters α, β (we want to reserve letters a and b) such that $\alpha\beta(\alpha^2 - 1)(4\beta^2 - \alpha^2)(4\beta^2 - 1)(4\beta^2 - \alpha^4)(4\beta^2 - 2\alpha^2 + 1) \neq 0$. Then $A = A_F$ is exact and $H_A = (1 + t)^{-1}(1 - t)^{-3}$.*

Proof. A scaling turns F into $F = x^4 + x^2y^{2\circlearrowright} + \frac{a}{2}xyxy^{\circlearrowright} + by^4$ with $a, b \in \mathbb{K}$ given by $a = \frac{2\beta}{\alpha}$ and $b = \frac{1}{\alpha^2}$. In terms of a and b, the assumption about α and β reads as follows: $a \neq 0$, $b \neq 1$, $a^2 \neq 1$, $b + a^2 \neq 2$, $a^2b \neq 1$, and $b \neq a^2$. Computing $\delta_x F$ and $\delta_y F$, we see that A is given by generators x and y and relations

$$x^3 = -xy^2 - ayxy - y^2x, \qquad x^2y = -axyx - yx^2 - by^3. \tag{3.1}$$

A direct computation allows to find all elements of the reduced Gröbner basis of the ideal of relations up to degree 5. They correspond to the relations

$$xyx^2 = \frac{1-b}{1-a^2}xy^3 + y^2xy - \frac{a(1-b)}{1-a^2}y^3x; \quad xyxyx = -\frac{1}{a}xy^2x^2 - \frac{1-a^2b}{a(1-a^2)}xy^4 - \frac{1}{a}y^2xy^2 + \frac{1-b}{1-a^2}y^3xy;$$

$$xy^2xy = -\frac{a(1-b)(2-b-a^2)}{(1-a^2)(1-a^2b)}xy^3x - \frac{a(1-b)}{1-a^2b}yxyxy + \frac{1-a^2}{1-a^2b}yxy^2x + \frac{(1-b)(2-a^2b-a^2)}{(1-a^2)(1-a^2b)}y^3x^2 + \frac{b(1-b)}{1-a^2b}y^5;$$

$$xyxy^2 = -\frac{2-b-a^2}{1-a^2b}xy^3x + \frac{a^2(1-b)}{1-a^2b}yxyxy - \frac{a(1-a^2)}{1-a^2b}yxy^2x - y^2xyx - \frac{a(1-b)}{1-a^2b}y^3x^2 - \frac{ab(1-b)}{1-a^2b}y^5.$$

This provides a multiplication table in A for degrees up to 5, which allows to verify that

$$g = -a(1 - b)xyxy + (1 - a^2)xy^2x + (1 - b)y^2x^2 - b(1 - a^2)y^4,$$

commutes with both x and y and therefore is central in A. Now we consider the algebra

$$B = A/I, \text{ where } I \text{ is the ideal in } A, \text{ generated by } g,$$

as well as the degree-graded right B-module

$$M = B/yB.$$

Note that using the above Gröbner basis elements for A, one easily sees that Hilbert series of M starts as $H_M = 1 + 2t + 2t^2 + 2t^3 + 2t^4 + 2t^5 + \ldots$ By the same token,

$$H_A = 1 + 2t + 4t^2 + 6t^3 + 9t^4 + 12t^5 + \ldots \tag{3.2}$$

According to Lemma 2.12,

$$H_A \geqslant (1 + t)^{-1}(1 - t)^{-3}. \tag{3.3}$$

By the same lemma, the proof will be complete if we show that

$$H_A = (1 + t)^{-1}(1 - t)^{-3} \text{ and } A \text{ has no non-trivial right annihilators.} \tag{3.4}$$

We start by proving the following two statements:

$$H_M(t) = 1 + \sum_{n=1}^{\infty} 2t^n \implies \text{(3.4) is satisfied}, \tag{3.5}$$

$$\text{if } k \in \mathbb{N} \text{ and } \dim M_j \leqslant 2 \text{ for } 1 \leqslant j \leqslant k, \text{ then } \dim M_j = 2 \text{ for } 1 \leqslant j \leqslant k. \tag{3.6}$$

Assume that $k \in \mathbb{N}$ and $\dim M_j \leqslant 2$ for $1 \leqslant j \leqslant k$. Clearly, $\dim B_j = \dim yB_{j-1} + \dim M_j$ for $j \in \mathbb{N}$. It follows that $\dim B_j \leqslant 2j$ for $1 \leqslant j \leqslant k$ and the inequalities turn into equalities if and only if $\dim M_j = 2$ for $1 \leqslant j \leqslant k$ and $yu \neq 0$ for every degree $< k$ homogeneous $u \in B$. Next, $\dim A_j = \dim gA_{j-4} + \dim B_j$ for all $j \geqslant 4$. Using this recurrent inequality and the initial data (3.2), we see that for $j \leqslant k$, $\dim A_j$ does not exceed the jth coefficient of $(1+t)^{-1}(1-t)^{-3}$ and the inequalities turn into equalities if and only if $\dim B_j = 2j$ for $1 \leqslant j \leqslant k$ and $gu \neq 0$ for every degree $\leqslant k-4$ homogeneous $u \in A$. However, by (3.3), turn into equalities they must. In particular, we must have $\dim M_j = 2$ for $1 \leqslant j \leqslant k$, which proves (3.6). In order to prove (3.5), we apply the above argument with arbitrarily large k. It follows that the equality $H_M(t) = 1 + \sum_{n=1}^{\infty} 2t^n$ not only yields $H_A = (1+t)^{-1}(1-t)^{-3}$, but also ensures that $yu \neq 0$ for every non-zero $u \in B$ and $gu \neq 0$ for every non-zero $u \in A$. In order to complete the proof, it suffices to show that $yu \neq 0$ for every non-zero $u \in A$. Assume the contrary. Then pick a non-zero homogeneous $u \in A$ of smallest possible degree such that $yu = 0$ in A. Since B is a quotient of A, $yu = 0$ in B. Hence, $u = 0$ in B. Then $u = gv$ in A for some $v \in A$. The equality $yu = 0$ yields $gyv = 0$ and therefore $yv = 0$ in A. Since the degree of v is smaller (by 4) than the degree of u, we have arrived to a contradiction. This concludes the proof of (3.5).

According to (3.5), the proof will be complete if we verify that $H_M(t) = 1 + \sum_{n=1}^{\infty} 2t^n$. By definition of B and the above formulas for the low degree elements of the Gröbner basis for A, we see that the following relations are satisfied in B:

$$x^3 = -xy^2 - ayxy - y^2x, \qquad x^2y = -axyx - yx^2 - by^3, \tag{3.7}$$

$$xyx^2 = \tfrac{1-b}{1-a^2}xy^3 + y^2xy - \tfrac{a(1-b)}{1-a^2}y^3x; \quad xyxy = \tfrac{1-a^2}{a(1-b)}xy^2x + \tfrac{1}{a}y^2x^2 - \tfrac{b(1-a^2)}{a(1-b)}y^4. \tag{3.8}$$

For each $k \in \mathbb{Z}_+$, consider the following property:

(Ω_k) $\dim M_j = 2$ for $1 \leqslant j \leqslant k+3$, M_{k+3} is spanned by xy^{k+2} and $xy^{k+1}x$ and there exist $a_k, b_k \in \mathbb{K}$ such that the equalities $xy^kx^2 = a_kxy^{k+2}$ and $xy^kxy = b_kxy^{k+1}x$ hold in M.

Note that if (Ω_k) is satisfied, then a_k and b_k are uniquely determined. Indeed, otherwise xy^{k+2} and $xy^{k+1}x$ would be linearly dependent in M. According to (3.7) and (3.8),

$$\Omega_0 \text{ and } \Omega_1 \text{ are satisfied with } a_0 = -1, \ b_0 = -a, \ a_1 = \tfrac{1-b}{1-a^2}, \text{ and } b_1 = \tfrac{1-a^2}{a(1-b)}. \tag{3.9}$$

Note also that

if $k \in \mathbb{Z}_+$, $\dim M_j = 2$ for $1 \leqslant j \leqslant k+1$, M_{k+2} is spanned by xy^{k+1} and xy^kx and
$xy^kx^2 = a_kxy^{k+2}$, $xy^kxy = b_kxy^{k+1}x$ in M for some $a_k, b_k \in \mathbb{K}$, then (Ω_k) holds. $\tag{3.10}$

Indeed, by (3.6), $\dim M_{k+2} = 2$. Since M_{k+2} is spanned by xy^{k+1} and xy^kx, M_{k+3} is spanned by xy^{k+2}, xy^kxy, $xy^{k+1}x$ and xy^kx^2. By the equations in (3.10), M_{k+3} is spanned by xy^{k+2}, and $xy^{k+1}x$ and $\dim M_{k+3} = 2$ by (3.6). Thus (Ω_k) holds.

Reducing the overlaps $xy^k x^2 y = (xy^k x^2) y = xy^k (x^2 y)$ and $xy^k x^3 = (xy^k x^2) x = xy^k (x^3)$ by means of (3.7) and the equations from (Ω_k), we obtain

if $k \in \mathbb{Z}_+$ and Ω_k is satisfied, then

$$(ab_k + 1) xy^{k+1} x^2 + (a_k + b) xy^{k+3} = (b_k + a) xy^{k+1} xy + (a_k + 1) xy^{k+2} x = 0 \ \text{in} \ M.$$
(3.11)

Reducing $xy^{k-1} xyxy = (xy^{k-1} xy) xy = xy^{k-1} (xyxy)$ and $xy^{k-1} xyx^2 = (xy^{k-1} xy) x^2 = xy^{k-1} (xyx^2)$ by means of (3.7) and (3.8) and the equations from (Ω_k) and (Ω_{k-1}), we get

if $k \in \mathbb{N}$ and both Ω_{k-1} and Ω_k are satisfied, then

$$\left(\tfrac{1}{a} + b_{k-1} + \tfrac{1-a^2 b}{a(1-b)} b_{k-1} b_k \right) xy^{k+1} x^2 + \left(bb_{k-1} - \tfrac{b(1-a^2)}{a(1-b)} \right) xy^{k+3} = 0,$$
(3.12)
$$\left(1 + ab_{k-1} + \tfrac{2-b-a^2}{1-a^2} b_{k-1} b_k \right) xy^{k+1} xy + \left(b_{k-1} - \tfrac{a(1-b)}{1-a^2} \right) xy^{k+2} x = 0 \ \text{in} \ M.$$

Assume now that $k \in \mathbb{N}$ and both Ω_{k-1} and Ω_k are satisfied. We consider the following three options:

(O1) $\left(ab_k + 1, \tfrac{1}{a} + b_{k-1} + \tfrac{1-a^2 b}{a(1-b)} b_{k-1} b_k \right) = (0, 0)$;

(O2) $\left(b_k + a, 1 + ab_{k-1} + \tfrac{2-b-a^2}{1-a^2} b_{k-1} b_k \right) = (0, 0)$;

(O3) $\left(ab_k + 1, \tfrac{1}{a} + b_{k-1} + \tfrac{1-a^2 b}{a(1-b)} b_{k-1} b_k \right) \neq (0, 0)$ and $\left(b_k + a, 1 + ab_{k-1} + \tfrac{2-b-a^2}{1-a^2} b_{k-1} b_k \right) \neq (0, 0)$,

which cover all possibilities. First observe that according to (3.11), (3.12), and (3.10) ,

$$\text{if (O3) holds, then } (\Omega_{k+1}) \text{ is satisfied.}$$
(3.13)

Assume now that (O2) holds. By the equalities in (O2), $b_k = -a$ and $b_{k-1} = \tfrac{1-a^2}{a(1-b)}$. The conditions $a^2 b \neq 1$ and $a^2 + b \neq 2$ allow to check that $b_{k-1} \neq -a$ and $b_{k-1} \neq -\tfrac{1}{a}$. Then (3.11) applied with $k-1$ instead of k yields

$$b_k = -\tfrac{a_{k-1}+1}{b_{k-1}+a} \ \text{and} \ a_k = -\tfrac{a_{k-1}+b}{ab_{k-1}-1}.$$
(3.14)

Plugging the above expressions for b_{k-1} and b_k into the first equation in (3.14), we get $a_{k-1} = \tfrac{b(1-a^2)}{1-b}$. Plugging this together with $b_{k-1} = \tfrac{1-a^2}{a(1-b)}$ into the second equation in (3.14), we get (after cancellations to perform which we need the assumptions about a and b) $a_k = -b$. Now plugging $b_k = -a$, $a_k = -b$, $a_{k-1} = \tfrac{b(1-a^2)}{1-b}$, and $b_{k-1} = \tfrac{1-a^2}{a(1-b)}$ into the equalities in (3.11) and (3.12), we see that $xy^{k+1} x^2 = xy^{k+2} x = 0$ in M. Then M_{k+4} is spanned by $xy^{k+1} xy$ and xy^{k+3}. Using (3.7), we reduce the overlaps $xy^{k+1} x^3 = (xy^{k+1} x^2) x = xy^{k+1} (x^3)$ and $xy^{k+1} x^2 y = (xy^{k+1} x^2) y = xy^{k+1} (x^2 y)$ to get

$$xy^{k+1} xy^2 + xy^{k+3} x = xy^{k+1} xyx + \tfrac{b}{a} xy^{k+4} = 0 \ \text{in} \ M.$$

Then M_{k+5} is spanned by $xy^{k+3} x$ and xy^{k+4}. Using the above display together with (3.8), we reduce the overlaps $xy^{k+1} xyx^2 = (xy^{k+1} xyx) x = xy^{k+1} (xyx^2)$ and $xy^{k+1} xyxy = (xy^{k+1} xyx) y = xy^{k+1} (xyxy)$ to get

$$xy^{k+3} x^2 + bxy^{k+5} = xy^{k+3} xy + \tfrac{1}{a} xy^{k+4} x = 0 \ \text{in} \ M.$$

By (3.10), we see that Ω_{k+3} is satisfied with $a_{k+3} = -b$ and $b_{k+3} = -\frac{1}{a}$. Dealing in a similar way with the overlaps $xy^{k+3}x^3 = (xy^{k+3}x^2)x = xy^{k+3}(x^3)$ and $xy^{k+2}xyxy = (xy^{k+2}xyx)y = xy^{k+2}(xyxy)$, we get

$$xy^{k+4}xy = \frac{a(1-b)}{1-a^2}xy^{k+5}x \text{ and } xy^{k+4}x^2 = \frac{b(1-a^2)}{1-b}xy^{k+6} \text{ in } M,$$

The last display together with (3.10) shows that Ω_{k+4} is satisfied. Hence,

$$\text{if (O2) holds, then } (\Omega_{k+3}) \text{ and } (\Omega_{k+4}) \text{ are satisfied.} \tag{3.15}$$

