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Nagoya Math. J.
Vol. 70 (1978), 25-40

AN EXAMPLE ON CANONICAL ISOMORPHISM

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A nonnegative locally Hölder continuous second order differential $P = P(z)dxdy$ $(z = x + iy)$ on a Riemann surface R is referred to as a *density* on R. A density P is said to be *finite* if P is integrable over R, i.e.

$$(1) \qquad \int_R P(z)dxdy < \infty .$$

Suppose that R is hyperbolic, i.e. there exists the harmonic Green's function $G(z, \zeta)$ on R. A density P on such a surface R is said to be *Green energy finite* provided the Green energy integral

$$(2) \qquad \iint_{R\times R} G(z, \zeta)P(z)P(\zeta)dxdyd\xi d\eta < \infty \qquad (\zeta = \xi + i\eta) .$$

Using a density P on a Riemann surface R we can consider a second order selfadjoint elliptic differential equation

$$(3) \qquad \Delta u(z) = P(z)u(z) \qquad (\text{i.e. } d*du = uP)$$

invariantly defined on R. Denote by $P(R)$ the space of C^2 solutions of (3) on R and by $PX(R)$ the space of $u \in P(R)$ with a certain boundedness condition X. As for X we take B to mean the boundedness, D the finiteness of the *Dirichlet integral*

$$D_R(u) = \int_R du \wedge *du ,$$

E the finiteness of the *energy integral* with respect to P:

$$E_R(u) = \int_R (du \wedge *du + u^2P) ,$$

and the combinations BD and BE with obvious meanings. We use the standard notations $H(R)$ and $HX(R)$ for $P(R)$ and $PX(R)$ with $P \equiv 0$,

Received March 17, 1975.

https://doi.org/10.1017/S0027763000021760 Published online by Cambridge University Press

and in this case of harmonic function, $E = D$.

The space $PX(R)$ $(X = B, D, E, BD, BE)$ consists of only constants for nonhyperbolic R, and avoiding such a trivial case we assume that R is hyperbolic. Then the operator T defined by

$$(4) \qquad Tu = u + \frac{1}{2\pi}\int_R G(\cdot, \zeta)P(\zeta)u(\zeta)d\xi d\eta$$

is an injective positive linear operator from $PX(R)$ to $HX(P)$ for $X = B, D, E, BD, BE$ (cf. e.g. [1]). We denote by T_X the operator T considered only on $PX(R)$, i.e.

$$T_X = T \,|\, PX(R) ,$$

and if $T_X : PX(R) \to HX(R)$ is surjective, then we say that $PX(R)$ is *canonically isomorphic* to $HX(R)$. For a systematic exposition on canonical isomorphisms we refer to [2].

It is known (cf. e.g. [1]) that: 1) The space $PBD(R)$ ($PBE(R)$, resp.) is dense in $PD(R)$ ($PE(R)$, resp.) with respect to the topology defined by the uniform convergence on each compact set of R and by the Dirichlet (energy, resp.) integral over R; 2) If P is a Green energy finite (finite, resp.) density on R, then T_{BD} (T_{BE}, resp.) is surjective. In view of these there naturally arises the *question*: Is T_D (T_E, resp.) surjective for any Green energy finite (finite, resp.) density P on R? The purpose of this paper is to answer *negatively* to this question by proving the following

THEOREM. *There exists a both finite and Green energy finite density P on the hyperbolic simply connected Riemann surface R such that T_D and T_E are not surjective.*

It has been known that there exists a density (a finite density, resp.) on the hyperbolic simply connected Riemann surface R such that T_{BD} (T_{BE}, resp.) is surjective and yet T_D (T_E, resp.) is not (Singer [6] ([3], resp.)). Our theorem contains the above as a special case. The simply connectedness of R in our theorem is not an essential restriction. Actually our theorem is true for *any* Riemann surface with the property $HD(R) - HBD(R) \neq \phi$. The only reason we put the restriction is to simplify the reasoning and to avoid inessential complications. At the end of the introduction the author should mention his indebtness to Professor Moses Glasner who gave him an important incentive to the present work.

https://doi.org/10.1017/S0027763000021760 Published online by Cambridge University Press

1. As a conformal representation of the hyperbolic simply connected Riemann surface R we take the unit disk in the complex plane $\boldsymbol{C}$. Thus we always mean in the sequel by R the unit disk $|z| < 1$ and by β the unit circle $|z| = 1$. Then a density P may be considered as a nonnegative locally Hölder continuous function $P(z)$ on R. The *P-unit* e is defined by

$$e(z) = \lim_{r\to 1} e_r(z)$$

where e_r is the solution of (3) on the disk $|z| < r < 1$ with boundary values 1 on $|z| = r$.

We state a sufficient condition for a given positive bounded solution u of (3) on R to be the P-unit e: If

$$(5) \qquad \lim_{r\to 1} u(re^{i\theta}) = 1$$

for almost every $e^{i\theta} \in \beta$, then u is the P-unit e. The harmonic Green's function $G(z, \zeta)$ on R is given by

$$G(z, \zeta) = \log\left|\frac{1 - \bar{\zeta}z}{z - \zeta}\right|$$

and, by (4), the function

$$Tu(z) = u(z) + \frac{1}{2\pi}\int_R \log\left|\frac{1 - \bar{\zeta}z}{z - \zeta}\right| d\mu(\zeta)$$

belongs to $HB(R)$, where $d\mu(\zeta) = P(\zeta)u(\zeta)d\xi d\eta$ is a measure on R with a finite total mass. By the Littlewood theorem (cf. e.g. Tsuji [7])

$$\lim_{r\to 1}\int_R \log\left|\frac{1 - \bar{\zeta}\, re^{i\theta}}{re^{i\theta} - \zeta}\right| d\mu(\zeta) = 0$$

for almost every $e^{i\theta} \in \beta$. Therefore (5) implies that $\lim_{r\to 1} Tu(re^{i\theta}) = 1$ for almost every $e^{i\theta} \in \beta$, and a fortiori the Fatou theorem implies that $Tu = 1$. In particular, $u \leq 1$ on R. By the maximum principle, $u \leq e_r \leq 1$ on $|z| < r$ for every $r \in (0, 1)$. Hence we conclude that $u \leq e \leq 1$ on R and the condition (5) is also satisfied by e. By the same reasoning as above we obtain $Te = 1$, i.e. $Tu = Te$. The injectiveness of T implies that $u = e$.

