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ON NON-UNIFORM AND GLOBAL DESCRIPTIONS OF THE RATE OF CONVERGENCE OF ASYMPTOTIC EXPANSIONS IN THE CENTRAL LIMIT THEOREM

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Abstract

The leading term approach to rates of convergence is employed to derive non-uniform and global descriptions of the rate of convergence in the central limit theorem. Both upper and lower bounds are obtained, being of the same order of magnitude, modulo terms of order n^{-r}. We are able to derive general results by considering only those expansions with an *odd* number of terms.

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1. Introduction and results

Using the leading term approach [1, 2] we obtain new bounds to rates of convergence in various metrics. These results are combined to complement earlier work of Heyde and Nakata [3]. By way of notation, let $X, X_1, X_2, \dots$ be i.i.d.r.v. with zero mean and unit variance, let $S_n = \sum_{j=1}^n X_j$, $F_n(x) = P(S_n \leqslant n^{1/2}x)$, and $(2\pi)^{1/2}\Phi(x) = \int_{-\infty}^x e^{-u^2/2}\,du$. Assume that $E(X^{2k+2}) < \infty$ for integer $k \geqslant 0$, let $\mu_j = E(X^j)$ for $1 \leqslant j \leqslant 2k+2$, define μ_{2k+3} arbitrarily, and define cumulants κ_j $(1 \leqslant j < \infty)$, polynomials P_j and Q_j, leading term function ${}_kL_n(x)$ and its order of magnitude

$$ {}_k\delta_n \equiv n^{-k}E\{X^{2k+2}I(|X| > n^{1/2})\} + n^{-(k+1)}E\{X^{2k+4}I(|X| \leqslant n^{1/2})\} + n^{-(2k+1)/2}\left|E\{X^{2k+3}I(|X| \leqslant n^{1/2})\} - \mu_{2k+3}\right| $$

as in [2].

Theorem 1 is a nonuniform description of error in Chebyshev-Edgeworth-Cramér expansions after the leading term has been accounted for, while Theorem 2 describes order of magnitude of the leading term.

THEOREM 1. *Assume* $E(X^{2k+2}) < \infty$ *for an integer* $k \geqslant 0$, *and that Cramér's condition*, $\limsup_{t\to\infty}|E(e^{itX})| < 1$, *holds. Then*

(1.1)
$$\sup_{-\infty<x<\infty}(1+|x|^{2k+2})\left|F_n(x)-\Phi(x)-\phi(x)\sum_{j=1}^{2k+1}Q_j(x)n^{-j/2}-{}_kL_n(x)\right|$$
$$= O\left(n^{-1/2}{}_k\delta_n + {}_k\delta_n^2 + n^{-(k+1)}\right).$$

THEOREM 2. *Assume* $E(X^{2k+2}) < \infty$, *and let* $E = (0,\varepsilon)$ *or* $(-\varepsilon, 0)$. *Then*

$$\liminf_{n\to\infty}\sup_{|x|\leqslant\varepsilon}|{}_kL_n(x)|/{}_k\delta_n, \qquad \liminf_{n\to\infty}\left\{\int_E|{}_kL_n(x)|^p\,dx\right\}^{1/p}\Big/{}_k\delta_n$$

are strictly positive for any $\varepsilon > 0$ *and* $p \geqslant 1$. *Moreover, for a constant* $C > 0$ *depending only on* k, *we have* $\sup_{-\infty<x<\infty}(1+x^{2k+2})|{}_kL_n(x)| < C_k\delta_n$.

Theorems 1 and 2 are readily combined to give various descriptions of rates of convergence in nonuniform and L^p metrics. For example, we extend a portion of Theorem 1 in [3]:

COROLLARY. *Assume* $E(X^{2k+2}) < \infty$, *Cramér's condition, and* $x^{2k+4}P(|X| > x) \to \infty$ *as* $x \to \infty$. *Then for any* $p \geqslant 1$ *and* $r < p(2k+2) - 1$, *the ratio*

$$\left\{\int_{-\infty}^{\infty}(1+|x|^r)\left|F_n(x)-\Phi(x)-\phi(x)\sum_{j=1}^{2k+1}Q_j(x)n^{-j/2}\right|^p dx\right\}^{1/p}\Big/{}_k\delta_n$$

is bounded away from zero and infinity as $n \to \infty$.