Finally, assume that (O1) holds. By the equalities in (O1), $b_k = -\frac{1}{a}$ and $b_{k-1} = \frac{a(1-b)}{1-a^2}$. The conditions $a^2b \neq 1$ and $a^2 + b \neq 2$ yield $b_{k-1} \neq -a$ and $b_{k-1} \neq -\frac{1}{a}$. As above, this means that (3.14) holds. Plugging the expressions for b_{k-1} and b_k into the first equation in (3.14), we get $a_{k-1} = \frac{1-b}{1-a^2}$. Plugging this together with $b_{k-1} = \frac{a(1-b)}{1-a^2}$ into the second equation in (3.14), we obtain $a_k = -1$. Plugging $a_{k-1} = \frac{1-b}{1-a^2}$, $b_{k-1} = \frac{a(1-b)}{1-a^2}$, $b_k = -\frac{1}{a}$, and $a_k = -1$ into the equalities from (3.11) and (3.12), we get $xy^{k+1}xy = xy^{k+3} = 0$ in M. Now M_{k+4} is spanned by $xy^{k+1}x^2$ and $xy^{k+2}x$. From this and (3.7), it follows that M_{k+5} is spanned by $xy^{k+2}x^2$ and $xy^{k+2}xy$. Now an inductive procedure (use (3.6)) shows that M_j is two-dimensional for every j. That is,

$$\text{if (O1) holds, then } M_j \text{ is two-dimensional for every } j \in \mathbb{N}. \tag{3.16}$$

Note that if (Ω_k) holds for infinitely many k, then M_j is two-dimensional for every $j \in \mathbb{N}$ as well. Applying (3.15) and (3.13) inductively ((3.9) serves as the basis of induction) and using (3.16), we see that no matter the case, M_j is two-dimensional for every $j \in \mathbb{N}$. This completes the proof. $\square$

3.1. Proof of Theorem 1.9. Combining Lemmas 3.2, 3.3, 3.4, and 3.5, we see that all statements of Theorem 1.9 hold with isomorphism of A_F and A_G condition replaced by equivalence of F and G (with respect to the $GL_2(\mathbb{K})$ action by linear substitutions). By Lemma 1.5, these two equivalences are the same for proper potentials. Thus, all that remains is to show that algebras from (P24–P28) are pairwise non-isomorphic. Since isomorphic graded algebras have the same Hilbert series, it remains to verify that three algebras from (P24–P26) are pairwise non-isomorphic. Now (P25) is singled out by being non-monomial (it is easy to see that it is not isomorphic as a graded algebra to a monomial one), while algebras from (P24) and (P26) are monomial. Algebras in (P24) and (P26) are non-isomorphic since the first one has cubes in the space of degree 3 relations, while the second one has no such thing.

4. Potential algebras A_F for $F \in \mathcal{P}_{3,3}$. Throughout this section, we equip the monomials in x, y, z with the left-to-right degree-lexicographical ordering assuming $x > y > z$. The following statement is elementary.

LEMMA 4.1. *The kernel of the canonical homomorphism from $\mathbb{K}\langle x, y, z \rangle$ onto $\mathbb{K}[x, y, z]$ (=abelianization) intersects $\mathcal{P}_{3,3}$ by the one-dimensional space spanned by $xyz^\circlearrowleft - xzy^\circlearrowleft$.*

LEMMA 4.2. *Let $F \in \mathcal{P}_{3,3}$. Then by a linear substitution, F can be turned into one of the following forms*:

(G1) $F = 0$;   (G2) $F = z^3$;   (G3) $F = yz^{2^\circlearrowleft}$;   (G4) $F = y^3 + z^3$;   (G5) $F = xyz^{\circlearrowleft}$;

(G6) $F = x^3 + y^3 + z^3$;   (G7) $F = xz^{2^\circlearrowleft} + y^3$;   (G8) $F = xz^{2^\circlearrowleft} + y^2 z^{\circlearrowleft}$;

(G9) $F = z^3 + xyz^{\circlearrowleft}$;   (G10) $F = xyz^{\circlearrowleft} + (y+z)^3$;   (G11) $F = yz^{2^\circlearrowleft} + xyz^{\circlearrowleft} - xzy^{\circlearrowleft}$;

(G12) $F = y^3 + z^3 + xyz^{\circlearrowleft} - xzy^{\circlearrowleft}$;   (G13) $F = y^3 + xz^{2^\circlearrowleft} + xyz^{\circlearrowleft} - xzy^{\circlearrowleft}$;

(G14) $F = xz^{2^\circlearrowleft} + y^2 z^{\circlearrowleft} + xyz^{\circlearrowleft} - xzy^{\circlearrowleft}$;   (G15) $F_a = xyz^{\circlearrowleft} - axzy^{\circlearrowleft}$ with $a \neq 0$;

(G16) $F_a = z^3 + xyz^{\circlearrowleft} + axzy^{\circlearrowleft}$ with $a \neq 0$;   (G17) $F_a = (y+z)^3 + xyz^{\circlearrowleft} + axzy^{\circlearrowleft}$

with $a \notin \{0, -1\}$;   (G18) $F_{a,b} = x^3 + y^3 + z^3 + axyz^{\circlearrowleft} + bxzy^{\circlearrowleft}$

with $(a+b)^3 \neq -1$, $(a,b) \neq (0,0)$ and $(a^3, b^3) \neq (1, 1)$,

where $a, b \in \mathbb{K}$. Moreover, to which of the above 18 forms F can be turned into is uniquely determined by F. For $F = F_a$ from (G15–G17) F_a can be obtained from F_b by a linear substitution if and only if $a = b$ or $ab = 1$. Finally, for $F = F_{a,b}$ from (G17), $F_{a,b}$ and $F_{a',b'}$ can be obtained from one another by a linear substitution if and only if they belong to the same orbit of the group action generated by two maps $(a, b) \mapsto (\theta a, \theta b)$ and $(a, b) \mapsto \left(\frac{1 + \theta a + \theta^2 b}{1 + a + b}, \frac{1 + \theta^2 a + \theta b}{1 + a + b} \right)$. This group has 24 elements and is isomorphic to $SL_2(\mathbb{Z}_3)$.

Proof. Let $F \in \mathcal{P}_{3,3}$. First, we show that F can be turned into exactly one of (G1–G18) by a linear sub. Let $G \in \mathbb{K}[x, y, z]$ be the abelianization of F. By Lemma 2.3 by means of a linear substitution, G can be turned into exactly one of the forms (W1–W10). Thus, we can assume from the start that G is in one of the forms (W1–W10). If G is given by (W1), Lemma 4.1 implies that $F = x^3 + y^3 + z^3 + axyz^{\circlearrowleft} + bxzy^{\circlearrowleft}$ with $(a + b)^3 \neq -1$. It is shown in [12] that if $a^3 = b^3 = 1$, then A_F is isomorphic to the potential algebra given by the potential from either (G5) or (G6), while these two potential algebras are non-isomorphic. If $a = b = 0$, F falls into (G6) directly. Otherwise F is from (G1). If G is given by (W10), then by Lemma 4.1, $F = c(xyz^{\circlearrowleft} - xzy^{\circlearrowleft})$ with $c \in \mathbb{K}$. If $c = 0$, F is given by (G1), while if $c \neq 0$ a scaling turns F into (G15) with $a = 1$. If G is given by (W6), then by Lemma 4.1, $F = (c + \frac{1}{3}) xyz^{\circlearrowleft} - cxzy^{\circlearrowleft}$ with $c \in \mathbb{K}$ and a scaling turns F into (G15) with $a \neq 1$ if $c \neq 0$ and into (G5) if $c = 0$. The required isomorphism statement for the potentials from (G1), (G5), (G6), (G15), and (G18) is verified in [12].

If G is given by (W9), by Lemma 4.1, $F = z^3 + s(xyz^{\circlearrowleft} - xzy^{\circlearrowleft})$ with $s \in \mathbb{K}$. If $s = 0$, F is given by (G2). If $s \neq 0$, a scaling brings F to the form (G16) with $a = -1$. If G is given by (W8), by Lemma 4.1, $F = \frac{1}{3} yz^{2^\circlearrowleft} + s(xyz^{\circlearrowleft} - xzy^{\circlearrowleft})$ with $s \in \mathbb{K}$. If $s = 0$, F acquires form (G3) after scaling. If $s \neq 0$, a scaling brings F to the form (G11). If G is given by (W7), by Lemma 4.1, $F = y^3 + z^3 + s(xyz^{\circlearrowleft} - xzy^{\circlearrowleft})$ with $s \in \mathbb{K}$. If $s = 0$, F is given by (G4). If $s \neq 0$, a scaling brings F to the form (G12). If G is given by (W2), by Lemma 4.1, $F = y^3 + \frac{1}{3} xz^{2^\circlearrowleft} + s(xyz^{\circlearrowleft} - xzy^{\circlearrowleft})$ with $s \in \mathbb{K}$. If $s = 0$, F acquires form (G7) after scaling. If $s \neq 0$, a scaling brings F to the form (G13). If G is given by (W4), by Lemma 4.1, $F = \frac{1}{3} xz^{2^\circlearrowleft} + \frac{1}{3} y^2 z^{\circlearrowleft} + s(xyz^{\circlearrowleft} - xzy^{\circlearrowleft})$ with $s \in \mathbb{K}$. If $s = 0$, F acquires form (G8) after scaling. If $s \neq 0$, a scaling brings F to the form (G14). If G is given by (W5), by Lemma 4.1, $F = z^3 + \left(s + \frac{1}{3}\right) xyz^{\circlearrowleft} - sxzy^{\circlearrowleft}$ with $s \in \mathbb{K}$. If $s = 0$, a scaling brings F to the form (G9). If $s = -\frac{1}{3}$, swapping x and y and a scaling brings F to the form (G9) again. If

$s \neq 0$ and $s \neq -\frac{1}{3}$, a scaling turns F into the form (G16) with $a \neq -1$. Finally, if G is of the form (W3), by Lemma 4.1, $F = (y+z)^3 + (s + \frac{1}{3})xyz^{\circlearrowleft} - sxzy^{\circlearrowleft}$ with $s \in \mathbb{K}$. Same as in the previous case, if $s = 0$ or $s = -\frac{1}{3}$ a scaling or the same together with swapping of y and z turns F into the form (G10). If $s \neq 0$ and $s \neq -\frac{1}{3}$, a scaling turns F into the form (G17) (automatically, $a \neq -1$).

The fact that F from the list (G1–G18) with different labels are non-equivalent (cannot be obtained from one another by a linear sub) follows from the non-equivalence of polynomials with different labels from the list (W1–W10), the isomorphism statements in the first paragraph (which also covers isomorphisms within (G15) and (G18)) as well as the trivial observation that a symmetric (not just cyclicly) element of $\mathcal{P}_{3,3}$ cannot be equivalent to a non-symmetric one. It remains to prove the statements about equivalence within each of (G16) and (G17). Let F_a, F_b be two potentials both from (G16). Their abelianizations are $G_a = z^3 + (1 + a)xyz$ and $G_b = z^3 + (1 + b)xyz$. A linear sub turning F_a to F_b must transform G_a into G_b. If $a = -1$, such a thing can obviously exist only if $b = -1$. Thus, we can assume that $a \neq -1$ and $b \neq -1$. Now it is straightforward to check that that such subs are among the scalings $x \to px$, $y \to qy$, and $z \to rz$ or scalings composed with the swap of x and y: $x \to py$, $y \to qx$, and $z \to rz$ with $p, q, r \in \mathbb{K}^*$, $r^3 = 1$. In order for an F_a to be transformed to any $F_{a'}$, we need additionally $pqr = 1$ in the first case and $pqra = 1$ in the second. Analyzing the way how these subs act on F_a, we see that F_a is transformed to itself if no swap is involved and to $F_{a^{-1}}$ otherwise. The situation with F_a from (G17) is similar. $\qquad\square$

LEMMA 4.3. *Let $F \in \mathcal{P}_{3,3}$ be given by* (G18) *of Lemma 4.2. Then the potential algebra $A = A_F$ is Koszul, exact, non-PBW and satisfies $H_A = (1 - t)^{-3}$.*

Proof. The fact that A, known also as a Sklyanin algebra, is Koszul and satisfies $H_A = (1 - t)^{-3}$ is proved in [2]. Different proofs are presented in [11] and [12]. In [11], it is shown that these algebras are non-PBW. Now, by Lemma 2.11, Koszulity of A yields its exactness. This exactness can be also seen in [4]. $\qquad\square$

LEMMA 4.4. *Let $F \in \mathcal{P}_{3,3}$ be given by one of* (G11–G17) *of Lemma 4.2. Then the potential algebra $A = A_F$ is PBW, Koszul, exact and satisfies $H_A = (1 - t)^{-3}$.*

Proof. Just apply Lemma 2.10: the potentials F from each of (G11–G17) satisfy the assumptions. $\qquad\square$

LEMMA 4.5. *Let $F \in \mathcal{P}_{3,3}$ be given by one of* (Gj) *of Lemma 4.2 with $1 \leqslant j \leqslant 8$. Then the potential algebra $A = A_F$ is PBW, Koszul, and non-exact. The Hilbert series of A is given by $H_A = (1 - 3t)^{-1}$ if $j = 1$, $H_A = \frac{1+t}{1-2t-2t^2}$ if $j = 2$, $H_A = \frac{1+t}{1-2t-t^2}$ if $j \in \{3, 4\}$ and $H_A = \frac{1+t}{1-2t}$ if $5 \leqslant j \leqslant 8$.*

Proof. An easy computation shows that the defining relations $\delta_x F$, $\delta_y F$, and $\delta_z F$ form a Gröbner basis in the ideal of relations of A. Hence, A is PBW and therefore Koszul. The computation of the Hilbert series is now easy and routine. $\qquad\square$

LEMMA 4.6. *Let $F \in \mathcal{P}_{3,3}$ be given by* (G9) *and $A = A_F$. Then A is non-Koszul, non-PBW, non-exact, non-proper and satisfies $H_A = \frac{1+t+t^2+t^3+t^4}{1-2t+t^2-t^3-t^4}$.*

Proof. Since $F = z^3 + xyz^{\circlearrowleft}$, the defining relations of A are yz, zx, and $xy + z^2$. The ideal of relations of A turns out to have a finite Gröbner basis comprising yz, zx,

$xy + z^2$, and z^3. This allows us to find an explicit expression for the Hilbert series of A: $H_A = \frac{1+t+t^2+t^3+t^4}{1-2t+t^2-t^3-t^4}$. Next, one easily checks that the Koszul dual $A^!$ has the Hilbert series $H_{A^!} = 1 + 3t + 3t^2 + 2t^3$. Then, the duality formula $H_A(t)H_{A^!}(-t) = 1$ fails. Hence, A is non-Koszul and therefore non-PBW. By Lemma 2.11, A is non-exact. The above formula for H_A yields $\dim A_3 = 11$ and therefore A is non-proper by Lemma 1.5. $\square$