A key lemma for the proof of our theorem is the following ingenious result of Singer [5]: For every $u \in PD(R)$ we have

$$(6) \qquad D_R(eT_D u) < \infty\ .$$

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Our *program of the proof* is to find an $h \in HD(R)$ and a positive C^∞ subharmonic function e on R with (5) such that $P(z) = \Delta e(z)/e(z)$ satisfies (1) and (2) and yet $D_R(e \cdot h) = \infty$. Then e is the P-unit and $h \notin T_D(PD(R))$ by (6), i.e. T_D is not surjective, from which nonsurjectiveness of T_E follows since $PE(R) \subset PD(R)$.

2. We start with constructing an $h \in HD(R)$. We use the notation $U(\zeta, r)$ to denote the open disk in $\boldsymbol{C}$ with center $\zeta \in \boldsymbol{C}$ and radius $r > 0$. Let $\{d_n\}$ $(n = 0, 1, \cdots)$ be the sequence determined by

$$d_0 = 1\,, \quad d_n/d_{n-1} = \exp(-2\pi n^4) \quad (n = 1, 2, \cdots)\,.$$

We define an $f_n \in C(\boldsymbol{C}) \cap H(U(1, d_{n-1}) - \overline{U(1, d_n)})$ by $f_n = 0$ on $\boldsymbol{C} - U(1, d_{n-1})$ and $f_n = 1$ on $U(1, d_n)$ $(n = 1, 2, \cdots)$. Then

$$D_C(f_n) = 2\pi/\log(d_{n-1}/d_n) = n^{-4}\,.$$

Let $h_n = H^R_{f_n}$, the harmonic function on R with boundary values f_n on β. Clearly $h_n > 0$ on R. By the Dirichlet principle

$$D_R(h_n) \leq D_R(f_n) < n^{-4}\,.$$

By the triangle inequality

$$D_R\Big(\sum_{n=m+1}^{m+p} h_n\Big) \leq \Big(\sum_{n=m+1}^{m+p} n^{-2}\Big)^2$$

for every m and $p = 1, 2, \cdots$. Hence the sequence $\{\sum_{n=1}^m h_n\}$ $(m = 1, 2, \cdots)$ in $HD(R)$ is convergent in the Dirichlet integral on R. On the other hand $\sum_{n=1}^m h_n = 0$ on the part β^- of β in the second and third quadrants. Therefore

$$h(z) = \sum_{n=1}^{\infty} h_n(z)$$

is convergent on R, $h \in HD(R) \cap C(\bar{R} - \{1\})$, and $h(z) \to \infty$ as $z \in \bar{R} - \{1\}$ tends to $z = 1$.

We denote by A_n the part of $\beta \cap [U(1, d_{n-1}) - \overline{U(1, d_n)}]$ in the first quadrant. Then A_n is an open arc in β such that

$$(7) \qquad h \mid A_n > n \qquad (n = 1, 2, \cdots)\,.$$

We denote by a_n the midpoint of A_n and by $2\sigma_n$ the length of A_n. The sequence $\{a_n\}$ $(n = 1, 2, \cdots)$ of points in β and $\{\sigma_n\}$ $(n = 1, 2, \cdots)$ of

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positive numbers will be used later in the construction of a subharmonic function e on R.

3. The construction of e is rather complicated and requires a bit delicate estimates. Therefore it is convenient to prepare several elementary lemmas first in nos. 3–7, and then the construction will be carried over in nos. 8–11. The proof of the theorem will be completed in the final no. 12.

Let λ be a cross cut in R, i.e. an analytic arc contained in R except end points joining two distinct points in β. We denote by F one of regions in $\boldsymbol{C}$ bounded by β and λ. We take one more cross cut γ in F joining two distinct points in $\beta \cap \bar{F}$ different from end points of λ. The region in F bounded by β and γ is denoted by V. Let Y be the empty set ϕ or the union of disks X_j $(j = 1, \cdots, k)$ such that $\bar{X}_j \subset V$ and $\bar{X}_i \cap \bar{X}_j = \phi$ $(i \neq j)$. Let $\mathscr{F} = \mathscr{F}(F; Y)$ be the class of functions u in $C(\bar{F}) \cap H(F - \bar{Y})$ such that $u | \bar{Y} \cup (\beta \cap \bar{F}) = 0$. The assertion is: There exists a positive constant $c_1 = c_1(F; Y, \gamma)$ such that

$$\sqrt{D_V(u)} \leq c_1 \max_{\lambda} |u| \tag{8}$$

for every $u \in \mathscr{F}$.