2. Proofs

We prove only Theorem 1. The proof of Theorem 2 parallels arguments in [1, Theorems 2.3 and 2.5] and [2]. Symbols C and p denote respectively a generic positive constant and a generic positive integer.

The characteristic function (Fourier-Stieltjes transform) of

$$ {}_kD_n(x) \equiv F_n(x) - \Phi(x) - \sum_{j=1}^{2k+1} P_j(-\Phi)(x) n^{-j/2} - {}_kL_n(x) $$

is given by

$$ {}_kd_n(t) \equiv \alpha^n(t/n^{1/2}) - e^{-t^2/2} \sum_{j=0}^{2k+1} P_j(it) n^{-j/2} $$
$$ - n\left\{\alpha(t/n^{1/2}) - \sum_{j=0}^{2k+3} \mu_j (it/n^{1/2})^j/j!\right\} e^{-t^2/2}. $$

The function $x^{2k+2}{}_kD_n(x)$ has Fourier-Stieltjes transform

$$ \chi(t) = it(d/dt)^{2k+2}\{(it)^{-1}{}_kd_n(t)\}. $$

Therefore, by [4, Lemma 8, page 155], we have

$$ \sup_{-\infty<x<\infty} (1+|x|^{2k+2})|{}_kD_n(t)| \tag{2.1} $$
$$ \leqslant C\left\{\int_0^T (t^{-1} + t^{-(2k+3)})|{}_kd_n(t)|\,dt \right. $$
$$ \left. + \sum_{r=0}^{2k+1} \int_0^T t^{-(r+1)}|{}_kd_n^{(2k+2-r)}(t)|\,dt + T^{-1}\right\} $$

for all $T > 1$, provided $\sup_{n,x}(1+|x|^{2k+2})|(d/dx)\{{}_kD_n(x) - F_n(x)\}| < \infty$. The latter inequality follows via a short algebraic argument, using techniques of [1, pages 30–33].

The remainder of the proof consists of estimating the terms on the right hand side of (2.1). We first estimate ${}_kd_n^{(l)}(t) = (d/dt)^l{}_kd_n(t)$ for t in the range $0 < t \leqslant n^\delta$, and for some $\delta \in (0, \frac{1}{2}]$. This is carried out in several stages. The following lemma will prove useful. Define

$$ A_{n1}(t) = \alpha(t/n^{1/2}) - 1 - \sum_{j=2}^{2k+3} \mu_j (it/n^{1/2})^j/j!. $$

LEMMA 2.1. *The following estimates are valid for all* $t > 0$, *and for all integers* $0 \leqslant l \leqslant 2k+2$:

$$ |A_{n1}^{(l)}(t)| \leqslant C(1+t^2)t^{2k+2-l}n^{-(k+1)}, \tag{2.2} $$

$$ n|A_{n1}^{(l)}(t)| \leqslant C(1+t^2)t^{2k+2-l}{}_k\delta_n. \tag{2.3} $$

Furthermore, for each $\varepsilon > 0$, *there exists* $\delta > 0$ *such that, whenever* $0 < t \leqslant \delta n^{1/2}$, *we have*

$$ n|A_{n1}(t/n^{1/2})| \leqslant \varepsilon t^2. \tag{2.4} $$

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PROOF. Since ${}_k\delta_n = o(n^{-k})$, it suffices to prove (2.3) and (2.4). In (2.3), we shall assume that $l = 2m + 1$ is odd; the case of even l may be treated similarly. Now,

$$\begin{aligned}|A_{n1}^{(l)}(t)| \leqslant & \left|(d/dt)^l E\left\{\cos(tX/n^{1/2}) - \sum_{j=0}^{k+1}(-1)^j (tX/n^{1/2})^{2j}/(2j)!\right\}\right| \\ & + \left|(d/dt)^l E\left\{\sin(tX/n^{1/2}) - \sum_{j=0}^{k}(-1)^j (tX/n^{1/2})^{2j+1}/(2j+1)!\right\}\right. \\ & \left. - (d/dt)^l(-1)^{k+1}\mu_{2k+3}(t/n^{1/2})^{2k+3}/(2k+3)!\right| \\ \leqslant & \, 2\Big[E\Big\{|X/n^{1/2}|^l |tX/n^{1/2}|^{2(k-m)+3} I(|X| \leqslant n^{1/2})\Big\} \\ & + E\Big\{|X/n^{1/2}|^l |tX/n^{1/2}|^{2(k-m)+1} I(|X| > n^{1/2})\Big\}\Big] \\ & + n^{-(2k+3)/2}\left|E\{X^{2k+3} I(|X| \leqslant n^{1/2})\} - \mu_{2k+3}\right| t^{2k+3-l} \\ \leqslant & \, 2(1+t^2)t^{2k+2-l}n^{-1}{}_k\delta_n,\end{aligned}$$

as required for (2.3). Result (2.4) follows from the fact that $|\alpha(t) - 1 + t^2/2| = o(t^2)$ as $t \to 0$.