LEMMA 4.7. *Let $F \in \mathcal{P}_{3,3}$ be given by* (G10) *and $A = A_F$. Then A is proper, non-Koszul, non-PBW, non-exact and satisfies $H_A = \frac{1+2t+3t^2+3t^3+2t^4+t^5}{1-t-t^3-2t^4}$.*

Proof. Since $F = (y+z)^3 + xyz^\circ$, the defining relations of A are $xy + (y+z)^2$, $zx + (y+z)^2$, and yz. The ideal of relations of B turns out to have a finite Gröbner basis comprising $xy - zx$, $y^2 + zx + zy + z^2$, yz, $xzx + xzy + xz^2 + z^2x$, zxz, and z^3. This allows us to find an explicit expression for the Hilbert series of A: $H_A = \frac{1+2t+3t^2+3t^3+2t^4+t^5}{1-t-t^3-2t^4}$. Next, the dual algebra $A^!$ is easily seen to have the Hilbert series $H_{A^!} = 1 + 3t + 3t^2 + t^3$. Clearly, the duality formula $H_A(t)H_{A^!}(-t) = 1$ fails and therefore A is non-Koszul. Hence, A is non-PBW. By Lemma 2.11, A is non-exact. The above formula for H_A yields $\dim A_3 = 10$ and therefore A is proper by Lemma 1.5. $\square$

4.1. Proof of Theorem 1.7.

Combining Lemmas 4.2, 4.3, 4.4, 4.5, 4.6, and 4.7, we see that all statements of Theorem 1.7 hold with isomorphism of A_F and A_G condition replaced by equivalence of F and G (with respect to the $GL_3(\mathbb{K})$ action by linear substitutions). By Remark 1.6, it remains to show that algebras (P10–P14) are pairwise non-isomorphic and algebras (P15–P18) are pairwise non-isomorphic. Since isomorphic graded algebras have the same Hilbert series, it remains to verify that four algebras from (P10–P13) are pairwise non-isomorphic and that the algebras from (P15) and (P16) are non-isomorphic. The latter holds because the algebra in (P15) is monomial, while the algebra in (P16) is not isomorphic to a monomial one. It remains to show that four algebras from (P10–P13) are pairwise non-isomorphic. The same argument on monomial algebras reduces the task to showing that the algebras in (P11) and (P12) are non-isomorphic and the algebras in (P10) and (P13) are non-isomorphic. The algebras in (P11) and (P12) are non-isomorphic since the (three-dimensional) space of quadratic relations for the first one is spanned by squares (of degree 1 elements), while the same space for the second algebra contains no squares at all. As for the algebras in (P10) and (P13), the second one sports just one (up to a scalar multiple) square in the space of quadratic relations, while the first one obviously has two linearly independent ones: x^2 and z^2.

5. Twisted potential algebras A_F with $F \in \mathcal{P}_{2,4}^*$.

We shall occasionally switch back and forth between denoting the generators x, y or x_1, x_2 meaning $x = x_1$ and $y = x_2$. The reasons are esthetic.

LEMMA 5.1. *For $a \in \mathbb{K}^*$, $a \neq 1$, let $F_a = x^3y + ax^2yx + a^2xyx^2 + a^3yx^3$ be the twisted potential of* (T34) *of Theorem 1.10 and $A^a = A_{F_a}$. Then the twisted potential algebras A^a are pairwise non-isomorphic, non-potential, non-proper and satisfy $H_{A^a} = \frac{1+t+t^2}{1-t-t^2}$.*

Proof. Clearly, A^a is presented by generators x, y and relations $x^2y + ayxy + a^2yx^2$ and x^3. It is easy to check that the defining relations of A^a form a Gröbner basis in the ideal of relations. Knowing the leading monomials x^3 and x^2y of the members of a Gröbner basis, we easily confirm that $H_{A^a} = \frac{1+t+t^2}{1-t-t^2}$. Next, we show that algebras A^a are pairwise non-isomorphic. Indeed, assume that a linear substitution facilitates an isomorphism between

A^a and A^b. As x^3 is the only cube (up to a scalar multiple) in the space of cubic relations for both A^a and A^b, our sub must map x to its own scalar multiple. Now it is elementary to check that that any such substitution leaves invariant each space R_a spanned by $x^2y + axyx + a^2yx^2$ and x^3. Since R_a are pairwise distinct, an isomorphism between A^a and A^b does exist only if $b = a$. Same type argument shows that each A^a is not isomorphic to any of three algebras from (P24–P26) of the already proven Theorem 1.9. Since these three algebras are the only cubic potential algebras on two generators with the Hilbert series $\frac{1+t+t^2}{1-t-t^2}$, it follows that A^a are non-potential. $\square$

LEMMA 5.2. *Each $F \in \mathbb{K}\langle x, y \rangle$ listed in* (T24–T33) *of Theorem 1.10 is a proper twisted potential such that the Jordan normal form of the corresponding twist is one block with eigenvalue -1 for F from* (T25), *one block with eigenvalue 1 for F from* (T26–27), *diagonalizable in all other cases with the two eigenvalues being a, a^{-1} for F from* (T24), *$a, -a^{-1}$ for F from* (T28), *$\theta, 1$ for F from* (T29), *$\theta^2, 1$ for F from* (T30), *$\xi_8, -\xi_8$ for F from* (T31), *$i\xi_8, -i\xi_8$ for F from* (T32), *$-1, -1$ for F from from* (T33). *Moreover, A_F is exact, non-potential and has the Hilbert series $(1 + t)^{-1}(1 - t)^{-3}$ for every F from* (T24–T33).

Proof. It is straightforward and elementary to check that each F is a twisted potential with the Jordan normal form of the twist being as specified. For F from (T24–T28) and (T33), a direct application of Lemma 2.9 shows that A_F is exact and has the Hilbert series $(1 + t)^{-1}(1 - t)^{-3}$. Assume now that $F = x^3y + yx^3 + axyx^2 + a^2x^2yx + y^4$ with $a^3 = 1 \neq a$. This covers (T29) and (T30). Then $A = A_F$ is presented by generators x, y and relations $x^3 + y^3$ and $x^2y + a^2xyx + ayx^2$. A direct computation shows that the defining relations together with $xy^3 - y^3x$ and $xyxy^2 - ayxyxy + a^2y^2xyx + 2ay^3x^2$ form a Gröbner basis in the ideal of relations of A. This allows to compute the Hilbert series $H_A = (1 + t)^{-1}(1 - t)^{-3}$ and to observe in the usual way that there are no non-trivial right annihilators in A. By Lemma 2.6, A is exact. Next, assume that $F = x^4 - ayx^3 - y^2x^2 + ay^3x + y^4 + xy^3 + x^2y^2 + x^3y$ with $a^2 = -1$. This covers (T31) and (T32). Again, the ideal of relations of $A = A_F$ has a finite Gröbner basis with the leading monomials of its members being x^3, x^2y, xyx^2, xy^4, xy^2x^2, $xyxy^2$, and xy^2xy^2 (seven members in total). We skip spelling out the exact formulas for the members of the basis this once since some of them turn out to be rather long. For instance, the last one is the sum of nine terms. Anyway, knowing the above leading terms allows to compute the Hilbert series $H_A = (1 + t)^{-1}(1 - t)^{-3}$ and to observe in the usual way that there are no non-trivial right annihilators in A. By Lemma 2.6, A_F is exact for F from (T29–T32). By Lemma 1.5, each A_F for F from (T24–T33) is proper. Since the corresponding twist (it is uniquely determined by A_F) is non-trivial, none of A_F is potential. $\square$

LEMMA 5.3. *Let $G \in \mathcal{P}_{2,4}^*$ be non-degenerate, $M \in GL_2(\mathbb{K})$ be the unique matrix providing the twist for G and assume that $A = A_G$ is non-potential. Assume also that the normal Jordan form of M consists of one block. If A is non-proper, then A is isomorphic to an A_F with F from* (T34) *of Theorem 1.10. If A is proper, then A is isomorphic to A_F for F from* (T25–T27) *of Theorem 1.10. Moreover, algebras A_F for F from* (T25–T27) *are pairwise non-isomorphic.*

Proof. By Remark 1.4, we can assume that $M = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}$. If $F = \sum\limits_{j,k,m,n=1}^{2} a_{j,k,m,n} x_j x_k x_m x_n$, then the inclusion $F \in \mathcal{P}_{2,4}(M)$ is equivalent to the following system of linear

equations on the coefficients of F:

$$a_{j,k,m,2} = \alpha a_{2,j,k,m} \text{ and } a_{j,k,m,1} = \alpha a_{1,j,k,m} + a_{2,j,k,m} \text{ for } 1 \leqslant j, k, m \leqslant 2. \qquad (5.1)$$

One easily sees that (5.1) has only zero solution unless $\alpha^4 = 1$. This leaves three cases to consider: $\alpha^2 = -1$, $\alpha = -1$ and $\alpha = 1$.

If $\alpha^2 = -1$, the space of solutions of (5.1) is one-dimensional, corresponding to $\mathcal{P}_{2,4}(M)$ being spanned by $G = yx^3 + \alpha x^3 y - \alpha xyx^2 - x^2 yx + \frac{1+\alpha}{2} x^4$. One easily sees that A_G is isomorphic to the algebra from (T34) with $a = \alpha$. By Lemma 5.1, it is non-proper.

If $\alpha = -1$, solving (5.1), we see that

$$\mathcal{P}_{2,4}(M) = \{F_{s,t} = s\left(\frac{1}{2}x^4 + yx^3 - x^3 y - xyx^2 + x^2 yx\right)$$
$$+ t(xyx^2 - x^2 yx - yx^2 y - xy^2 x + x^2 y^2 + y^2 x^2) : s, t \in \mathbb{K}\}.$$

If $t = 0$, A_F with $F = F_{s,t}$ is easily seen to be isomorphic to the algebra from (T34) with $a = -1$. Thus, we can assume that $t \neq 0$. A scaling turns t into 1, leaving us to deal with $F_{a,1}$ for $a \in \mathbb{K}$. These are twisted potentials from (T25). By Lemma 5.2, the corresponding algebras are proper. Using Remark 1.4 and Lemma 1.5, we see that only substitutions of the form $y \to py + qx, x \to px$ with $p \in \mathbb{K}^*$ and $q \in \mathbb{K}$ can provide an isomorphism between algebras in (T25). However, none of these substitutions changes the parameter a. Hence, algebras in (T25) are pairwise non-isomorphic.

If $\alpha = 1$, solving (5.1), we see that

$$\mathcal{P}_{2,4}(M) = \{F_{s,t,r} = s(x^2 y^{2\circlearrowleft} - xyxy^\circlearrowleft - xyx^2 + x^2 yx) + tx^3 y^\circlearrowleft + rx^4 : s, t, r \in \mathbb{K}\}.$$

If $s = 0$, $F_{s,t,r}$ is a potential. Thus, we can assume that $s \neq 0$. A substitution $x \to x, y \to y + bx$ with an appropriate $b \in \mathbb{K}$ kills r. Now, a scaling turns F into one of the twisted potentials from (T26) if $t \neq 0$ or to the twisted potential from (T27) if $t = 0$. By Lemma 5.2, the corresponding algebras are proper. Same argument as above shows that they are pairwise non-isomorphic. Algebras from (T25) and (T26–T27) cannot be isomorphic since they are proper and the Jordan normal forms of the twists do not match. $\qquad \square$

LEMMA 5.4. *Let $G \in \mathcal{P}_{2,4}^*$ be non-degenerate, $M \in GL_2(\mathbb{K})$ be the unique matrix providing the twist for G and assume that $A = A_G$ is non-potential. Assume also that M is diagonalizable. If A is non-proper, then A is isomorphic to an A_F with F from (T34) of Theorem 1.10. If A is proper, then A is isomorphic to A_F for F from (T24) or (T28–T33) of Theorem 1.10. Moreover, the corresponding algebras A_F are pairwise non-isomorphic.*

Proof. By Remark 1.4, we can assume that $M = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$. If $F = \sum_{j,k,m,n=1}^{2} a_{j,k,m,n} x_j x_k x_m x_n$, then the inclusion $F \in \mathcal{P}_{2,4}(M)$ is equivalent to the following system of linear equations on the coefficients of F:

$$a_{j,k,m,2} = \beta a_{2,j,k,m} \text{ and } a_{j,k,m,1} = \alpha a_{1,j,k,m} \text{ for } 1 \leqslant j, k, m \leqslant 2. \qquad (5.2)$$

One easily sees that (5.2) has only zero solution unless $1 \in \{\alpha, \beta, \alpha^3 \beta, \alpha^2 \beta^2, \alpha\beta^3\}$. The case $\alpha = \beta = 1$ is excluded since $F \notin \mathcal{P}_{2,4}$. If $1 \notin \{\alpha^3 \beta, \alpha^2 \beta^2, \alpha\beta^3\}$, then F is both degenerate and a potential. Indeed, F is either zero or is the fourth power of a degree 1 element. The cases $\alpha^3 \beta = 1$ and $\alpha\beta^3 = 1$ transform to one another when we swap x and y. Thus, we have just two options to consider: $\alpha^3 \beta = 1$ or $\alpha^2 \beta^2 = 1$.

First, assume that $\alpha^3\beta = 1$. Then $x^3y + \alpha x^2yx + \alpha^2xyx^2 + \alpha^3yx^3$ is in $\mathcal{P}_{2,4}(M)$. Moreover, $\mathcal{P}_{2,4}(M)$ is spanned by this one element unless $\alpha^3 = 1 \neq \alpha$, or $\alpha^8 = 1 \neq \alpha$. If F is a scalar multiple of $x^3y + \alpha x^2yx + \alpha^2xyx^2 + \alpha^3yx^3$, we fall into (T34) after scaling and A_F is non-proper by Lemma 5.1.

Next, assume that $\alpha\beta = -1$. By (5.2), $x^2y^2 + \alpha^2y^2x^2 + \alpha xy^2x - \alpha yx^2y \in \mathcal{P}_{2,4}(M)$ and $\mathcal{P}_{2,4}(M)$ is spanned by this one element unless $\alpha^4 = 1$. If F is a scalar multiple of $x^2y^2 + \alpha^2y^2x^2 + \alpha xy^2x - \alpha yx^2y$, a scaling sends F to (T28). By Lemma 5.2, the corresponding algebras are proper. Using Remark 1.4 and Lemma 1.5, we see that only scalings can provide an isomorphism between algebras in (T28) (unless $\alpha = \pm i$ in which case a separate easy argument is needed; we skip it now since we study this case below in detail anyway). Now one sees that algebras in (T28) are pairwise non-isomorphic.