To show this let β_0 be a Jordan arc joining two end points of λ outside $\bar{F}$ such that the region F_0 bounded by λ and β_0 contains F and let Y_0 be the union of disks X_{j0} $(j = 1, \cdots, k)$ concentric to X_j such that $\bar{X}_{j0} \subset X_j$. Since $u | (\partial(F - \bar{Y}) - \lambda) = 0$ for every $u \in \mathscr{F}$ which is harmonic on $F - \bar{Y}$, the symmetry principle on harmonic functions assures that every $u | (F - \bar{Y})$ simultaneously has the harmonic extension $\tilde{u}$ to $F_0 - \bar{Y}_0$ if we take β_0 and Y_0 close enough to β and Y, respectively. By the maximum principle, $|u| \leq \max_\lambda |u|$ on $\bar{F}$ and thus

$$|\tilde{u}| \leq \max_{\lambda} |u|$$

on $F_0 - \bar{Y}_0$ for every $u \in \mathscr{F}$. Let $a \in \bar{V} - Y$, $r(a) > 0$ be such that $\overline{U(a, 2r(a))} \subset F_0 - \bar{Y}_0$, and $P(z, 2r(a)e^{i\theta})$ be the Poisson kernel on $U(a, 2r(a))$, i.e.

$$v(z) = \int_0^{2\pi} P(z, 2r(a)e^{i\theta}) v(a + 2r(a)e^{i\theta}) d\theta$$

for every v harmonic on $\overline{U(a, 2r(a))}$. Since

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$$\frac{\partial}{\partial t}v(z) = \int_0^{2\pi} \frac{\partial}{\partial t}P(z, 2r(a)e^{i\theta})v(a + 2r(a)e^{i\theta})d\theta$$

for $t = x$ and y, we have

$$\left|\frac{\partial}{\partial t}v(z)\right| \leq K(a) \sup_{|z-a|=2r(a)} |v|$$

where

$$K(a) = \sum_{t=x,y} \sup_{|\zeta-a|=2r(a),\ |z-a|\leq r(a)} \left|\frac{\partial}{\partial t}P(z,\zeta)\right|$$

which is seen to be finite by using the concrete representation of $P(z,\zeta)$. Since $\bar{V} - Y$ is compact, it can be covered by a finite number of disks $V(a_\nu, r(a_\nu))$ $(a_\nu \in \bar{V} - Y; \nu = 1, \cdots, \ell)$. Then the required c_1^2 is $(\max_{1\leq\nu\leq\ell} K(a_\nu))^2$ multiplied by the area of $F - \bar{Y}$.

4. Let λ, γ, V, and F be as in no. 3. This time we assume $Y \neq \phi$, i.e. Y is the union of disks X_j $(j = 1, \cdots, k)$ such that $\bar{X}_j \subset V$ and $\bar{X}_i \cap \bar{X}_j = \phi$ $(i \neq j)$. By exactly the same consideration as in no. 3, we obtain: There exists a positive constant $c_2 = c_2(F; Y, \gamma)$ such that

$$\sum_{j=1}^{k} \left|\int_{\partial Y_j} *du\right| \leq c_2 \max_\lambda |u| \tag{9}$$

for every $u \in \mathscr{F} = \mathscr{F}(F; Y)$ and

$$\left|\int_\gamma f * du\right| \leq c_2\Big(\max_\gamma |f|\Big)\Big(\max_\lambda |u|\Big) \tag{10}$$

for every $f \in C(\gamma)$ and every $u \in \mathscr{F} = \mathscr{F}(F; Y)$.

5. We introduce the notation $V(\zeta, r)$ for $\zeta \in \beta$ and $r > 0$ to mean $V(\zeta, r) = R \cap U(\zeta, r)$. Let Y be the empty set ϕ or the union of disks X_j $(j = 1, \cdots, k)$ such that $\bar{X}_j \subset R$ and $\bar{X}_i \cap \bar{X}_j = \phi$ $(i \neq j)$. Let $\sigma = \sigma(1, Y)$ be the distance $\mathrm{dis}(Y, 1)$ between the point $z = 1$ and Y if $Y \neq \phi$ and $\sigma = 2$ if $Y = \phi$. We fix Y and take one more variable disk X such that $\bar{X} \subset R - \bar{Y}$. Consider the harmonic measure $w = w(\cdot, Y \cup X)$ of $Y \cup X$ in R, i.e. $w \in C(\bar{R})$ such that $w|\overline{X \cup Y} = 1$ and $w|\beta = 0$. The third lemma is: For any numbers η_1, η_2 and s with $0 < \eta_1 < \eta_2$ and $0 < s < \sigma(1, Y)$ there exists a number $\rho \in (0, s)$ and a disk X with $\bar{X} \subset V(1, \rho)$ such that

$$\eta_1 < \int_{-\partial X} * dw(\cdot, Y \cup X) < \eta_2\ , \tag{11}$$

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where ∂X is positively oriented with respect to X, and

(12) $$\eta_1 < D_{V(1,\rho)}(w(\cdot, Y \cup X)) < \eta_2 .$$

Actually we can choose $U(t, \varepsilon)$ as X where t is on the real axis.

6. To prove the assertion in no. 5 we first consider the function

$$f(t, \varepsilon) = \int_{-\partial U(t,\varepsilon)} * dw(\cdot, Y \cup U(t, \varepsilon))$$

for $t \in [\sigma_0, 1)$ $(\sigma < \sigma_0 < 1)$ and $\varepsilon \in (0, 1 - t)$. As auxiliary results to prove (11) and (12) we assert the following:

(13) $$\lim_{\varepsilon \to 0} f(\sigma_0, \varepsilon) = 0 ;$$

For any fixed $\varepsilon_0 \in (0, \min [(\sigma_0 - \sigma), 1 - \sigma_0])$

(14) $$f(\cdot, \varepsilon_0) \in C[\sigma_0, 1 - \varepsilon_0) ,$$

i.e. $f(t, \varepsilon_0)$ is continuous on the interval $[\sigma_0, 1 - \varepsilon_0)$ as a function of t, and