Choose $\varepsilon > 0$ so small that $|\alpha(t) - 1| \leqslant 1/2$ for $0 < t \leqslant 2\varepsilon$. Define

$$A_{n2}(t) \equiv n \log \alpha(t/n^{1/2}) = n \sum_{j=1}^{\infty} (-1)^{j+1}\{\alpha(t/n^{1/2}) - 1\}^j/j.$$

Then for $0 < t < \varepsilon n^{1/2}$, we have

$$\begin{aligned}A_{n3}(t) \equiv & \left|(d/dt)^l\left[A_{n2}(t) - n\sum_{j=1}^{3k+3}(-1)^{j+1}\{\alpha(t/n^{1/2}) - 1\}^j/j\right]\right| \\ \leqslant & \, n \sum_{j=3k+4}^{\infty} j^{-1}\left|(d/dt)^l\{\alpha(t/n^{1/2}) - 1\}^j\right|.\end{aligned}$$

Since α has $2k+2$ bounded derivatives, and since $|\alpha(t) - 1| \leqslant t^2/2$, it follows that for $j \geqslant 3k+4$ and $0 \leqslant l \leqslant 2k+2$, we have

$$\left|(d/dt)^l\{\alpha(t) - 1\}^j\right| \leqslant Cj^l(t^2/2)^{k+2}|\alpha(t) - 1|^{j-l-(k+2)},$$

where C does not depend on j. Therefore, if $0 < t < \varepsilon n^{1/2}$, we have

(2.5)
$$A_{n3}(t) \leqslant C_1 n n^{-l/2}(t^2/2n)^{k+2}\sum_{j=0}^{\infty}(j+3k+4)^l(1/2)^j \leqslant C_2 t^{2(k+2)}n^{-(k+1)}.$$

Next, observe from (2.2) that

(2.6)

$$n^{-1}A_{n4}(t) \equiv \left|(d/dt)^l\left[\sum_{r=2}^{3k+3}(-1)^{r+1}\{\alpha(t/n^{1/2})-1\}^r/r - \sum_{r=2}^{3k+3}(-1)^{r+1}\left\{\sum_{j=2}^{2k+3}\mu_j(it/n^{1/2})^j/j!\right\}^r/r\right]\right|$$

$$\leqslant C_1\sum_{r=2}^{3k+3}\sum_{s=1}^{r}\sum_{a=0}^{l}\left|(d/dt)^a\{A_{n1}(t)\}^s\right|\left|(d/dt)^{l-a}\left\{\sum_{j=2}^{2k+3}\mu_j(it/n^{1/2})^j/j!\right\}^{r-s}\right|$$

$$\leqslant C_2(1+t^p)t^{2k+2-l}n^{-(k+2)}.$$

It follows from the definition of cumulants κ_j that

$$(2.7)\quad A_{n5}(t) \equiv n\left|(d/dt)^l\left[\sum_{r=1}^{3k+3}(-1)^{r+1}\left\{\sum_{j=2}^{2k+3}\mu_j(it/n^{1/2})^j/j!\right\}^r/r - \sum_{r=2}^{2k+3}\kappa_j(it/n^{1/2})^j/j!\right]\right|$$

$$\leqslant C_2(1+t^p)t^{2(k+2)-l}n^{-(k+1)}$$

for all $t > 0$. Combining (2.5), (2.6) and (2.7), we see that if $0 < t < \varepsilon n^{1/2}$ and $0 \leqslant l \leqslant 2k+2$, then

$$(2.8)\quad \left|(d/dt)^l\left[n\log\alpha(t/n^{1/2}) - n\left\{\alpha(t/n^{1/2}) - 1 - \sum_{j=2}^{2k+3}\mu_j(it/n^{1/2})^j/j!\right\} - n\sum_{j=2}^{2k+3}\kappa_j(it/n^{1/2})^j/j!\right]\right|$$

$$\leqslant A_{n3}(t) + A_{n4}(t) + A_{n5}(t) \leqslant C(1+t^p)t^{2(k+2)-l}n^{-(k+1)}.$$