Next, assume that $\alpha\beta = 1$. By (5.2), $x^2y^2 + \alpha^2y^2x^2 + \alpha xy^2x + \alpha yx^2y, xyxy + \alpha yxyx \in \mathcal{P}_{2,4}(M)$ and $\mathcal{P}_{2,4}(M)$ is spanned by these two elements unless $\alpha = -1$. That is,

$$F = s(x^2y^2 + \alpha^2y^2x^2 + \alpha xy^2x + \alpha yx^2y) + t(xyxy + \alpha yxyx) \text{ with } s, t \in \mathbb{K}.$$

If $s = 0$, A_F coincides with the potential algebra from (P26), which contradicts the assumptions. Thus, $s \neq 0$. Now a scaling turns F into the form (T24). By Lemma 5.2, the corresponding algebras are proper. Using Remark 1.4 and Lemma 1.5, we see that only scalings can provide an isomorphism between algebras in (T28) (unless $\alpha = \pm 1$ in which case a separate easy argument is needed; we skip it now since we study this case below in detail). Now one sees that algebras in (T24) are pairwise non-isomorphic.

It remains to consider the following finite set of options for (α, β): $(1, -1)$, $(-1, -1)$, $(a, 1)$ with $a^3 = 1 \neq a$, (a, a) with $a^2 = -1$ and $(a, -a)$ with $a^4 = -1$.

If $(\alpha, \beta) = (a, 1)$ with $a^3 = 1 \neq a$, then solving (5.2), we see that

$$\mathcal{P}_{2,4}(M) = \{F_{s,t} = s(x^3y + yx^3 + axyx^2 + a^2x^2yx) + ty^4 : s, t \in \mathbb{K}\}.$$

If $s = 0$, $F_{s,t}$ is a potential. If $t = 0$, $F_{s,t}$ falls into (T34). Thus, we can assume that $st \neq 0$. Now a scaling transforms $F_{s,t}$ into $F_{1,1}$, which is the twisted potential from (T29) if $a = \theta$ and (T30) if $a = \theta^2$. In both cases, the corresponding algebra is proper by Lemma 5.2.

Now assume that $(\alpha, \beta) = (b, -b)$ with $b^4 = -1$. Since changing b by $-b$ corresponds to swapping x and y, we can assume that $b \in \{\xi_8, i\xi_8\}$. Solving (5.2), we see that

$$\mathcal{P}_{2,4}(M) = \{G_{s,t} = s(x^2yx - b^3x^3y + b^2yx^3 + bxyx^2)$$
$$+ t(y^3x - b^3xy^3 + b^2yxy^2 - by^2xy) : s, t \in \mathbb{K}\}$$

If $st = 0$, the corresponding algebra is easily seen to fall into (T34): just scale or swap x and y and scale. By Lemma 5.1, the corresponding algebra is non-proper if $st = 0$. If $st \neq 0$, a scaling turns both s and t into 1. That is, if $st \neq 0$, $F_{s,t}$ is equivalent to $F_{1,1}$, which is easily seen to be proper. That is, $F_{s,t}$, if proper, is isomorphic to one of two algebras $F_{1,1}$ for $b = \xi_8$ or $F_{1,1}$ for $b = i\xi_8$. Denote $a = b^2$. That is, $a = i$ if $b = \xi_8$ and $a = -i$ if $b = i\xi_8$. In both cases, $a^2 = -1$. Note that the matrix $N = \begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix}$ is conjugate to M. It is easy to see that

$G = x^4 - ayx^3 - y^2x^2 + ay^3x + y^4 + xy^3 + x^2y^2 + x^3y$ belongs to $\mathcal{P}_{2,4}(N)$ and therefore G is equivalent to a member of $\mathcal{P}_{2,4}(M)$. This produces two algebras from (T31) and (T32). By Lemma 5.2, they are proper and therefore they must be isomorphic to algebras given by $F_{1,1}$ for $b = \xi_8$ or $F_{1,1}$ for $b = i\xi_8$.

If $(\alpha, \beta) = (1, -1)$, then solving (5.2), we see that

$$\mathcal{P}_{2,4}(M) = \{F_{s,t} = s(x^2y^2 + y^2x^2 - xy^2x + yx^2y) + tx^4 : s, t \in \mathbb{K}\}.$$

If $s = 0$, $F_{s,t}$ is a potential. If $s \neq 0$, a substitution given by $x \to px$, $y \to py + qx$ with appropriate $p \in \mathbb{K}^*$ and $q \in \mathbb{K}$ turns (s, t) into $(1, 0)$. Then F falls into (T28) with $a = 1$.

Assume now that $(\alpha, \beta) = (a, a)$ with $a^2 = -1$. Solving (5.2), we see that $\mathcal{P}_{2,4}(M)$ consists of

$$G_{s,t,r} = s(x^3y + ayx^3 - xyx^2 - ax^2yx) + t(x^2y^2 + ayx^2 - y^2x^2 - axy^2x)$$
$$+ r(y^3x + axy^3 - yxy^2 - ay^2xy),$$

with $s, t, r \in \mathbb{K}$. If we perform a (non-degenerate) linear substitution $x \to \lambda x + \mu y$, $y \to \gamma x + \delta y$ with $D = \lambda\delta - \mu\gamma \in \mathbb{K}^*$, $G_{s,t,r}$ transforms into $G_{s',t',r'}$ with

$$s' = D(\lambda^2 s - \gamma^2 r + (1+a)\lambda\gamma t), \quad r' = D(-\mu^2 s + \delta^2 r - (1+a)\mu\delta t),$$
$$t' = D((1-a)\lambda\mu s - (1-a)\gamma\delta r + (\lambda\delta + \mu\gamma)t).$$

Now it is routine to show that if $at^2 + 2sr \neq 0$, the substitution can be chosen in such a way that $s' = r' = 0$ and $t' = 1$ (note that F is non-degenerate and therefore $(s, t, r) \neq (0, 0, 0)$). Otherwise, a substitution can be chosen in such a way that $r' = t' = 0$ and $r' = 1$. In the first case, F falls to (T24) with $b = 0$, while in the second case it falls into (T34).

Finally, assume that $(\alpha, \beta) = (-1, -1)$. Solving (5.2), we see that $\mathcal{P}_{2,4}(M)$ consists of

$$G_{s,t,r,u} = s(x^3y - yx^3 + xyx^2 - x^2yx) + t(x^2y^2 - yx^2y + y^2x^2 - xy^2x)$$
$$+ r(y^3x - xy^3 + yxy^2 - y^2xy) + u(xyxy - yxyx).$$

with $s, t, r, u \in \mathbb{K}$. If we perform a (non-degenerate) linear substitution $x \to \lambda x + \mu y$, $y \to \gamma x + \delta y$ with $D = \lambda\delta - \mu\gamma \in \mathbb{K}^*$, $G_{s,t,r,u}$ transforms into $G_{s',t',r',u'}$ with

$$s' = D(\lambda^2 s - \gamma^2 r + \lambda\gamma u), \quad r' = D(-\mu^2 s + \delta^2 r - \mu\delta u),$$
$$t' = D^2 t, \quad u' = D(2\lambda\mu s - 2\gamma\delta r + (\lambda\delta + \mu\gamma)u).$$

It is routine to show that if $u^2 + 4sr \neq 0$, the substitution can be chosen in such a way that $s' = r' = 0$ and F transforms into the form (T24) after an appropriate scaling. It remains to consider the case $u^2 + 4sr = 0$. Note that this property is invariant under linear substitutions (one easily sees that $u'^2 + 4s'r' = 0$). Clearly, a substitution can be chosen in such a way that $s' = 0$. If it happens that $t' = 0$, then F falls into (T34) after swapping x and y and scaling. Thus, we can assume that $t' \neq 0$. The equation $u'^2 + 4s'r' = 0$ yields $u' = 0$. Then a scaling turns F into the twisted potential from (T33), which is proper by Lemma 5.2.

As for algebras from (T24) and (T28–T33) being pairwise non-isomorphic, it is a consequence of the following observations. By Lemma 1.5, the question reduces to pairwise non-equivalence of corresponding twisted potentials, which certainly holds when the twists have different Jordan normal form. As for two F's from the same $\mathcal{P}_{2,4}(M)$, M being in Jordan form, they are equivalent precisely when a linear substitution with a matrix whose transpose commutes with M transforms one F to the other. In the case of one Jordan block this leaves us to consider substitutions of the form $x \to px$, $y \to py + qx$ with $p \in \mathbb{K}^*$ and $q \in \mathbb{K}$. If M is diagonal but not scalar, we are left only with scalings. Finally, if M is scalar, we have to deal with the entire $GL_2(\mathbb{K})$. Fortunately, this only concerns the cases $(\alpha, \beta) = (a, a)$ with $a^2 = -1$ or $a = -1$ in which we gave the explicit formulae for how

the substitutions act on $\mathcal{P}_{2,4}(M)$. Now the stated non-equivalence is a matter for a direct verification. $\qquad \square$

5.1. Proof of Theorem 1.10. Recall that any two proper (degree-graded) twisted potential algebras with non-equivalent twisted potentials are non-isomorphic and a proper twisted potential algebra cannot be isomorphic to a non-proper one. Next, if $F \in \mathcal{P}_{2,4}^*$ is degenerate, then it is easily seen that F is either 0 or is a fourth power of a degree 1 element. In particular, F is a potential and therefore A_F does not satisfy the assumptions. Taking this into account, we see that Lemmas 5.1, 5.2, 5.3, and 5.4 imply Theorem 1.10.

6. Twisted potential algebras A_F with $F \in \mathcal{P}_{3,3}^*$. This section is devoted to the proof of Theorem 1.8. We shall occasionally switch back and forth between denoting them x, y, z or x_1, x_2, x_3 meaning $x = x_1$, $y = x_2$, and $z = x_3$. In this section, we always use the left-to-right degree-lexicographical order on monomials in x, y, z assuming $x > y > z$.

LEMMA 6.1. *The algebras A given by* (T19–T21) *of Theorem 1.8 are non-proper, non-degenerate, non-potential twisted potential algebras. They are pairwise non-isomorphic, PBW, Koszul and have Hilbert series $H_A = \frac{1+t}{1-2t}$ as specified in* (T19–T21).

Proof. Algebras A_a from (T20) are presented by the defining relations x^2, $xy + ayx$, and $xz + a^2zx + y^2$ with $a \neq 0$ and $a \neq 1$. Algebras B_a from (T19) are presented by x^2, z^2, and $xy + ayx$ with $a \neq 0$ and $a \neq 1$. Finally, the defining relations of the algebra C from (T21) are $x^2 + z^2$, $xz - zx$, and y^2. For all these algebras, the defining relations form a Gröbner basis in the ideal of relations. Hence, all algebras in question are PBW and therefore Koszul. Knowing the leading monomials for elements of a Gröbner basis, we can easily compute the Hilbert series: $H_A = \frac{1+t}{1-2t}$ in all cases. By Lemma 1.5, all these algebras are non-proper. By Theorem 1.7, in order to show that none of these algebras is potential, it is enough to verify that none of them is isomorphic to any of the four algebras (P10–P13) (the only potential algebras with the Hilbert series $\frac{1+t}{1-2t}$), which is an elementary exercise.

Since each B_a has two linearly independent squares in the space of quadratic relations, while none of A_a or C has such a thing, B_a is non-isomorphic to any A_b or C. The latter is singled out by the existence of a decomposition $V = V_1 \oplus V_2$ with one-dimensional V_1 for which the space of quadratic relations lies in $V_1^2 + V_2^2$. It remains to verify that A_a are pairwise non-isomorphic and that B_a are pairwise non-isomorphic. Assume that A_a is isomorphic to A_b. Since x^2 is the only square (up to a scalar multiple) in the space of quadratic relations for both algebras, a linear substitution providing an isomorphism must map x to its scalar multiple. Without loss of generality, x is mapped to x. For both algebras, the quotient by the ideal generated by x is presented by generators y, z and one relation y^2. Hence, our substitution must map y to $\alpha y + \beta x$ with $\alpha, \beta \in \mathbb{K}$, $\alpha \neq 0$. It easily follows that $xy + ayx$ is mapped to a scalar multiple of itself plus a scalar multiple of x^2. Thus, $xy + ayx$ must be a relation of A_{F_b}, which yields $a = b$. Finally, assume that B_a is isomorphic to B_b. Since x^2 and z^2 are the only squares (up to scalar multiples) in the space of quadratic relations for both algebras, a linear substitution providing an isomorphism must either map x and z to their own scalar multiples or map x to a scalar multiple of z and z to a scalar multiple of x. In the second case, B_b has a relation of the form $zu + auz$ for some homogeneous degree 1 u non-proportional to z, which is obviously nonsense (u is the image of y under our substitution). Hence, x and z are mapped to their own scalar multiples. Now B_b has a relation of the form $xu + aux$ with $u = y + \alpha z$ with $\alpha \in \mathbb{K}$. This is only possible if $b = a$ (and $\alpha = 0$). $\qquad \square$

LEMMA 6.2. *Each $F \in \mathbb{K}\langle x, y, z\rangle$ listed in* (T1–T18) *of Theorem* 1.8 *is a proper twisted potential such that the Jordan normal form of the corresponding twist is one block with eigenvalue* 1 *for F from* (T3–T4), *two blocks of sizes* 2 *and* 1 *with eigenvalues b and b^{-2}, respectively, for F from* (T5), *two blocks of sizes* 2 *and* 1 *with eigenvalues* -1 *and* 1, *respectively, for F from* (T6), *two blocks of sizes* 2 *and* 1 *with both eigenvalues* 1 *for F from* (T7), *diagonalizable in all other cases with the three eigenvalues being $\frac{a}{b}$, $\frac{b}{c}$, $\frac{c}{a}$ for F from* (T1), *$\frac{a}{b}$, $\frac{b}{a}$,* 1 *for F from* (T2), *a, a, a^{-2} for F from* (T9), *$1, -1, -1$ for F from* (T10), *$a, -a, a^{-2}$ for F from* (T11), $-1, 1, 1$ *for F from* (T16–T18), *$i, -1, 1$ for F from* (T12), *$-i, -1, 1$ for F from* (T13), *ξ_9, ξ_9^4, ξ_9^7 for F from* (T14) *and $\xi_9^2, \xi_9^5, \xi_9^8$ for F from* (T15). *Moreover A_F is Koszul, exact, non-potential and has the Hilbert series $(1-t)^{-3}$ for every F from* (T1–T18) *and A_F is PBW for F from* (T1–T11) *and* (T16–T17).

Proof. It is straightforward to check that each F is a twisted potential with the Jordan normal form of the twist being as specified. For F from (T1–T10), the defining relations as given in Theorem 1.8 form a Gröbner basis in the ideal of relations. Thus, for such F, A_F are PBW and therefore Koszul and we immediately get $H_A = (1-t)^{-3}$. For F from (T11), we perform the substitution $x \to x, y \to y + ix, z \to z$, which turns the defining relations into $xz + azx, yz - azy - 2aizx$, and $xy + yx - iy^2$. For F from (T16), we perform the same substitution $x \to x, y \to y + ix, z \to z$, which turns the defining relations into $xz - zx, yz + zy - 2izx$, and $xy + yx - iy^2 - iz^2$. For F from (T17), we first swap y and z turning the defining relations into $y^2 + z^2, xy - yx$, and $xz + zx + z^2$. Next, we follow up with the substitution $x \to x, y \to y$, and $z \to z + iy$, turning the defining relations into $xy - yx, yz + zy - iz^2$, and $xz + zx + 2iyx - y^2$. In all three cases, the new set of defining relations is a Gröbner basis in the ideal of relations, showing that A_F is PBW, Koszul and has the Hilbert series $(1-t)^{-3}$. Note also that for F from (T1–T11) and (T16–T17) A_F has no non-trivial right annihilators as no leading monomial of an element of the above quadratic Greöbner bases starts with z.