(15) $$\lim_{t \to 1 - \varepsilon_0} f(t, \varepsilon_0) = \infty .$$

We fix an $\varepsilon_1 \in (0, \min [\sigma_0 - \sigma, 1 - \sigma_0])$. Then $w(\cdot, Y \cup U(\sigma_0, \varepsilon))$ together with its first derivatives converge to $w(\cdot, Y)$ and its first derivative uniformly on each compact subset of $R - \bar{Y} - \{\sigma_0\}$, and in particular on $\partial U(\sigma_0, \varepsilon_1)$, and therefore

$$\lim_{\varepsilon \to 0} \int_{\partial U(\sigma_0, \varepsilon_1)} * dw(\cdot, X \cup U(\sigma_0, \varepsilon)) = \int_{\partial U(\sigma_0, \varepsilon_1)} * dw(\cdot, X) = 0 .$$

Since $f(\sigma_0, \varepsilon) = -\int_{\partial U(\sigma_0, \varepsilon_1)} * dw(\cdot, X \cup U(\sigma_0, \varepsilon))$ for $\varepsilon \in (0, \varepsilon_1)$, we deduce (13).

Fix a $t_0 \in [\sigma_0, 1 - \varepsilon_0)$ and an $\varepsilon_1 \in (\varepsilon_0, \min [\sigma_0 - \sigma, 1 - \sigma_0])$. It is easily seen that $w(\cdot, Y \cup U(t, \varepsilon_0))$ together with its first derivatives converge to $w(\cdot, Y \cup U(t_0, \varepsilon_0))$ and its first derivatives uniformly on each compact subset of $R - Y - \overline{U(t_0, \varepsilon_0)}$ and in particular on $\partial U(t_0, \varepsilon_1)$ as $t \to t_0$. In view of

$$f(t, \varepsilon_0) = -\int_{\partial U(t_0, \varepsilon_1)} * dw(\cdot, Y \cup U(t, \varepsilon_0))$$

for t in $|t - t_0| < \varepsilon_1 - \varepsilon_0$, we conclude (14) as in the proof of (13).

Let u_0 be the harmonic measure of $Y \cup U(1 - \varepsilon_0, \varepsilon_0)$ with respect to

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R, i.e. $u_0 \in C(\bar{R} - \{1\}) \cap H(R - \overline{Y \cup U(1-\varepsilon_0, \varepsilon_0)})$ such that $u = 1$ on $\bar{Y} \cup (\overline{U(1-\varepsilon_0, \varepsilon_0)} - \{1\})$ and $u = 0$ on $\beta - \{1\}$. Suppose $D_R(u_0) < +\infty$. By the Dirichlet principle, $D_R(u_0) > D_R(v_t)$, where $v_t = w(\cdot, U(t, \varepsilon))$ with a fixed $\varepsilon \in (0, \varepsilon_0)$ and a $t \in (1-\varepsilon_0, 1-\varepsilon)$. Observe that $D_R(v_t) = 2\pi/\log \mu_t$ where μ_t is the modulus of the annulus $R - \overline{U(t, \varepsilon)}$. We know that $\lim_{t \to 1-\varepsilon} \mu_t = 1$ (cf. e.g. Sario-Nakai [4; p. 28]), which implies a contradiction. Thus we must have

$$D_R(u_0) = \infty .$$

On the other hand, $w_t = w(\cdot, Y \cup U(t, \varepsilon_0))$ together with its first derivatives converge to u_0 and its first derivatives uniformly on each compact subset of $R - Y - \overline{U(1-\varepsilon_0, \varepsilon_0)}$ as $t \to 1 - \varepsilon_0$. Therefore

$$\lim_{t \to 1-\varepsilon_0} \left| \int_{\partial Y} * dw_t \right| = \left| \int_{\partial Y} * du_0 \right| < \infty .$$

By the Fatou lemma, $D_R(u_0) \leq \liminf_{t \to 1-\varepsilon_0} D_R(w_t)$ and thus

$$\lim_{t \to 1-\varepsilon_0} D_R(w_t) = \infty .$$

Observe that

$$D_R(w_t) = f(t, \varepsilon_0) + \int_{-\partial Y} * dw_t .$$

Hence we see that (15) is valid.

7. We proceed to the proof of the assertion in no. 5. First we choose and fix a $\rho \in (0, s)$ so small that

$$D_{V(1,\rho)}(w(\cdot, Y)) < (\eta_2 - \eta_1)/4 .$$

Let $\gamma = R \cap \partial U(1, \rho)$ and $\lambda = R \cap \partial U(1, \rho/2)$. The region bounded by β and γ (λ, resp.) containg Y in its interior is denoted by V (F, resp.). The above inequality means that

$$\left| \int_{\gamma} w(\cdot, Y) * dw(\cdot, Y) \right| < (\eta_2 - \eta_1)/4 . \tag{16}$$

We consider auxiliary functions $w_r \in C(\bar{R} - \beta \cap \partial U(1, r)) \cap H(R - \bar{Y} \cup \overline{V(1, r)})$ with $w_r = 1$ on $\bar{Y} \cup [\overline{V(1, r)} - \beta \cap \partial U(1, r)]$ and $w_r = 0$ on $\beta - \beta \cap \overline{U(1, r)}$ for $r \in (0, \rho/4)$. For any disk X with $\bar{X} \subset V(1, r)$, the maximum principle yields

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$$w(\cdot, Y) \leq w(\cdot, Y \cup X) \leq w_r$$

on R, and $\sup_\lambda |w(\cdot, Y) - w(\cdot, Y \cup X)| \leq \sup_\lambda |w(\cdot, Y) - w_r| \equiv \varepsilon(r)$. Since w_r converges to $w(\cdot, Y)$ uniformly on each compact of $\bar{R} - \{1\}$, we have $\lim_{r \to 0} \varepsilon(r) = 0$. Observe that

$$\left| \int_r w(\cdot, Y) * dw(\cdot, Y) - \int_r w(\cdot, Y \cup X) * dw(\cdot, Y \cup X) \right|$$

is dominated by the sum of

$$\sup_r |w(\cdot, Y) - w(\cdot, Y \cup X)| \cdot \int_r |* dw(\cdot, Y)| \leq \varepsilon(r) \int_r |* dw(\cdot, Y)|$$

and

$$\left| \int_r w(\cdot, Y \cup X) * d(w(\cdot, Y \cup X) - w(\cdot, Y)) \right| .$$