Let

$$A_{n6}(t) = n\left\{\alpha(t/n^{1/2}) - 1 - \sum_{j=2}^{2k+3}\mu_j(it/n^{1/2})^j/j! + \sum_{j=2}^{2k+3}\kappa_j(it/n^{1/2})^j/j!\right\},$$

and let $A_{n7}(t) = n\log\alpha(t/n^{1/2}) - A_{n6}(t)$. Then result (2.8) may be written in the form

$$(2.9)\qquad |(d/dt)A_{n7}(t)| \leqslant C(1+t^p)t^{2(k+2)-l}n^{-(k+1)}.$$

We may deduce from (2.6) that

(2.10) $$\left|(d/dt)^l A_{n6}(t)\right| \leqslant C(1+t^p).$$

Let $\Sigma_{(l.1)}$ denote summation over vectors $(i_1,\ldots,i_r)$ with $1 \leqslant r \leqslant l$, $i_a \geqslant 1$ for each a, and $\Sigma_{a=1}^r i_a = l$. Let $\Sigma_{(l.2)}$ denote summation over vectors $(i_1,\ldots,i_r)$ and $(j_1,\ldots,j_s)$ with $0 \leqslant r \leqslant l-1$, $1 \leqslant s \leqslant l$, $i_a \geqslant 1$ and $j_a \geqslant 1$ for each a, and $\Sigma_{a=1}^r i_a + \Sigma_{a=1}^s j_a = l$. Combining (2.9) and (2.10), we see that

(2.11)
$$\begin{aligned} A_{n8}(t) &\equiv \left|(d/dt)^l\left[\alpha^n(t/n^{1/2}) - \exp\{A_{n6}(t)\}\right]\right| \\ &\leqslant C_1\Bigg(\sum_{(l.1)} \left|A_{n6}^{(i_1)}(t)\cdot\ldots\cdot A_{n6}^{(i_r)}(t)\right|\cdot\left|\exp\{A_{n6}(t)\}\left[1-\exp\{A_{n7}(t)\}\right]\right| \\ &\qquad + \sum_{(l.2)} \left|A_{n6}^{(i_1)}(t)\cdot\ldots\cdot A_{n6}^{(i_r)}(t) A_{n7}^{(j_1)}(t)\cdot\ldots\cdot A_{n7}^{(j_s)}(t)\right| \\ &\qquad\qquad \cdot\left|\exp\{A_{n6}(t)+A_{n7}(t)\}\right|\Bigg) \\ &\leqslant C_2(1+t^p)\left|\alpha^n(t/n^{1/2})\right|\left[\left|1-\exp\{-A_{n7}(t)\}\right| + t^{2(k+2)-l}n^{-(k+1)}\right]. \end{aligned}$$

In view of (2.9) (with $l = 0$), there exists $\delta \in (0, 1/2)$ such that $|A_{n7}(t)| \leqslant Ct$ whenever $0 < t < n^\delta$ and n is sufficiently large. For such values of t, we have

(2.12) $$\left|1-\exp\{-A_{n7}(t)\}\right| \leqslant C_1 t^{2(k+2)-l}\exp(C_2 t)n^{-(k+1)}.$$

An elementary argument shows that there exist $\varepsilon_1, \varepsilon_2 > 0$ such that

(2.13) $$\left|\alpha^n(t/n^{1/2})\right| \leqslant \exp(-2\varepsilon_1 t^2)$$

if $0 < t \leqslant \varepsilon_2 n^{1/2}$. Combining (2.11), (2.12) and (2.13), we see that if $0 < t < n^\delta$ and n is sufficiently large, then

(2.14) $$A_{n8}(t) \leqslant Ct^{2(k+2)-l}\exp(-\varepsilon_1 t^2)n^{-(k+1)}.$$

Next, observe that

(2.15)
$$\begin{aligned} A_{n9}(t) &\equiv \left|(d/dt)^l\left[\exp\{A_{n6}(t)\} - \exp\{nA_{n1}(t)\}\left\{1+\sum_{j=1}^{2k+1} P_j(it)n^{-j/2}\right\}e^{-t^2/2}\right]\right| \\ &\leqslant \sum_{a=0}^{l}\binom{l}{a}\left|(d/dt)^a\exp\{nA_{n1}(t)\}\right| \\ &\quad \times\left|(d/dt)^{l-a}\left(\left[\exp\left\{n\sum_{j=3}^{2k+3}\kappa_j(it/n^{1/2})^j/j!\right\}\right.\right.\right. \\ &\qquad\qquad \left.\left.\left. -\left\{1+\sum_{j=1}^{2k+1}P_j(it)n^{-j/2}\right\}\right]e^{-t^2/2}\right)\right|. \end{aligned}$$