Now we shall show that for F from (T12–T15) and (T18), A_F has the Hilbert series $(1-t)^{-3}$ and has no non-trivial right annihilators. For F from (T12–T13), we swap x and y to bring the defining relations to the form $x^2 + y^2, xy - yx + z^2$, and $xz \pm izx$. A direct computation shows that the defining relations together with $y^2z + zy^2$ and $yz^2 + z^2y$ form a Gröbner basis in the ideal of relations of A_F. The cases of F from (T14) and (T15) are identical (just swap θ and θ^2). The substitution $x \to x, y \to y - \alpha x, z \to z$ turns the defining relations of A_F for F from (T14) into $x^2 - \theta^2 yx - zy, xy + \theta^2 xz - yx - \theta zx + (1-\theta)zy$, and $yz - \theta zy$. This time the defining relations together with $xzy + \theta xz^2 - \theta^2 yxz - zxz + (1-\theta)z^2y, \; xzx + xz^2 - \theta yxz - \theta^2 zxz - z^2y, \; xz^3 - \theta yxz^2 - \theta^2 zxz^2 + \theta^2 z^2 yx + (1-\theta)z^3y$, and $xz^2y - \theta^2 z^2 yx$ form a Gröbner basis in the ideal of relations of A_F. It remains to consider A_F with F from (T18). After swapping y and z, the defining relations of A_F for F from (T18) take shape $xy - yx, x^2 + ay^2 + yz + zy$, and $y^2 + z^2$. A direct computation shows that the defining relations together with $yz^2 - z^2y, xz^2 - z^2x$, and $xzy + yxz - yzx - zyx$ form a Gröbner basis in the ideal of relations of A_F. In each case, the Gröbner basis is finite, which allows to confirm that $H_{A_F} = (1-t)^{-3}$. Since none of the leading monomials of elements of the above Gröbner bases starts with z, A_F has non-trivial right annihilators. Now Lemma 2.6 implies that A_F for F from (T1–T18) is exact and Koszul, while Lemma 1.5 says that they is proper. As each A_F is proper and the corresponding twist is non-trivial, none of A_F is potential. $\square$

LEMMA 6.3. *Let F be the twisted potential from* (T12–T15) *or* (T18) *of Theorem* 1.8 *and $A = A_F$. Then A is non-PBW.*

Proof. Since the PBW property is preserved when one passes to the opposite multiplication and the algebras from (T12) and (T13) as well as the algebras from (T14) and (T15) are isomorphic to each other's opposites, it is enough to deal with F from (T12), (T14), and (T18). That is, A is presented by the generators x, y, z and quadratic relations r_1, r_2, r_3 from the following list:

(1) $r_1 = x^2 + y^2$, $r_2 = xy - yx + z^2$, and $r_3 = xz + izx$;
(2) $r_1 = xz - zx$, $r_2 = x^2 + yz + zy + az^2$, and $r_3 = y^2 + z^2$, where $a \in \mathbb{K}$, $a^2 + 4 \neq 0$;
(3) $r_1 = yx + \theta zy + \theta^2 zx$, $r_2 = xy + zy + \theta^2 xz$, and $r_3 = yx + yz + \theta xz$.

Assume the contrary: A is PBW. By Lemma 6.2, $H_A = (1 - t)^{-3}$ and therefore $\dim A_1 = 3$, $\dim A_2 = 6$, and $\dim A_3 = 10$. By Lemma 2.1, there is a well-ordering $\leqslant$ on the x, y, z monomials compatible with multiplication and satisfying $x > y > z$ (this we can acquire by permuting the variables) and a non-degenerate linear substitution $x \mapsto ux + \alpha_1 y + \beta_1 z$, $y \mapsto vx + \alpha_2 y + \beta_2 z$, $z \mapsto wx + \alpha_3 y + \beta_3 z$ such that the leading monomials m_1, m_2, m_3 of the new space of defining relations satisfy

$$\{m_1, m_2, m_3\} \in \big\{\{xy, xz, yz\}, \{xy, xz, zy\}, \{xy, zx, zy\}, \{yx, yz, xz\}, \{yx, yz, zx\}, \{yx, zy, zx\}\big\}. \tag{6.1}$$

Note that we do not assume that $m_1 > m_2 > m_3$ here. Since x^2 is the biggest degree 2 monomial,

$$x^2 \text{ is absent in each of } r_j \text{ after the substitution.} \tag{6.2}$$

Since the order satisfies $x > y > z$ and is compatible with multiplication,

$$\text{four biggest degree 2 monomials are either } x^2, xy, yx, xz \text{ or } x^2, xy, yx, zx, \tag{6.3}$$

(not necessarily in this order).

Case 1. r_j are given by (1). Then (6.2) reads $0 = uw = w^2 = u^2 + v^2$. Since our substitution is non-degenerate $(u, v, w) \neq (0, 0, 0)$. By scaling x (this does not effect the leading monomials), we can assume that $u = 1$. Then $w = 0$ and $v^2 = -1$. The following table gives some coefficients in r_j in front of certain monomials:

	x^2	xy	yx	xz	zx	y^2
r_1	0	$\alpha_1 + v\alpha_2$	$\alpha_1 + v\alpha_2$	$\beta_1 + v\beta_2$	$\beta_1 + v\beta_2$	$\alpha_1^2 + \alpha_2^2$
r_2	0	$\alpha_2 - v\alpha_1$	$-v\alpha_1 - \alpha_2$	$\beta_2 - v\beta_1$	$-v\beta_1 - \beta_2$	α_3^2
r_3	0	α_3	$i\alpha_3$	β_3	$i\beta_3$	$\alpha_1\alpha_3(1 + i)$

Since our sub is non-degenerate, we see that if $\alpha_1 + v\alpha_2 \neq 0$, then the both 3×3 matrices of coefficients of xy, yx, and xz and of xy, yx, and zx are non-degenerate. By (6.3), in the case $\alpha_1 + v\alpha_2 \neq 0$, the set of leading monomials of the relations is either $\{xy, yx, xz\}$ or $\{xy, yx, zx\}$, contradicting (6.1). Hence, we must have $\alpha_1 + v\alpha_2 = 0$. Since $v = \pm i$, the above table takes the following form:

	x^2	xy	yx	xz	zx	y^2
r_1	0	0	0	$\beta_1 + v\beta_2$	$\beta_1 + v\beta_2$	0
r_2	0	0	0	$\beta_2 - v\beta_1$	$-v\beta_1 - \beta_2$	α_3^2
r_3	0	α_3	$i\alpha_3$	β_3	$i\beta_3$	$\alpha_1\alpha_3(1 + i)$

Then $\alpha_3 \neq 0$ (otherwise both xy and yx are not among the leading monomials, contradicting (6.1)) and $\beta_1 + v\beta_2 \neq 0$ (otherwise both xz and zx are not among the leading monomials, contradicting (6.1)). Now, one easily sees that the y^2-column of the above matrix is not in the linear span of any of the following pair of columns: xy and xz, xy and zx, yx and xz or yx and zx. Since $y^2 > yz$, $y^2 > zy$ and $y^2 > z^2$, it follows that y^2 is among the leading monomials of the relations, which contradicts (6.1). This contradiction completes the proof in Case 1.

Case 2. r_j are given by (2). In this case (6.2) reads

$$0 = uv + \theta vw + \theta^2 uw = uv + vw + \theta^2 uw = uv + vw + \theta uw,$$

which is equivalent to $uv = vw = uw = 0$. Hence, exactly two of u, v, and w are zero and we can normalize to make the third equal 1. Since cyclic permutations of the variables composed with appropriate scalings provide automorphisms of our algebra, we can without loss of generality assume that $u = 1$ and $v = w = 0$. The following table gives the coefficients in r_j in front of certain monomials:

	x^2	xy	yx	xz	zx	y^2
r_1	0	0	$\alpha_2 + \theta^2\alpha_3$	0	$\beta_2 + \theta^2\beta_3$	$\alpha_1\alpha_2 + \theta\alpha_2\alpha_3 + \theta^2\alpha_1\alpha_3$
r_2	0	$\alpha_2 + \theta^2\alpha_3$	0	$\beta_2 + \theta^2\beta_3$	0	$\alpha_1\alpha_2 + \alpha_2\alpha_3 + \theta^2\alpha_1\alpha_3$
r_3	0	$\theta\alpha_3$	α_2	$\theta\beta_3$	β_2	$\alpha_1\alpha_2 + \alpha_2\alpha_3 + \theta\alpha_1\alpha_3$

Since our sub is non-degenerate, we see that if $\alpha_2 + \theta^2\alpha_3 \neq 0$, then the both 3×3 matrices of coefficients of xy, yx, and xz and of xy, yx, and zx are non-degenerate. By (6.3), in the case $\alpha_2 + \theta^2\alpha_3 \neq 0$, the set of leading monomials of the relations is either $\{xy, yx, xz\}$ or $\{xy, yx, zx\}$, contradicting (6.1). Hence, we must have $\alpha_2 + \theta^2\alpha_3 = 0$. The above table takes the following form.

	x^2	xy	yx	xz	zx	y^2
r_1	0	0	0	0	$\beta_2 + \theta^2\beta_3$	$-\theta\alpha_3^2$
r_2	0	0	0	$\beta_2 + \theta^2\beta_3$	0	$-\theta^2\alpha_3^2$
r_3	0	$\theta\alpha_3$	α_2	$\theta\beta_3$	β_2	$*$

Now unless $\alpha_3(\beta_2 + \theta^2\beta_3) = 0$, all six 3×3 matrices of coefficients of xy, xz, and y^2; yx, xz, and y^2; xy, zx, and y^2; yx, zx, and y^2; xy, xz, and zx; yx, xz, and y^2 are non-degenerate. The latter means that either xz and zx or y^2 are among the leading monomials of the defining relations, which contradicts (6.1). Thus, we must have $\alpha_3(\beta_2 + \theta^2\beta_3) = 0$ and $\alpha_2 + \theta^2\alpha_3 = 0$, which contradicts the fact that our substitution is non-degenerate. This contradiction completes the proof in Case 2.

Case 3. r_j are given by (3).

Since this class of algebras is closed (up to an isomorphism) with respect to passing to the opposite multiplication and the two options in (6.3) reduce to one another via passing to the opposite multiplication, for the rest of the proof, we can assume that

$$\text{four biggest degree 2 monomials are } x^2, xy, yx, xz. \tag{6.4}$$

In the current case, (6.2) reads $0 = u^2 + 2vw + aw^2 = v^2 + w^2$. Since $(u, v, w) \neq (0, 0, 0)$, we have $v \neq 0$, which allows to normalize: $v = 1$. Then $w \in \{i, -i\}$ and $a = 2w - u^2$. Since $a^2 + 4 \neq 0$ and $w^2 = -1$, we have $u \neq 0$. It is easy to see that we can split our substitution into two consecutive substitutions: first, $x \mapsto ux$, $y \mapsto x + y$, $z \mapsto wx + z$ and, second, $x \mapsto x + \alpha_1 y + \beta_1 z$, $y \mapsto \alpha_2 y + \beta_2 z$, $z \mapsto \alpha_3 y + \beta_3 z$ (α_j and β_j are not the same as before). After the first substitution, the defining relations are spanned by $r_1 = xz - zx$, $r_2 = u^2(xz + zx) + w(yz + zy) + y^2 - (1 + wu^2)z^2$, and $r_3 = u^2(xy + yx) + (u^2 - w)y^2 + (yz + zy) + wz^2$. The following table gives the coefficients in r_j in front of certain monomials after the second substitution:

	x^2	xy	yx	xz	zx
r_1	0	α_3	$-\alpha_3$	β_3	$-\beta_3$
r_2	0	$u^2\alpha_3$	$u^2\alpha_3$	$u^2\beta_3$	$u^2\beta_3$
r_3	0	$u^2\alpha_2$	$u^2\alpha_2$	$u^2\beta_2$	$u^2\beta_2$

Using the fact that our sub is non-degenerate, we easily see that if $\alpha_3 \neq 0$, then the 3×3 matrix of coefficients of xy, yx, and xz is non-degenerate. By (6.4), both xy and yx are among the leading monomials, contradicting (6.1). Hence, we must have $\alpha_3 = 0$. The above table with the extra y^2-column takes the following form:

	x^2	xy	yx	xz	zx	y^2
r_1	0	0	0	β_3	$-\beta_3$	0
r_2	0	0	0	$u^2\beta_3$	$u^2\beta_3$	α_2^2
r_3	0	$u^2\alpha_2$	$u^2\alpha_2$	$u^2\beta_2$	$u^2\beta_2$	$(u^2 - w)\alpha_2^2$

Same way as in Case 2, it follows that unless $\alpha_2\beta_3 = 0$, either xz and zx or y^2 feature among the leading monomials. Thus, $\alpha_2\beta_3 = \alpha_3 = 0$, which contradicts the fact that our substitution is non-degenerate. This contradiction completes the proof in the final Case 3. $\square$

Now we deal with the possibilities for the Jordan normal form of the twist for $F \in \mathcal{P}_{3,3}^*$ one by one.

LEMMA 6.4. *Let $G \in \mathcal{P}_{3,3}^*$ be non-degenerate, $M \in GL_3(\mathbb{K})$ be the unique matrix providing the twist for G and assume that $A = A_G$ is non-potential. Assume also that the normal Jordan form of M consists of one block. If A is non-proper, then A is isomorphic to A_F with F from (T20) of Theorem 1.8 with $a^3 = 1 \neq a$. If A is proper, then A is isomorphic to A_F for F from (T3) or (T4) of Theorem 1.8. Moreover, algebras A_F for F from (T3) and (T4) are pairwise non-isomorphic.*

Proof. By Remark 1.4, we can assume that $M = \begin{pmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 1 \\ 0 & 0 & \alpha \end{pmatrix}$ with $\alpha \in \mathbb{K}^*$. If $G = \sum_{j,k,m=1}^{3} a_{j,k,m} x_j x_k x_m$, then the inclusion $G \in \mathcal{P}_{3,3}(M)$ is equivalent to the following system of linear equations on the coefficients of G:

$$a_{j,k,3} = \alpha a_{3,j,k}, \quad a_{j,k,2} = \alpha a_{2,j,k} + a_{3,j,k} \text{ and } a_{j,k,1} = \alpha a_{1,j,k} + a_{2,j,k} \text{ for } 1 \leqslant j, k \leqslant 3. \quad (6.5)$$

One easily sees that (6.5) has only zero solution unless $\alpha^3 = 1$. This leaves two cases to consider: $\alpha^3 = 1 \neq \alpha$ and $\alpha = 1$.