Observe once more that $w(\cdot, Y \cup X) - w(\cdot, Y) \in \mathscr{F}(F; Y)$ in the sense of no. 4. Thus the last term is dominated by

$$\begin{aligned} &c_2\Big(\sup_r |w(\cdot, Y \cup X)|\Big)\Big(\sup_r |w(\cdot, Y \cup X) - w(\cdot, Y)|\Big) \\ &\quad \leq c_2\Big(\max_r |w_r|\Big)\varepsilon(r) \leq c_2\Big(\max_r |w(\cdot, Y)| + \varepsilon(r)\Big)\varepsilon(r) \end{aligned}$$

as a consequence of (10). By fixing $r \in (0, \rho/4)$ so small that

$$\varepsilon(r) \int_r |* dw(\cdot, Y)| + c_2\Big(\max_r |w(\cdot, Y)| + \varepsilon(r)\Big)\varepsilon(r) < (\eta_2 - \eta_1)/4$$

we conclude that

$$\left| \int_r w(\cdot, Y) * dw(\cdot, Y) - \int_r w(\cdot, Y \cup X) * dw(\cdot, Y \cup X) \right| < (\eta_2 - \eta_1)/4$$

whenever $\bar{X} \subset U(1, r) \cap R = V(1, r)$. This with (16) gives

$$\left| \int_r w(\cdot, Y \cup X) * dw(\cdot, Y \cup X) \right| < (\eta_2 - \eta_1)/2 \tag{17}$$

for every disk X with $\bar{X} \subset V(1, r)$.

We next take $f(t, \varepsilon)$ considered for the present $w(\cdot, Y \cup U(t, \varepsilon))$ as in no. 6 for $t \in [r/2, 1)$ and $\varepsilon \in (0, 1 - t)$. By (13) we can choose $\varepsilon_0 \in (0, r/2)$ with $f(r/2, \varepsilon_0) < (\eta_1 + \eta_2)/2$. By (14) and (15), the mean value theorem applied to $f(\cdot, \varepsilon_0) \in C[r/2, 1 - \varepsilon_0)$ yields the existence of $t_0 \in (r/2, 1 - \varepsilon_0)$

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such that $f(t_0, \varepsilon_0) = (\eta_1 + \eta_2)/2$. Finally we prove that $X = U(t_0, \varepsilon_0)$ is the required. First, $\bar{X} \subset U(1, r) \subset U(1, \rho)$. The (11) is clearly satisfied since $\eta_1 < (\eta_1 + \eta_2)/2 < \eta_2$. Observe that

$$D_{V(1,\rho)}(w(\cdot, Y \cup X)) = f(t_0, \varepsilon_0) + \int_{\gamma} w(\cdot, Y \cup X) * dw(\cdot, Y \cup X)$$

where γ is oriented in the direction of $\partial V(1, \rho)$. In view of (17) and $f(t_0, \varepsilon_0) = (\eta_1 + \eta_2)/2$, we conclude that (12) is true.

8. Having finished preparations in nos. 3–7, we proceed to the construction of e as announced at the end of no. 1. The construction will be carried over related to the $h \in HD(R)$ constructed in no. 2 and in particular related to the sequences $\{a_n\} \subset \beta$ and $\{\sigma_n\}$ defined in no. 2. First choose an $s_1 \in (0, \sigma_1)$ such that $h | U(a_1, s_1) > 1$. By no. 5, we can find a $\rho_1 \in (0, s_1)$ and a disk X_1 with $\bar{X}_1 \subset U(a_1, \rho_1)$ such that

$$(18) \qquad \begin{cases} 2^{-1} \cdot 1^{-2} < \int_{-\partial X_1} * dw(\cdot, X_1) < 2 \cdot 1^{-2} \,; \\ 2^{-1} \cdot 1^{-2} < D_{V(a_1, \rho_1)}(w(\cdot, X_1)) < 2 \cdot 1^{-2} \,. \end{cases}$$

Next choose $s_2 \in (0, \sigma_2)$ such that $h | U(a_2, s_2) > 2$. Let $\gamma = \bar{R} \cap \partial U(a_2, s_2)$ and $\lambda = \bar{R} \cap \partial U(a_2, s_2/2)$ and V (F, resp.) be the region bounded by β and γ (λ, resp.) containing $\bar{X}_1$. Let w_r be the function in the class $C(\bar{R} - \beta \cap \partial U(a_2, r)) \cap H(R - \bar{X}_1 - \overline{V(a_2, r)})$ such that $w_r = 1$ on $\bar{X}_1 \cup [\overline{V(a_2, r)} - \beta \cap \partial U(a_2, r)]$ and $w_r = 0$ on $\beta - \overline{U(a_2, r)}$ for $r \in (0, s_2/2)$. Take any disk X with $\bar{X} \subset U(a_2, r)$. The maximum principle yields $u_1 \le u \le w_r$ on R where $u_1 = w(\cdot, X_1)$ and $u = w(\cdot, X_1 \cup X)$. Observe that $u_1 - u \in \mathscr{F}(F, X_1)$ in the sense of nos. 3 and 4. Since $|u - u_1| \le |u_1 - w_r|$,

$$\left| \int_{-\partial X_1} * du - \int_{-\partial X_1} * du_1 \right| \le c_2 \max_{\lambda} |u_1 - w_r| \,,$$

where c_2 is the constant in (9), and

$$\begin{aligned} |\sqrt{D_{V(a_1,\rho_1)}(u)} - \sqrt{D_{V(a_1,\rho_1)}(u_1)}| &\le \sqrt{D_{V(a_1,\rho_1)}(u - u_1)} \\ &\le \sqrt{D_V(u - u_1)} \le c_1 \max_{\lambda} |u_1 - w_r| \end{aligned}$$

where c_1 is the constant in (8). Since w_r converges to u_1 uniformly on each compact subset of $\bar{R} - \{a_2\}$ as $r \to 0$, by using (18) and the above we can find an $r_2 \in (0, s_2/2)$ such that