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Now,

$$\exp\left\{n\sum_{j=3}^{2k+3}\kappa_j\left(it/n^{1/2}\right)^j/j!\right\} = 1 + \sum_{j=1}^{2k+1}P_j(it)n^{-j/2}$$
$$+\sum_{r=2k+2}^{\infty}\left\{n\sum_{j=3}^{2k+3}\kappa_j\left(it/n^{1/2}\right)^j/j!\right\}^r/r$$
$$+n\left(it/n^{1/2}\right)^{2(k+2)}A_{n,10}(t),$$

where $A_{n,10}(t)$ denotes a polynomial in it of degree not exceeding $(2k+1)\times(2k+3)-2(k+2)$, and all of whose coefficients are uniformly bounded. Therefore,

$$A_{n,11}(t) \equiv \left|(d/dt)^l\left[\exp\left\{n\sum_{j=3}^{2k+3}\kappa_j\left(it/n^{1/2}\right)^j/j!\right\} - \left\{1+\sum_{j=1}^{2k+1}P_j(it)n^{-j/2}\right\}\right]\right|$$
$$\leqslant \sum_{r=2k+2}^{\infty}\left|(d/dt)^l\left\{n\sum_{j=3}^{2k+3}\kappa_j\left(it/n^{1/2}\right)^j/j!\right\}^r\right|/r!$$
$$+C(1+t^p)t^{2(k+2)-l}n^{-(k+1)}.$$

The series on the right hand side is dominated by $C(1+t^p)t^{6(k+1)}n^{-(k+1)}$ uniformly in $0 < t < n^{1/2}$. Consequently,

$$A_{n,11}(t) \leqslant C(1+t^p)t^{2(k+2)-l}n^{-(k+1)}. \tag{2.16}$$

It follows from (2.2) and (2.4) that if $\varepsilon \in (0,1]$ is sufficiently small, and if $0 < t < \varepsilon n^{1/2}$, then

$$\left|(d/dt)^l\exp\{nA_{n1}(t)\}\right| \leqslant Ce^{-t^2/4}. \tag{2.17}$$

Substituting (2.16) and (2.17) into (2.15), we see that if $0 < t < \varepsilon n^{1/2}$, then

$$A_{n9}(t) \leqslant Ct^{2(k+2)-l}\exp(-t^2/5)n^{-(k+1)}. \tag{2.18}$$

Since

$$\left|(d/dt)^l\left[\left\{1+\sum_{j=1}^{2k+1}P_j(it)n^{-j/2}\right\}e^{-t^2/2}\right]\right| \leqslant C(1+t^p)e^{-t^2/2},$$

it follows that

$$(2.19)\quad A_{n,12}(t) \equiv \left|(d/dt)^l\left[\exp\{nA_{n1}(t)\}\left\{1+\sum_{j=1}^{2k+1}P_j(it)n^{-j/2}\right\}e^{-t^2/2} - \{1+nA_{n1}(t)\}\left\{1+\sum_{j=1}^{2k+1}P_j(it)n^{-j/2}\right\}e^{-t^2/2}\right]\right|$$

$$\leqslant C_1(1+t^p)e^{-t^2/2}\sum_{r=2}^{\infty}(r!)^{-1}\sum_{a=0}^{l}\left|(d/dt)^a\{nA_{n1}(t)\}^r\right|.$$

Now,

$$A_{n,13}(t) \equiv \sum_{a=0}^{l}\left|(d/dt)^a\{nA_{n1}(t)\}^r\right|$$

$$\leqslant Cr^l\left\{|nA_{n1}(t)|^{r-2} + |nA_{n1}(t)|^{\max(0,r-2-l)}\right\}$$

$$\times\left[|nA_{n1}(t)|^2 + |nA_{n1}(t)|\sum_{a=1}^{l}|nA_{n1}^{(a)}(t)| + \sum_{a=2}^{l}\Sigma_{(a.3)}\left|nA_{n1}^{(j_1)}(t)\cdot\ldots\cdot nA_{n1}^{(j_s)}(t)\right|\right],$$