If $\alpha^3 = 1 \neq \alpha$, solving (6.5), we see that G belongs to $\mathcal{P}_{3,3}(M)$ precisely when

$$G = sx^2z + \alpha^2 szx^2 + \alpha sxzx - \alpha^2 sxy^2 - sy^2x - \alpha syxy + tx^2y + \alpha(\alpha t - s)yx^2 + \alpha(t - s)xyx$$
$$+ \frac{\alpha t - s}{\alpha^2 - 1}x^3,$$

with $s, t \in \mathbb{K}$. Since G is non-degenerate, $s \neq 0$. A scaling turns s into 1. Now the space of quadratic relations of $A = A_G$ is spanned by x^2, $xy + \alpha^2 yx$, and $xz + \alpha zx - \alpha^2 y^2 + pyx$, where $p = (\alpha - \alpha^2)t - \alpha$. A substitution $x \to x$, $y \to y$, and $z \to vz + uy$ with appropriate $u \in \mathbb{K}$ and $v \in \mathbb{K}^*$ turns the defining relations of A into x^2, $xy + \alpha^2 yx$, and $xz + \alpha zx + y^2$. Thus, A is isomorphic to A_F with F from (T20) of Theorem 1.8 with $a = \alpha^2$. By Lemma 6.1, A is non-proper.

It remains to consider the case $\alpha = 1$. Solving (6.5), we see that

$$\mathcal{P}_{3,3}(M) = \left\{ G_{s,t,r} = sxyz^{\circlearrowleft} - sxzy^{\circlearrowleft} + tx^2z^{\circlearrowleft} - sxzx + \frac{s - t}{2}xy^{2\circlearrowleft} - syxy \right.$$
$$\left. + tx^2y + \frac{t - s}{2}xyx + rx^3 : s, t, r \in \mathbb{K} \right\}.$$

Clearly, $G_{s,t,r}$ is non-degenerate precisely when $(s, t) \neq (0, 0)$. By Remark 1.4, two such twisted potentials are equivalent if and only if they are obtained from one another by a linear substitution with the matrix, whose transpose commutes with M. That is, we have to look only at substitutions $x \to ux$, $y \to u(vx + y)$, $z \to u(wx + vy + z)$ with $u \in \mathbb{K}^*$, $v, w \in \mathbb{K}$. A direct computation shows that this substitution transforms $G_{s,t,r}$ to $G_{s',t',r'}$ with $s' = u^3 s$, $t' = u^3 t$ and $r' = u^3(r + (3t - s)(w + \frac{1}{2}v - \frac{1}{2}v^2))$. If $s = 0$, $G_{s,t,r}$ is cyclically invariant and therefore A is potential. Since this contradicts the assumptions, $s \neq 0$. Now the above observation shows that $G_{s,t,r}$ is equivalent to precisely one of the following: $G_{1,t,0}$ for $t \neq \frac{1}{3}$ or $G_{1,1/3,r}$ for $r \in \mathbb{K}$. Swapping of x and z brings the latter to the forms (T3) or (T4) of Theorem 1.8. By Lemma 6.2, these algebras are proper. Since we already know that these twisted potentials are pairwise non-equivalent, Lemma 1.5 implies that the corresponding algebras are pairwise non-isomorphic. $\qquad\square$

LEMMA 6.5. *Let $G \in \mathcal{P}^*_{3,3}$ be non-degenerate, $M \in GL_3(\mathbb{K})$ be the unique matrix providing the twist for G and assume that $A = A_G$ is non-potential and the normal Jordan form of M consists of two blocks. If A is non-proper, then A is isomorphic to A_F with F from (T20) of Theorem 1.8. If A is proper, then A is isomorphic to A_F for F from (T5–T8) of Theorem 1.8. Moreover, algebras A_F for F with different labels from (T5–T8) are non-isomorphic and the isomorphism conditions of Theorem 1.8 concerning each of (T5–T8) are satisfied.*

Proof. By Remark 1.4, we can assume that $M = \begin{pmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix}$ with $\alpha, \beta \in \mathbb{K}^*$. If $G =$

$\sum\limits_{j,k,m=1}^{3} a_{j,k,m}x_j x_k x_m$, then the inclusion $F \in \mathcal{P}_{3,3}(M)$ is equivalent to the following system of linear equations on the coefficients of G:

$$a_{j,k,3} = \beta a_{3,j,k}, \quad a_{j,k,2} = \alpha a_{2,j,k}, \quad \text{and} \quad a_{j,k,1} = \alpha a_{1,j,k} + a_{2,j,k} \text{ for } 1 \leqslant j, k \leqslant 3. \quad (6.6)$$

One easily sees that (6.6) has only zero solution if $1 \notin \{\alpha, \beta, \alpha^2\beta, \alpha\beta^2\}$. Furthermore, $\mathcal{P}_{3,3}(M)$ contains no non-degenerate elements unless $\alpha^2\beta = 1$. Indeed, if $\alpha^2\beta \neq 1$, y does not feature at all in members of $\mathcal{P}_{3,3}(M)$. Thus, for the rest of the proof, we can assume that $\alpha^2\beta = 1$. That is, $\beta = \alpha^{-2}$. By (6.6),

$$F_{s,t} = s(xyz + \alpha yzx + \alpha^2 zxy - \alpha xzy - yxz - \alpha^2 zyx - xzx)$$
$$+ t(x^2z + \alpha^2 zx^2 + \alpha xzx) \in \mathcal{P}_{3,3}(M) \text{ for } s, t \in \mathbb{K}.$$

Furthermore, there are no other elements in $\mathcal{P}_{3,3}(M)$ unless $\alpha^3 = 1$ or $\alpha^2 = 1$. If $s = 0$, $F_{s,t}$ is degenerate. Thus, we can assume that $s \neq 0$. By scaling, we can make $s = 1$. By Remark 1.4, two twisted potentials $F_{1,t}$ and $F_{1,t'}$ are equivalent precisely when they are obtained from one another by a linear substitution with the matrix, whose transpose commutes with M. That is, in the case $\alpha \neq \beta$ (equivalently, $\alpha^3 \neq 1$), we have to look only at substitutions $x \to ux$, $y \to u(vx + y)$, $z \to wz$ with $u, w \in \mathbb{K}^*$, $v \in \mathbb{K}$. A direct computation shows that this sub transforms $F_{1,t}$ to $F_{1,t'}$ if and only if $t = t'$. That is, in the case $\alpha^3 \neq 1$, $F_{1,t}$ are pairwise non-equivalent. Swapping of x and z turns $F_{1,a}$ into

$$G_{a,b} = zyx + byxz + b^2xzy - bzxy - yzx - b^2xyz + (ab - 1)zxz + az^2x + ab^2xz^2$$
$$\text{with } a, b \in \mathbb{K}, b^3 \neq 1,$$

where $b = \alpha$ and $a = t$, which is precisely the twisted potential from (T5) with $b^3 \neq 1$. By Lemma 6.2, these algebras are proper. Since we already know that the corresponding twisted potentials are non-equivalent, Lemma 1.5 implies that the algebras themselves are pairwise non-isomorphic.

It remains to consider the cases $\alpha = -1$, $\alpha = 1$, and $\alpha^3 = 1 \neq \alpha$. If $\alpha = -1$, then $\beta = 1$. By (6.6),

$$\mathcal{P}_{3,3}(M) = \{F_{s,t,r} = s(xyz - yzx + zxy + xzy - yxz - zyx - xzx) + t(x^2z + zx^2 + xzx)$$
$$+ rz^3 : s, t, r \in \mathbb{K}\}.$$

Since $F_{s,t,r}$ is degenerate for $s = 0$, we can assume that $s \neq 0$. If $r = 0$, we are back to the previous considerations (with $\alpha = -1$). Thus, we can assume that $r \neq 0$. By scaling, we can make $s = r = 1$, which leaves us with $F_{1,t,1}$. Same argument as above shows that $F_{1,t,1}$ are pairwise non-equivalent. Swapping of x and y turns $F_{1,a,1}$ into

$$G_a = yxz - xzy + zyx + yzx - xyz - zxy + (a - 1)yzy + ay^2z + azy^2 + z^3 \text{ with } a \in \mathbb{K},$$

which is precisely the twisted potential from (T6). By Lemma 6.2, the corresponding algebras are proper. Since we already know that these twisted potentials are pairwise non-equivalent, Lemma 1.5 implies that the corresponding algebras are pairwise non-isomorphic.

Next, consider the case $\alpha^3 = 1 \neq \alpha$. Then $\beta = \alpha$. Solving (6.6), we see that $\mathcal{P}_{3,3}(M)$ consists of

$$F_{s,t,p,q} = s(xyz + \alpha yzx + \alpha^2 zxy - \alpha xzy - yxz - \alpha^2 zyx - xzx) + p(x^2z + \alpha^2 zx^2 + \alpha xzx)$$

$$+ t(xz^2 + \alpha^2 zxz + \alpha z^2x) + q(x^2y + \alpha^2 yx^2 + \alpha xyx + \frac{\alpha^2}{1-\alpha}x^3) \text{ with } s, t, p, q \in \mathbb{K}.$$

The only linear substitutions with the matrix, whose transpose commutes with M have the form $x \to ux$, $y \to vx + uy + wz$, $z \to cz + dx$ with $u, c \in \mathbb{K}^*$ and $v, w, d \in \mathbb{K}$. A direct computation shows that this substitution transforms $F_{s,t,p,q}$ to $F_{s',t',p',q'}$ with $s' =$

su^2c, $t' = tuc^2 + s(1-\alpha)ucw$, $q' = qu^3 + s(\alpha^2 - \alpha)u^2d$ and $p' = pu^2c + qu^2w - t\alpha udc + s(\alpha^2 - \alpha)udw$. If $s = 0$ to begin with, a substitution of the above form allows to kill p (make $p' = 0$) unless $q = t = 0$. In the latter case, $F_{s,t,p,q} = F_{0,0,p,0}$ is degenerate. This leaves $F_{0,t,0,q}$. If $tq = 0$, then again F is degenerate. Thus, $tq \neq 0$. A sub of the above form then allows to turn t and q into 1. For $G = F_{0,1,0,1}$, the space of defining relations is spanned by x^2, $xz + \alpha^2zx$, and $xy + \alpha yx + z^2$. Swapping y and z now provides an isomorphism of A and an algebra from (T20) with $a = \alpha^2$. It remains to consider the case $s \neq 0$. A substitution of the above form now can be chosen in such a way that $s' = 1$ and $q' = t' = 0$. Thus, it remains to consider the case $G = F_{1,0,a,0}$ with $a \in \mathbb{K}$. It is easy to see that the above substitutions cannot transform $F_{1,0,a,0}$ into $F_{1,0,a',0}$ with $a \neq a'$: $F_{1,0,a,0}$ are pairwise non-equivalent. Swapping x and z turns $F_{1,0,a,0}$ into

$$G = zyx + \alpha^2yxz + \alpha^2xzy - \alpha zxy - yzx - \alpha^2xyz + (\alpha a - 1)zxz + az^2x + \alpha^2axz^2,$$

which are exactly the twisted potentials from (T5) with $b^3 = 1 \neq b$.

It remains to consider the case $\alpha = \beta = 1$. By (6.6),

$$\mathcal{P}_{3,3}(M) = \{F_{s,t,p,q,r} = s(xyz^{\circlearrowleft} - xzy^{\circlearrowleft} - xzx) + px^2z^{\circlearrowleft}$$
$$+ txz^{2^{\circlearrowleft}} + qx^3 + rz^3 : s, t, p, q, r \in \mathbb{K}\}.$$

As above, the only linear substitutions with the matrix, whose transpose commutes with M have the form $x \to ux$, $y \to vx + uy + wz$, $z \to cz + dx$ with $u, c \in \mathbb{K}^*$ and $v, w, d \in \mathbb{K}$. A direct computation shows that this substitution transforms $F_{s,t,p,q,r}$ to $F_{s',t',p',q',r'}$ with $s' = su^2c$, $t' = tuc^2 + rc^2d$, $q' = qu^3 + rd^3 + (3p - s)u^2d + 3tud^2$, $p' = pu^2c + 2tudc + rcd^2$, and $r' = rc^3$. If $s = 0$, $F_{s,t,p,q,r}$ is cyclically invariant and therefore the corresponding algebra is potential. Thus, we can assume $s \neq 0$. If $r \neq 0$, we can find a substitution of the above shape such that $t' = 0$ and $s' = r' = 1$. Thus, we have to consider $F_{1,0,p,q,1}$. Analyzing the action of the above substitutions on these, we see that $F_{1,0,p,q,1}$ and $F_{1,0,p',q',1}$ are equivalent if and only if $p' = p$ and $q' = \pm q$. After swapping x and y, we arrive to twisted potentials

$$G_{a,b} = xzy^{\circlearrowleft} - xyz^{\circlearrowleft} - yzy + ay^2z^{\circlearrowleft} + by^3 + z^3,$$

with $a, b \in \mathbb{K}$, which are precisely the twisted potentials from (T7). By Lemma 6.2, the corresponding twisted potential algebras are proper. Since we already know when their twisted potentials are equivalent, Lemma 1.5 implies the isomorphism condition for (T7). It remains to consider the case $s \neq 0$ and $r = 0$. The case $t = 0$ yields algebras from (T5) (with $b = 1$, $a \neq 0$) (after scaling and swapping x with z). Thus, we can assume that $t \neq 0$. Now we can easily find a substitution of the above form for which $s' = t' = 1$ and $r' = p' = 0$. Thus, we have to consider $F_{1,1,0,q,0}$. Analyzing the action of the above substitutions on these, wee see that $F_{1,1,0,q,0}$ and $F_{1,1,0,q',0}$ are pairwise non-equivalent. After swapping x and y, we arrive to twisted potentials

$$G_a = xzy^{\circlearrowleft} - xyz^{\circlearrowleft} - yzy + yz^{2^{\circlearrowleft}} + ay^3,$$

with $a \in \mathbb{K}$. If $a = 0$, we are back to (T5). Thus, we can assume that $a \neq 0$. Now we have precisely the twisted potentials from (T8). By Lemma 6.2, the corresponding algebras are proper. Since we already know that their twisted potentials are pairwise non-equivalent, Lemma 1.5 implies these algebras are pairwise non-isomorphic. $\square$

LEMMA 6.6. *Let $G \in \mathcal{P}_{3,3}^*$ be non-degenerate, $M \in GL_3(\mathbb{K})$ be the unique matrix providing the twist for G and assume that $A = A_G$ is non-potential. Assume also that M is diagonalizable and has determinant 1. If A is non-proper, then A is isomorphic to A_F with*

F from (T20) *of Theorem* 1.8. *If A is proper, then A is isomorphic to A_F for F from* (T1–T2) *or* (T9–T10) *of Theorem* 1.8 *with different labels corresponding to non-isomorphic algebras. Furthermore, the relevant isomorphism statements of Theorem* 1.8 *hold.*

Proof. By Remark 1.4, we can assume that $M = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}$ with $\alpha, \beta, \gamma \in \mathbb{K}^*$. If $G = \sum_{j,k,m=1}^{3} a_{j,k,m} x_j x_k x_m$, then the inclusion $F \in \mathcal{P}_{3,3}(M)$ is equivalent to the following system of linear equations on the coefficients of G:

$$a_{j,k,3} = \gamma a_{3,j,k}, \quad a_{j,k,2} = \beta a_{2,j,k} \text{ and } a_{j,k,1} = \alpha a_{1,j,k} \text{ for } 1 \leqslant j, k \leqslant 3. \qquad (6.7)$$

Since M has determinant 1, we have $\alpha\beta\gamma = 1$. Since we are not interested in potentials, $(\alpha, \beta) \neq (1, 1)$.