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$$(19)\qquad \begin{cases} 2^{-1}\cdot 1^{-2} < \int_{-\partial X_1} * dw(\cdot, X_1 \cup X) < 2\cdot 1^{-2} ; \\ 2^{-1}\cdot 1^{-2} < D_{V(a_1,\rho_1)}(w(\cdot, X_1 \cup X)) < 2\cdot 1^{-2} \end{cases}$$

for any disk X with $\bar{X} \subset U(a_2, r_2)$. Again using the results in no. 5, we can find a $\rho_2 \in (0, r_2)$ and a disk X_2 with $\bar{X}_2 \subset U(a_2, \rho_2)$ such that

$$\begin{cases} 2^{-1}\cdot 2^{-2} < \int_{-\partial X_2} * dw(\cdot, X_1 \cup X_2) < 2\cdot 2^{-2} ; \\ 2^{-1}\cdot 2^{-2} < D_{V(a_2,\rho_2)}(w(\cdot, X_1 \cup X_2)) < 2\cdot 2^{-2} . \end{cases}$$

Combining this with (19) we have

$$(20)\qquad \begin{cases} 2^{-1}\cdot j^{-2} < \int_{-\partial X_j} * dw(\cdot, X_1 \cup X_2) < 2\cdot j^{-2} \qquad (j = 1, 2) ; \\ 2^{-1}\cdot j^{-2} < D_{V(a_j,\rho_j)}(w(\cdot, X_1 \cup X_2)) < 2\cdot j^{-2} \qquad (j = 1, 2) . \end{cases}$$

9. Repeating the process as in no. 8, we can find a sequence $\{\rho_n\}$ with $\rho_n \in (0, \sigma_n)$ and a sequence $\{X_n\}$ of disks X_n with $\bar{X}_n \subset U(a_n, \rho_n)$ $(n = 1, 2, \cdots)$ such that

$$(21)\qquad \begin{cases} 2^{-1}\cdot j^{-2} < \int_{-\partial X_j} * dw\left(\cdot, \bigcup_{k=1}^{n} X_k\right) < 2\cdot j^{-2} \qquad (j = 1, \cdots, n) ; \\ 2^{-1}\cdot j^{-2} < D_{V(a_j,\rho_j)}\left(w\left(\cdot, \bigcup_{k=1}^{n} X_k\right)\right) < 2\cdot j^{-2} \qquad (j = 1, \cdots, n) ; \\ h \,|\, V(a_n, \rho_n) > n \end{cases}$$

for every $n = 1, 2, \cdots$. Although it should be clear by no. 8, we show how to find ρ_{n+1} and X_{n+1} when $\{\rho_\nu\}$ $(\nu = 1, \cdots, n)$ and $\{W_\nu\}$ $(\nu = 1, \cdots, n)$ satisfying (21) have already been found. Choose $s_{n+1} \in (0, \sigma_{n+1})$ such that $h \,|\, U(a_{n+1}, s_{n+1}) > n+1$. Let $\gamma = \bar{R} \cap \partial U(a_{n+1}, s_{n+1})$ and $\lambda = \bar{R} \cap \partial U(a_{n+1}, s_{n+1}/2)$ and V (F, resp.) be the region bounded by β and γ (λ, resp.) containing $\bigcup_{k=1}^{n} \bar{X}_k$. Let w_r be the function in the class

$$C(\bar{R} - \beta \cup \partial U(a_{n+1}, r)) \cap H\left(R - \bigcup_{k=1}^{n} \bar{X}_k - \overline{V(a_n, r)}\right)$$

such that $w_r = 1$ on $(\bigcup_{k=1}^{n} \bar{X}_k) \cup [\overline{V(a_{n+1}, r)} - \beta \cap \partial U(a_{n+1}, r)]$ and $w_r = 0$ on $\beta - \overline{U(a_{n+1}, r)}$ for $r \in (0, s_{n+1}/2)$. Take any disk X with $\bar{X} \subset U(a_{n+1}, r)$. The maximum principle yields

$$u_n \le u \le w_r$$

on R where $u_n = w(\cdot, \bigcup_{k=1}^{n} X_k)$ and $u = w(\cdot, (\bigcup_{k=1}^{n} X_k) \cup X)$. Observe