where C does not depend on r, and where $\Sigma_{(a\cdot 3)}$ denotes summation over vectors $(j_1,\ldots,j_s)$ with $2 \leqslant s \leqslant a$, each $j_b \geqslant 1$, and $\Sigma_{b=1}^{s} j_b = a$. Using (2.3) and (2.4), we see that if $0 < t < \varepsilon n^{1/2}$, and if ε is sufficiently small, then

$$A_{n,13}(t) \leqslant Cr^l\left\{(t^2/10)^{r-2} + (t^2/10)^{\max(0,r-2-l)}\right\}(1+t^p)t^{2k+3-l}{}_k\delta_n^2,$$

where C does not depend on r. Substituting into (2.19), we obtain

$$(2.20)\qquad A_{n,12}(t) \leqslant Ct^{2k+3-l}\exp(-t^2/4)_k\delta_n^2.$$

In view of (2.3), we have

$$(2.21)\quad A_{n,14}(t) \equiv \left|(d/dt)^l\left[\{1+nA_{n1}(t)\}\left\{1+\sum_{j=1}^{2k+1}P_j(it)n^{-j/2}\right\}e^{-t^2/2} - \left\{1+\sum_{j=1}^{2k+1}P_j(it)n^{-j/2}+nA_{n1}(t)\right\}e^{-t^2/2}\right]\right|$$

$$\leqslant C(1+t^p)t^{2k+3-l}\exp(-t^2/2)n^{-1/2},$$

uniformly in t. (Note that the polynomials $P_j(it)$ satisfy $P_j(0) = 0$.) Combining (2.14), (2.18), (2.20) and (2.21), we see that if $0 < t < n^{\delta}$, and if n is sufficiently large, then

$$(2.22)\qquad \left|{}_k d_n^{(l)}(t)\right| \leqslant A_{n8}(t) + A_{n9}(t) + A_{n,12}(t) + A_{n,14}(t)$$
$$\leqslant Ct^{2k+3-l}\exp(-\varepsilon t^2)\left(n^{-(k+1)} + {}_k\delta_n^2 + n^{-1/2}{}_k\delta_n\right).$$

To treat the case where $t > n^{\delta}$, we observe that

$$(2.23)\qquad \left|{}_k d_n^{(l)}(t)\right| \leqslant \left|(d/dt)^l \alpha^n(t/n^{1/2})\right|$$
$$+\left|(d/dt)^l\left[\left\{1 + \sum_{j=1}^{2k+1} P_j(it)n^{-j/2} + nA_{n1}(t)\right\}e^{-t^2/2}\right]\right|$$
$$\leqslant C_1(1 + t^p)n^p\left\{\left|\alpha(t/n^{1/2})\right|^{n-l} + e^{-t^2/2}\right\}$$

for all $t > 0$. It follows from (2.13) that for some $\varepsilon \in (0, 1/4)$ and $\eta > 0$, we have $|\alpha(t/n^{1/2})|^{n-l} \leqslant C\exp(-2\varepsilon t^2)$ whenever $0 < t < \eta n^{1/2}$. Consequently, $|{}_k d_n^{(l)}(t)| \leqslant Ct^{2k+3-l}\exp(-\varepsilon t^2)n^{-(k+1)}$ uniformly in $n^{\delta} < t < \eta n^{1/2}$. Therefore (2.22) holds uniformly in $0 < t < \eta n^{1/2}$.

Let

$$\rho = \sup_{t > \eta}|\alpha(t)| < 1.$$

If $\eta n^{1/2} < t < n^{k+1}$, then it follows from (2.23) that

$$(2.24)\qquad \left|{}_k d_n^{(l)}(t)\right| \leqslant C_2(1 + t^p)n^p\left(\rho^n + e^{-t^2/2}\right) < C_3 r^n$$

for some $r < 1$.

Returning to (2.1), taking $T = n^{k+1}$, and estimating $|{}_k d_n^{(l)}(t)|$ using (2.22) if $0 < t < \eta n^{1/2}$, or using (2.24) if $\eta n^{1/2} < t < n^{k+1}$, we see that

$$\sup_{-\infty < x < \infty}(1 + |x|^{2k+2})\left|{}_k D_n(t)\right| = O\left(n^{-(k+1)} + {}_k\delta_n^2 + n^{-1/2}{}_k\delta_n\right),$$

which proves Theorem 1.

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