Analyzing (6.7), we see that

$$F_{s,t} = s(xyz + \alpha yzx + \alpha\beta zxy) + t(yxz + \beta xzy + \alpha\beta zyx) \in \mathcal{P}_{3,3}(M) \text{ for } s, t \in \mathbb{K}.$$

Furthermore, there are no other elements in $\mathcal{P}_{3,3}(M)$ unless either 1 is among the eigenvalues or at least two of the eigenvalues are equal.

If $s = t = 0$, then $F_{s,t}$ is degenerate. If $st = 0$ and $(s, t) \neq (0, 0)$, then the corresponding twisted potential algebra is easily seen to be isomorphic to the algebra from (P12) and therefore is potential. Thus, we can assume that $st \neq 0$. By scaling, we can make $s = 1$. Then, $G = F_{1,t}$ acquires the form (T1) with $a = t$, $b = \frac{t}{\alpha}$, and $c = \beta t$. Since $(\alpha, \beta) \neq (1, 1)$, we have the condition $(a - b, a - c) \neq (0, 0)$ of (T1). By Lemma 6.2, the algebras from (T1) are proper. By Lemma 1.5, two algebras from (T1) are isomorphic precisely when their twisted potentials are equivalent. Using Remark 1.4, we see that if the eigenvalues of M are pairwise distinct, then the only substitutions transforming a corresponding F from (T1) to another F from (T1) are scalings composed with permutations of the variables. The isomorphism condition in (T1) is now easily verified. The case when some of the eigenvalues coincide leads to a bigger group of eligible substitutions; however, the result in terms of isomorphic members of (T1) is easily seen to be the same.

It remains to consider two options for the triple (α, β, γ) of the eigenvalues of M to which all the remaining options are reduced by a permutation of the variables: $(\alpha, \alpha^{-1}, 1)$ and $(\alpha^{-2}, \alpha, \alpha)$ with $\alpha \in \mathbb{K}^*$, $\alpha \neq 1$.

Consider the case when the eigenvalues of M are $(\alpha, \alpha^{-1}, 1)$. Solving (6.7), we see that

$$F_{s,t,r} = s(xyz + \alpha yzx + zxy) + t(yxz + \alpha^{-1}xzy + zyx) + rz^3 \in \mathcal{P}_{3,3}(M) \text{ for } s, t, r \in \mathbb{K}.$$

Furthermore, there are no other elements in $\mathcal{P}_{3,3}(M)$ unless $\alpha = -1$ (the case $\alpha = 1$ is already off the table). If $r = 0$, we fall back into the previous case. Thus, we can assume $r \neq 0$. If $s = t = 0$, $F_{s,t,r}$ is degenerate (and potential to boot). Thus, $(s, t) \neq (0, 0)$. If $st = 0$, a scaling (if $t = 0$) or a scaling composed with the swap of x and y turns $F_{s,t,r}$ into $xyz + \alpha yzx + zxy + z^3$. Now the corresponding twisted potential algebra is easily seen to be isomorphic to the potential algebra from (P14). Hence, $str \neq 0$ and by scaling we can turn s and r into 1, leaving us with $F_{1,t,1}$, which is a scalar multiple of the twisted potential from (T2) with $a = \frac{t}{\alpha}$ and $b = t$. By Lemma 6.2, the algebras from (T2) are proper. By Lemma 1.5, algebras from (T2) are isomorphic if and only if their twisted potentials are equivalent. By Remark 1.4, this happens precisely when they can be transformed into one

another by a linear substitution with the matrix whose transpose commutes with M. Now it is easy to verify the isomorphism condition from (T2).

Next, consider the case when the eigenvalues of M are $(\alpha^{-2}, \alpha, \alpha)$. Solving (6.7), we see that

$$\begin{aligned} F_{s,t,p,q} = {} & s(xyz + \alpha^{-2}yzx + \alpha^{-1}zxy) + t(xzy + \alpha^{-2}zyx + \alpha^{-1}yxz) \\ & + p(xy^2 + \alpha^{-2}y^2x + \alpha^{-1}yxy) \\ & + q(xz^2 + \alpha^{-2}z^2x + \alpha^{-1}zxz) \in \mathcal{P}_{3,3}(M) \ \text{ for } s, t, p, q \in \mathbb{K}. \end{aligned}$$

Furthermore, there are no other elements in $\mathcal{P}_{3,3}(M)$ unless $\alpha = -1$ or $\alpha^3 = 1 \neq \alpha$ (again, the case $\alpha = 1$ is off). Applying Lemma 2.2 to $syz + tzy + py^2 + qz^2$, we see that by a linear substitution, which leaves both x and the linear span of y, z invariant, (s, t, p, q) can be transformed into exactly one of the following forms: $(0, 0, 0, 0)$, $(0, 0, 0, 1)$, $(1, t, 0, 0)$ with $t \in \mathbb{K}$ or $(1, -1, 0, 1)$. All cases except for the last one either give a degenerate twisted potential or one that has been already dealt with earlier in this proof. This leaves us with $(s, t, p, q) = (1, -1, 0, 1)$:

$$G = xyz + \alpha^{-2}yzx + \alpha^{-1}zxy - xzy - \alpha^{-2}zyx - \alpha^{-1}yxz + xz^2 + \alpha^{-2}z^2x + \alpha^{-1}zxz,$$

which is the twisted potential from (T9) with $a = \alpha$. By Lemma 6.2, these twisted potentials are proper. Using Lemma 1.5, as above on a number of occasions, we see that the corresponding twisted potential algebras are pairwise non-isomorphic.

At this point, it remains to deal with three specific triples of eigenvalues of M: $(-1, -1, 1)$ and (α, α, α) with $\alpha^3 = 1 \neq \alpha$. We start with the case when the eigenvalues of M are $(-1, -1, 1)$. Solving (6.7), we see that $\mathcal{P}_{3,3}(M)$ is the space of

$$\begin{aligned} F_{s,t,p,q,r} = {} & s(xyz - yzx + zxy) + t(yxz - xzy + zyx) + p(x^2z - xzx + zx^2) \\ & + q(y^2z - yzy + zy^2) + rz^3, \end{aligned}$$

with $s, t, p, q, r \in \mathbb{K}$. Applying Lemma 2.2 to $sxy + tyx + px^2 + qy^2$, we see that by a linear substitution, which leaves both z and the linear span of x, y invariant, (s, t, p, q) can be transformed into exactly one of the following forms: $(0, 0, 0, 0)$, $(0, 0, 0, 1)$, $(1, t, 0, 0)$ with $t \in \mathbb{K}$ or $(1, -1, 0, 1)$. If either $r = 0$ or any of the first three of the last four cases occurs, then either our F is degenerate or it is a twisted potential that has been already dealt with earlier in this proof (up to a possible permutation of variables). An additional scaling allows to turn r into 1 leaving us with

$$G = xyz - yzx + zxy - yxz + xzy - zyx + y^2z - yzy + zy^2 + z^3,$$

which is the twisted potential from (T10). By Lemmas 6.2, the corresponding twisted potential algebra is proper.

This leaves us with the final case when the eigenvalues of M are (α, α, α) with $\alpha^3 = 1 \neq \alpha$. For the sake of convenience, we use the following notation: $uvw^{@} = uvw + \alpha vwu + \alpha^2 wuv$. Note that $u^{3@} = 0$ since $1 + \alpha + \alpha^2 = 0$. Solving (6.7), we see that

$$F_u = u_1 x^2 y^{@} + u_2 x^2 z^{@} + u_3 y^2 x^{@} + u_4 y^2 z^{@} + u_5 z^2 x^{@} + u_6 z^2 y^{@} + u_7 xyz^{@} + u_8 xzy^{@},$$

for $u = (u_1, \ldots, u_8) \in \mathbb{K}^8$ comprise $\mathcal{P}_{3,3}(M)$. Since M is central in $GL_3(\mathbb{K})$, every linear substitution preserves this general form of a twisted potential, changing the coefficients however. First, we shall verify that there always is a linear substitution, which kills u_1

and u_2 (= turns both of them into 0). If $u_5 = u_6 = 0$, then swapping x and z achieves the objective. Thus, we can assume that $(u_5, u_6) \neq (0, 0)$. If $u_5 = 0$, the substitution $x \to x$, $y \to y + x$, $z \to z$ makes both u_5 and u_6 non-zero. If $u_6 = 0$, the substitution $x \to x + y$, $y \to y, z \to z$ makes both u_5 and u_6 non-zero. Thus, we can assume $u_5 u_6 \neq 0$. Now the substitution $x \to x, y \to y, z \to z + \frac{u_2}{u_5}x + \frac{u_4}{u_6}y$ is easily seen to kill both u_2 and u_4, while leaving u_5 and u_6 unchanged. Thus, we can assume that $u_2 = u_4 = 0$ and $u_5 u_6 \neq 0$. If $u_1 = 0$, the job is already done. Thus, we can assume $u_1 \neq 0$. If $u_3 = 0$, then swapping x and y we turn both u_1 and u_2 into zero. Thus, we can assume that $u_3 \neq 0$. Performing a scaling, we can turn both u_1 and u_3 into 1, while the conditions $u_2 = u_4 = 0$ and $u_5 u_6 \neq 0$ remain unaffected. Thus, we have $u_1 = u_3 = 1$, $u_2 = u_4 = 0$, and $u_5 u_6 \neq 0$. Using the fact that $\mathbb{K}$ is algebraically closed, we can find $s, t \in \mathbb{K}$ such that $w_1 = 1 - \alpha^2 s + (u_8 + \alpha^2 u_7)t + u_6 t^2 = 0$ and $w_2 = -u_6 st - u_5 t + (u_7 + \alpha^2 u_8) = 0$. Indeed, the first equation amounts to expressing s in terms of t. Plugging this into the second equation yields a genuinely cubic equation on t: the t^3-coefficient is $-\alpha^2 u_6 \neq 0$. Now the substitution $x \to x, y + sx, z \to z + tx$ transforms (u_1, u_2) into (w_1, w_2) thus killing both u_1 and u_2. That is, no matter the case, a linear substitution kills both u_1 and u_2. By Lemma 2.2 applied to $f = u_3 y^2 + u_5 z^2 + u_7 \alpha yz + u_8 \alpha zy$, there is a linear substitution on the variables y, z turning f into (exactly) one of the following four forms: 0, z^2, $yz - azy$ with $a \in \mathbb{K}$ or $yz - zy + z^2$. The same substitution augmented by $x \to x$ transforms F_u into one of the following forms: $G_1 = py^2 z^{\circledcirc} + qz^2 y^{\circledcirc}$, $G_2 = z^2 x^{\circledcirc} + py^2 z^{\circledcirc} + qz^2 y^{\circledcirc}$, $G_3 = yzx^{\circledcirc} - azyx^{\circledcirc} + py^2 z^{\circledcirc} + qz^2 y^{\circledcirc}$ or $G_4 = yzx^{\circledcirc} - zyx^{\circledcirc} + z^2 x^{\circledcirc} + py^2 z^{\circledcirc} + qz^2 y^{\circledcirc}$ with $a, p, q \in \mathbb{K}$. We can disregard G_1, since it is degenerate. The twisted potential G_2 is degenerate if $p = 0$. Thus, we can assume that $p \neq 0$. The substitution $x \to x - qy, y \to y, z \to z$ kills q in G_2. A scaling turns p into 1 yielding the twisted potential $z^2 x^{\circledcirc} + y^2 z^{\circledcirc}$, which falls into (T20) after a permutation of variables. As for G_3, if $a \neq \alpha$, a substitution $x \to x + sy, y \to y, z \to z$ with an appropriate $s \in \mathbb{K}$ kills p, while if $a \neq \alpha^2$, a substitution $x \to x + sz, y \to y, z \to z$ with an appropriate $s \in \mathbb{K}$ kills q. In any case, we can assume that $pq = 0$, which lands us (up to a permutation of variables) into cases already considered above in this very proof. Finally, a substitution $x \to x + sy + tz, y \to y, z \to z$ with an appropriate $s, t \in \mathbb{K}$, applied to G_4, kills both p and q and again we arrive to a situation already dealt with earlier. Annoyingly, the last case required quite a bit of work while producing no extra twisted potentials. $\square$

LEMMA 6.7. *Let $G \in \mathcal{P}_{3,3}^*$ be non-degenerate, $M \in GL_3(\mathbb{K})$ be the unique matrix providing the twist for G and assume that $A = A_G$ is non-potential. Assume also that M is diagonalizable and has determinant different from 1. If A is non-proper, then A is isomorphic to A_F with F from (T19–T21) of Theorem 1.8. If A is proper, then A is isomorphic to A_F for F from (T11–T18) of Theorem 1.8 with different labels corresponding to non-isomorphic algebras. Furthermore, the relevant isomorphism statements of Theorem 1.8 hold.*

Proof. Applying Remark 1.4 in the same way as in the last proof, we can assume that M is diagonal with $\alpha, \beta, \gamma \in \mathbb{K}^*$ on the main diagonal. Since the determinant of M is different from 1, we have $\alpha\beta\gamma \neq 1$. Analyzing (6.7), we see that $\mathcal{P}_{3,3}(M)$ contains only degenerate twisted potentials unless the eigenvalues of M in some order are $(\alpha, \alpha^{-2}, 1)$ with $\alpha \neq 1$, or $(\alpha, -\alpha, \alpha^{-2})$, or $(\alpha, \alpha^{-2}, \alpha^4)$ with $\alpha^3 \neq 1$ (everywhere $\alpha \in \mathbb{K}^*$).

First, assume that the eigenvalues of M are $(\alpha, \alpha^{-2}, 1)$ with $\alpha \neq 1$. Solving (6.7), we see that

$$F_{s,t} = s(x^2 y + \alpha xyx + \alpha^2 yx^2) + tz^3 \in \mathcal{P}_{3,3}(M) \text{ for } s, t \in \mathbb{K}.$$

Furthermore, there are no other elements in $\mathcal{P}_{3,3}(M)$ unless $\alpha^4 = 1$ or $\alpha^3 = 1$. If $st = 0$, then $F_{s,t}$ is degenerate. Thus, we can assume that $st \neq 0$. By scaling, we can make $s = t = 1$. That is, G is equivalent to $F_{1,1}$, which falls into (T19) and is non-proper according to Lemma 6.1.