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that $u-u_n\in\mathscr{F}(F,\bigcup_{k=1}^n X_k)$ in the sense of nos. 3 and 4. Since $|u-u_n|\leq u_n-w_r$,

$$\sum_{j=1}^n\left|\int_{-\partial X_j}*du-\int_{-\partial X_j}*du_n\right|\leq c_2\max_\lambda|u_n-w_r|\,,$$

where c_2 is the constant in (9), and

$$\begin{aligned}|\sqrt{D_{V(a_j,\rho_j)}(u)}-\sqrt{D_{V(a_j,\rho_j)}(u_n)}|&\leq\sqrt{D_{V(a_j,\rho_j)}(u-u_n)}\\&\leq\sqrt{D_V(u-u_n)}\leq c_1\max_\lambda|u_n-w_r|\end{aligned}$$

for $j=1,2,\cdots,n$. Since w_r converges to u_n uniformly on each compact of $\bar{R}-\{a_{n+1}\}$ as $r\to 0$, by using (18) and the above, we can find an $r_{n+1}\in(0,s_2/2)$ such that

$$\text{(22)}\quad\begin{cases}2^{-1}\cdot j^{-2}<\displaystyle\int_{-\partial X_j}*dw\left(\cdot,\left(\bigcup_{k=1}^n X_k\right)\cup X\right)<2\cdot j^{-2}\qquad(j=1,\cdots,n)\;;\\2^{-1}\cdot j^{-2}<D_{V(a_j,\rho_j)}\left(w\left(\cdot,\left(\bigcup_{k=1}^n X_k\right)\cup X\right)\right)<2\cdot j^{-2}\qquad(j=1,\cdots,n)\end{cases}$$

for every disk X with $\bar{X}\subset U(a_{n+1},r_{n+1})$. Again using the results in no. 5 we can find a $\rho_{n+1}\in(0,r_{n+1})$ and a disk X_{n+1} with $\bar{X}_{n+1}\subset U(a_{n+1},\rho_{n+1})$ such that

$$\begin{cases}2^{-1}\cdot(n+1)^{-2}<\displaystyle\int_{-\partial X_{n+1}}*dw\left(\cdot,\bigcup_{k=1}^{n+1}X_k\right)<2\cdot(n+1)^{-2}\\2^{-1}\cdot(n+1)^{-2}<D_{V(a_{n+1},\rho_{n+1})}\left(w\left(\cdot,\bigcup_{k=1}^{n+1}X_k\right)\right)<2\cdot(n+1)^{-2}\end{cases}$$

and clearly $h\,|\,U(a_{n+1},\rho_{n+1})>n+1$. Combining this with (22) for $X=X_{n+1}$, we deduce that $\{\rho_k\}$ $(k=1,\cdots,n+1)$ and $\{X_k\}$ $(k=1,\cdots,n+1)$ satisfies (21).

10. Since $w(\cdot,\bigcup_{k=1}^n X_k)$ increases as n increases and is bounded by 1,

$$\text{(23)}\qquad w\left(\cdot,\bigcup_{k=1}^\infty X_k\right)=\lim_{n\to\infty}w\left(\cdot,\bigcup_{k=1}^n X_k\right)$$

exists on R, which is continuous on $\bar{R}-\{1\}$, 1 on $\bigcup_{k=1}^\infty X_k$, 0 on $\beta-\{1\}$, harmonic on $R-\bigcup_{k=1}^\infty\bar{X}_k$, and superharmonic on R. Set $u_n=w(\cdot,\bigcup_{k=1}^n X_k)$ and $u_\infty=w(\cdot,\bigcup_{k=1}^\infty X_k)$. Then

$$D_R(u_{n+p}-u_n)=\sum_{k=n+1}^{n+p}\int_{-\partial X_k}(1-u_n)*d(u_{n+p}-u_n)$$

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$$= \sum_{k=n+1}^{n+p} \left(\int_{-\partial X_k} (1-u_n) * du_{n+p} - D_{X_k}(u_n) \right) .$$

Here $0 < 1 - u_n < 1$ and $* du_{n+p} > 0$ on $-\partial X_k$ for $k = n+1, \cdots, n+p$ and a fortiori by (21)

$$D_R(u_{n+p} - u_n) \leq 2 \sum_{k=n+1}^{n+p} k^{-2} .$$

On letting $p \to \infty$ and by using the Fatou lemma

$$D_R(u_\infty - u_n) \leq 2 \sum_{k=n+1}^{\infty} k^{-2}$$

for every n. Thus

$$\lim_{n\to\infty} D_R\left(w\left(\cdot, \bigcup_{k=1}^{\infty} X_k\right) - w\left(\cdot, \bigcup_{k=1}^{\infty} X_k\right)\right) = 0 . \tag{24}$$

On the other hand,

$$D_R\left(w\left(\cdot, \bigcup_{k=1}^{n} X_k\right)\right) = \sum_{k=1}^{n} \int_{-\partial X_k} * dw\left(\cdot, \bigcup_{k=1}^{n} X_k\right)$$
$$\leq 2 \sum_{k=1}^{n} k^{-2}$$

and we conclude with (24) that

$$D_R\left(w\left(\cdot, \bigcup_{k=1}^{\infty} X_k\right)\right) \leq 2 \sum_{k=1}^{\infty} k^{-2} < \infty . \tag{25}$$

Passing to the limit in (21) by using (23) and (24) we obtain the following:

$$\begin{cases} 2^{-1}\cdot j^{-2} \leq \int_{-\partial X_j} * dw\left(\cdot, \bigcup_{k=1}^{\infty} X_k\right) \leq 2\cdot j^{-2} & (j = 1, 2, \cdots) ; \\ 2^{-1}\cdot j^{-2} \leq D_{V(a_j,\rho_j)}\left(w\left(\cdot, \bigcup_{k=1}^{\infty} X_k\right)\right) \leq 2\cdot j^{-2} & (j = 1, 2, \cdots) ; \\ h \,|\, V(a_j, \rho_j) > j \quad (j = 1, 2, \cdots) . \end{cases} \tag{26}$$

11. Take two concentric disks W_n and Z_n to X_n such that $\bar{Z}_n \subset X_n \subset \bar{X}_n \subset W_n \subset \bar{W}_n \subset U(a_n, \rho_n)$ $(n = 1, 2, \cdots)$. By applying the regularization (cf. e.g. Yosida [8], Tsuji [7], Sario-Nakai [4; p. 150]) to $w(\cdot, \bigcup_{n=1}^{\infty} X_n)$ on each $W_n - \bar{Z}_n$ $(n = 1, 2, \cdots)$, the resulting C^∞ superharmonic function on R will be denoted by g. Then

$$g \,|\, \left(R - \bigcup_{n=1}^{\infty} (W_n - \bar{Z}_n)\right) = w\left(\cdot, \bigcup_{n=1}^{\infty} X_k\right) .$$