Assume now that the eigenvalues of M are $(\alpha, -\alpha, \alpha^{-2})$. According to (6.7),

$$F_{s,t} = s(x^2z + \alpha xzx + \alpha^2 zx^2) + t(y^2z - \alpha yzy + \alpha^2 zy^2) \in \mathcal{P}_{3,3}(M) \quad \text{for } s, t \in \mathbb{K}.$$

Furthermore, there are no other elements in $\mathcal{P}_{3,3}(M)$ unless $\alpha^6 = 1$. If $st = 0$, then $F_{s,t}$ is degenerate and we can assume that $st \neq 0$. By scaling, we can make $s = t = 1$. That is, G is equivalent to $F_{1,1}$, which falls into (T11) with $a = \alpha$. By Lemma 6.2, the corresponding algebras are proper. Since the isomorphic proper twisted potential algebras must have conjugate twists, the isomorphism of two algebras from (T11) corresponding to parameters a and a' is only possible if $a' = a$ or $a' = -a$. In the latter case, the swap of x and y provides a required isomorphism.

Next, assume that the eigenvalues of M are $(\alpha, \alpha^{-2}, \alpha^4)$ with $\alpha^3 \neq 1$. Solving (6.7), we see that

$$F_{s,t} = s(x^2y + \alpha xyx + \alpha^2 yx^2) + t(\alpha^4 y^2z + \alpha^2 yzy + zy^2) \in \mathcal{P}_{3,3}(M) \quad \text{for } s, t \in \mathbb{K}.$$

Furthermore, there are no other elements in $\mathcal{P}_{3,3}(M)$ unless $\alpha^6 = 1$ or $\alpha^4 = 1$ or $\alpha^9 = 1$. If $st = 0$, then $F_{s,t}$ is degenerate and we can assume that $st \neq 0$. By scaling, we can make $s = t = 1$. That is, G is equivalent to $F_{1,1}$, which falls into (T20) after a permutation of variables. By Lemma 6.1, the corresponding algebra is non-proper.

It remains to deal with few specific triples of eigenvalues of M: $(-1, 1, 1)$, $(\alpha, \alpha, -\alpha)$ with $\alpha^3 = 1 \neq \alpha$, $(\alpha, \alpha, 1)$ with $\alpha^3 = 1 \neq \alpha$, $(\alpha, -1, 1)$ with $\alpha^2 = -1$, and $(\alpha, \alpha^4, \alpha^7)$ with $\alpha^9 = 1 \neq \alpha^3$. These are the triples for which there are more solutions than in the generic cases considered above.

First, assume that the eigenvalues of M are $(\alpha, \alpha, 1)$ with $\alpha^3 = 1 \neq \alpha$. By (6.7),

$$\mathcal{P}_{3,3}(M) = \{F_{s,t,r} = s(x^2y + \alpha xyx + \alpha^2 yx^2) + t(y^2x + \alpha yxy + \alpha^2 xy^2) + rz^3 : s, t, r \in \mathbb{K}\}.$$

One easily sees that a linear substitution leaving both z and the space spanned by x and y invariant can be chosen to kill t. This places the twisted potential within the framework of the very first case considered in this proof.

Next, assume that the eigenvalues of M are $(\alpha, \alpha, -\alpha)$ with $\alpha^3 = 1 \neq \alpha$. Solving (6.7), we see that

$$\begin{aligned}F_{s,t,p,q} = {}&s(x^2y + \alpha xyx + \alpha^2 yx^2) + t(y^2x + \alpha yxy + \alpha^2 xy^2) + p(z^2x - \alpha zxz + \alpha^2 xz^2) \\ &+ q(z^2y - \alpha zyz + \alpha^2 yz^2),\end{aligned}$$

with $s, t, p, q \in \mathbb{K}$ comprise the space $\mathcal{P}_{3,3}(M)$. One can easily verify that a linear substitution leaving both z and the space spanned by x and y invariant can be chosen to kill either t and p or s and p. In both cases, we fall into situations already dealt with in this proof.

Now assume that the eigenvalues of M are $(\alpha, -1, 1)$ with $\alpha^2 = -1$. According to (6.7),

$$\mathcal{P}_{3,3}(M) = \{F_{s,t,r} = s(x^2y + \alpha xyx - yx^2) + t(y^2z - yzy + zy^2) + rz^3 : s, t, r \in \mathbb{K}\}.$$

If $rt = 0$, we are back to the already considered cases. If $s = 0$, our twisted potential is degenerate. Thus, we can assume $str \neq 0$. By scaling, we can make $s = t = r = 1$. That is,

G is equivalent to $F_{1,1,1}$. By swapping x and z, we see that G is equivalent to the twisted potential

$$F = z^2 y + \alpha zyz - yz^2 + y^2 x - yxy + xy^2 + x^3 \quad \text{with } \alpha = \pm i,$$

which are the two twisted potentials from (T12) and (T13). By Lemma 6.2, they are proper.

Next, assume that the eigenvalues of M are $(\alpha, \alpha^4, \alpha^7)$ with $\alpha^9 = 1 \neq \alpha^3$. By (6.7),

$$\mathcal{P}_{3,3}(M) = \{ F_{s,t,r} = s(x^2 z + \alpha xzx + \alpha^2 zx^2) + t(y^2 x + \alpha^4 yxy + \alpha^8 xy^2)$$
$$+ r(z^2 y + \alpha^7 zyz + \alpha^5 yz^2) : s, t, r \in \mathbb{K} \}.$$

If $str = 0$, we are back to the already considered cases and we know that the corresponding twisted potential algebra is non-proper. If $str \neq 0$, by scaling we can make $s = t = r = 1$. Thus, G in this case is equivalent to $F_{1,1,1}$. An easy computation shows that this time the corresponding twisted potential algebra is proper. Note that the assumption $\alpha^9 = 1 \neq \alpha^3$ is the same as $\alpha \in \{\xi_9, \xi_9^2, \xi_9^4, \xi_9^5, \xi_9^7, \xi_9^8\}$. Since cyclic permutations of x, y, and z provide equivalence of $F_{1,1,1}$ for $\alpha \in \{\xi_9, \xi_9^4, \xi_9^7\}$ as well as for $\alpha \in \{\xi_9^2, \xi_9^5, \xi_9^8\}$, we have just two twisted potentials to deal with in this case: $F_{1,1,1}$ for $\alpha = \xi_9$ and $F_{1,1,1}$ for $\alpha = \xi_9^2$. By Lemma 6.2, the algebras in (T14) and (T15) are proper and their respective twists have eigenvalues ξ_9, ξ_9^4, ξ_9^7 and $\xi_9^2, \xi_9^5, \xi_9^8$. Thus, they are isomorphic to $F_{1,1,1}$ for $\alpha = \xi_9$ and $\alpha = \xi_9^2$, respectively.

It remains to deal with the final case when the eigenvalues of M are $(-1, 1, 1)$. By (6.7),

$$\mathcal{P}_{3,3}(M) = \{ G_w = w_1 y^3 + w_2 y^2 z^{\circlearrowleft} + w_3 yz^{2\circlearrowleft} + w_4 z^3$$
$$+ w_5(x^2 y - xyx + yx^2) + w_6(x^2 z - xzx + zx^2) : w \in \mathbb{K}^6 \}.$$

If $w_5 = w_6 = 0$, G_w is degenerate. Thus, we can assume that $(w_5, w_6) \neq (0, 0)$. Now it is easy to see that a substitution leaving both x and the linear span of y, z intact preserves the form of G_w and turns (w_5, w_6) into $(0, 1)$. Now we have only to consider

$$F_u = u_1 y^3 + u_2 y^2 z^{\circlearrowleft} + u_3 yz^{2\circlearrowleft} + u_4 z^3 + x^2 z - xzx + zx^2 \quad \text{with } u = (u_1, \dots, u_4) \in \mathbb{K}^4.$$

The only substitutions which preserve this general shape of a twisted potential are given by $x \to sx$, $z \to s^{-2} z$, $y \to py + qz$ with $s, p \in \mathbb{K}^*$, $q \in \mathbb{K}$. This substitution transforms F_u into $F_{u'}$ with $u_1' = p^3 u_1$, $u_2' = p^2 s^{-2} u_2 + p^2 qu_1$, $u_3' = ps^{-4} u_3 + 2pqs^{-2} u_2 + pq^2 u_1$ and $u_4' = s^{-6} u_4 + 3s^{-4} qu_3 + 3s^{-2} q^2 u_2 + q^3 u_1$. Now it is easy to see that a general F_u can be transformed into one of the following forms $F_{1,0,1,a}$ with $a \in \mathbb{K}$, $F_{1,0,0,1}$, $F_{1,0,0,0}$, $F_{0,1,0,1}$, $F_{0,1,0,0}$, $F_{0,0,1,0}$, $F_{0,0,0,1}$, and $F_{0,0,0,0}$ among which there are no equivalent ones except for $F_{1,0,1,a}$ being equivalent to $F_{1,0,1,-a}$ for $a \in \mathbb{K}$. Among these $F_{0,0,0,1}$ and $F_{0,0,0,0}$ are degenerate, while $F_{0,1,0,0}$, $F_{0,0,1,0}$, and $F_{1,0,0,0}$ fall into the cases already considered in this proof. This leaves us to deal with $F_{1,0,1,a}$ with $a \in \mathbb{K}$, $F_{1,0,0,1}$ and $F_{0,1,0,1}$. First, $F_{1,0,0,1} = y^3 + z^3 + x^2 z - xzx + zx^2$ is non-proper and features as (T21). Next,

$$G = F_{0,1,0,1} = y^2 z^{\circlearrowleft} + z^3 + x^2 z - xzx + zx^2,$$

features as (T16) and is proper by Lemma 6.2. Since $F_{1,0,1,a}$ and $F_{1,0,1,-a}$ are equivalent, the case of $G = F_{1,0,1,a}$ with $a^2 + 4 = 0$ reduces to $G = F_{1,0,1,2i} = y^3 + yz^{2\circlearrowleft} + 2iz^3 + x^2 z - xzx + zx^2$. The substitution $x \to z$, $z \to ix$, $y \to x + y$ followed by an appropriate scaling

turns the latter into the twisted potential $y^3 + xy^2{}^\circlearrowleft + z^2x - zxz + xz^2$ of (T17), which is proper by Lemma 6.2. This leaves only $G = F_{1,0,1,a}$ with $a^2 + 4 \neq 0$:

$$G = F_{1,0,1,a} = y^3 + yz^2{}^\circlearrowleft + az^3 + x^2z - xzx + zx^2 \quad \text{with } a \in \mathbb{K}, a^2 + 4 \neq 0.$$

The latter are twisted potentials from (T18). By Lemmas 6.2, they are proper. Knowing which of them are equivalent justifies the isomorphism condition in (T18).

Finally, the absence of isomorphism for algebras with different labels follows from the fact that proper twisted potential algebras with non-conjugate twist cannot be isomorphic. $\qquad\square$

6.1. Proof of Theorem 1.8. Theorem 1.8 is just an amalgamation of Lemmas 2.15, 6.1, 6.2, 6.3, 6.4, 6.5, 6.6, and 6.7.

7. Concluding remarks.

REMARK 7.1. Note that according to Theorem 1.7, there is only one (up to an isomorphism) proper quadratic potential algebra on three generators, which fails to be exact. Namely, it is the algebra given by (P9). Furthermore, there are exactly two non-Koszul (up to an isomorphism) quadratic potential algebras on three generators: (P9) and (P14). By Theorem 1.9, there is only one (up to an isomorphism) proper cubic potential algebra on two generators, which fails to be exact: it features with the label (P23). However, we do not expect this pattern to extend to higher degrees or higher numbers of generators.

REMARK 7.2. By Theorems 1.7 and 1.9, both sets $\{H_{A_F} : F \in \mathcal{P}_{3,3}\}$ and $\{H_{A_F} : F \in \mathcal{P}_{2,4}\}$ are finite. Indeed, the first set has seven elements, while the second has five elements. By Proposition 2.13, $\{H_{A_F} : F \in \mathcal{P}_{2,3}\}$ is a three-element set. This leads to the following question (we expect an affirmative answer).

QUESTION 7.3. Let $n \geqslant 2$ and $k \geqslant 3$. Is it true that the set $\{H_{A_F} : F \in \mathcal{P}_{n,k}\}$ is finite?

REMARK 7.4. By Theorems 1.7 and 1.9, H_{A_F} is rational for every $F \in \mathcal{P}_{3,3}$ as well as for every $F \in \mathcal{P}_{2,4}$. Proposition 2.13, the same holds for $F \in \mathcal{P}_{2,3}$. This prompts the following question (again, we believe the answer to be affirmative).

QUESTION 7.5. Is it true that the Hilbert series of every degree-graded potential algebra is rational?

The above question resonates with the following issue. It was believed at some point that graded finitely presented algebras must have rational Hilbert series. This conjecture was disproved by Shearer [17], who produced an example of a quadratic algebra with non-rational Hilbert series. However, his algebra as well as any of the later examples fail to be potential or Koszul. Note that the question whether the Hilbert series of a Koszul algebra must be rational is a long-standing open problem, see, for instance, [16].

REMARK 7.6. By Theorems 1.8 and 1.10, every non-potential proper twisted potential algebra A_F with $F \in \mathcal{P}_{3,3}^* \cup \mathcal{P}_{2,4}^*$ is exact. Furthermore, every non-potential twisted potential algebra A_F with $F \in \mathcal{P}_{3,3}^*$ is Koszul.

REMARK 7.7. Note, that in quadratic case, we examined twisted potential algebras for Koszulity and PBW property. Both concepts have no sense for cubic algebras; however, one

can consider analogous notions and study 3-Koszul and 3-PBW for cubic twisted potential algebras. We expect similar results for cubic case and plan to present these elsewhere.

REMARK 7.8. From the obtained classification, one can see that an algebra is Artin–Shelter regular iff it is an exact twisted potential algebra given by cubic twisted potential in three generators or degree 4 twisted potential in two generators. This can be also deduced using known connections to twisted Calabi–Yau (see [4, 3]). In fact, it follows in the above cases that the three notions of exactness, twisted 3-Calabi–Yau, and Artin–Shelter regularity are equivalent.

REMARK 7.9. Using our classification, one can observe that for $F \in \mathcal{P}_{n,k}^*$ with $(n, k) \in \{(3, 3), (2, 4)\}$, A_F is a domain (=has no zero divisors) if and only if A_F is exact if and only if $H_{A_F} = (1 - nt + nt^{k-1} - t^k)^{-1}$. We wonder if this equivalences hold for other pairs (n, k).

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