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The first inequality of (26) is also valid if the integrating curve $-\partial X_j$ is replaced by $-\partial W_j$ and a fortiori we have

(27) $$2^{-1}\cdot j^{-2} \leq \int_{-\partial W_j} *dg \leq 2\cdot j^{-2} \qquad (j=1,2,\cdots) .$$

We can also make $D_{W_j-\bar{Z}_j}(g - w(\cdot, \bigcup_{n=1}^{\infty} X_n))$ as small as we wish by choosing the regularization g close enough to $w(\cdot, \bigcup_{n=1}^{\infty} X_n)$ (cf. e.g. Sario-Nakai [4; p. 150]) in each $W_j - \bar{Z}_j$ $(j=1,2,\cdots)$ and thus the second inequality of (26) yields

(28) $$4^{-1}\cdot j^{-2} \leq D_{V(a_j,\rho_j)}(g) \leq 4\cdot j^{-2} \qquad (j=1,2,\cdots)$$

and we stress here once more the following

(29) $$h\,|\,V(a_j, \rho_j) > j \qquad (j=1,2,\cdots) .$$

Finally we set $e(z) = 1 - g(z)/2$ and observe that

(30) $$1/2 \leq e(z) \leq 1$$

on R and that $e(z)$ is C^∞ subharmonic on R. As the counter parts of (27)–(29) we obtain

(31) $$\begin{cases} 4^{-1}\cdot j^{-2} \leq \int_{\partial W_j} *de \leq j^{-2} \qquad (j=1,2,\cdots) ; \\ 16^{-1}\cdot j^{-2} \leq D_{V(a_j,\rho_j)}(e) \leq j^{-2} \qquad (j=1,2,\cdots) ; \\ h\,|\,V(a_j,\rho_j) > j \qquad (j=1,2,\cdots) . \end{cases}$$

12. The required density P in the theorem is given by

$$P(z) = \Delta e(z)/e(z)$$

on R. Then e is a bounded solution of (3) with this P. Since e has boundary values 1 on $\beta - \{1\}$, the condition (5) in no. 1 is satisfied by e and therefore e is the *P-unit* for this P. Therefore

(32) $$1 = Te = e + \frac{1}{2\pi}\int_R G(\cdot,\zeta)P(\zeta)e(\zeta)d\xi d\eta$$

where $G(z,\zeta)$ is the harmonic Green's function on R.

In view of (30) and (31), we deduce

$$\begin{aligned} \int_R P(z)dxdy &\leq 2\int_R \Delta e(z)dxdy = 2\sum_{j=1}^{\infty}\int_{W_j}\Delta e(z)dxdy \\ &= 2\sum_{j=1}^{\infty}\int_{\partial W_j} *de \leq 2\sum_{j=1}^{\infty} j^{-2} < \infty , \end{aligned}$$

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i.e. P is a *finite density* on R. By (30) and (32), we have

$$\int_R G(\cdot, \zeta)P(\zeta)d\xi d\eta \leq 4\pi$$

on R. Then by the Fubini theorem

$$\iint_{R\times R} G(z,\zeta)P(z)P(\zeta)dxdyd\xi d\eta = \int_R P(z)\left(\int_R G(z,\zeta)P(\zeta)d\xi d\eta\right)dxdy$$
$$\leq 4\pi\int_R P(z)dxdy \leq 8\pi\sum_{j=1}^{\infty} j^{-2} < \infty\,,$$

i.e. P is a *Green energy finite density* on R.

The last and the most delicate part of the proof of the theorem is to show that T_D for the present P is *not surjective*. For the aim we shall show that the h in $HD(R)$ defined in no. 2 does not have the counter image of T_D in $PD(R)$, i.e.

$$h \notin T_D(PD(R))\,.$$

To prove this we estimate $D_R(eh)$. Take a concentric disk Ω to R with $\bar{\Omega} \subset R$. Then

$$D_\Omega(eh) - \int_\Omega e^2 dh \wedge * dh - 2\int_\Omega edh \wedge * hde = \int_\Omega h^2 de \wedge * de$$

and hence

$$\int_\Omega h^2 de \wedge * de \leq D_\Omega(eh) + \int_\Omega e^2 dh \wedge * dh + 2\left|\int_\Omega edh \wedge * hde\right|\,.$$

By the Schwarz inequality, the last term is dominated by

$$2\left(\int_\Omega e^2 dh \wedge * dh\right)^{1/2}\cdot\left(\int_\Omega h^2 de \wedge * de\right)^{1/2}\,.$$

In view of $e^2 \leq 1$, on setting $\ell_\Omega = \left(\int_\Omega h^2 de \wedge * de\right)^{1/2}$, we have

$$\ell_\Omega^2 \leq D_\Omega(eh) + D_\Omega(h) + 2D_\Omega(h)^{1/2}\cdot \ell_\Omega$$

or

$$(\ell_\Omega - D_\Omega(h)^{1/2})^2 \leq D_\Omega(eh) + 2D_\Omega(h)\,.$$

Since $D_\Omega(h) < D_R(h) < \infty$, on letting $\Omega \to R$, we obtain

$$\left[\left(\int_R h^2 de \wedge * de\right)^{1/2} - D_R(h)^{1/2}\right]^2 \leq D_R(eh) + 2D_R(h)\,. \tag{33}$$

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On the other hand, by (29) and (31), we deduce

$$\begin{aligned}\int_R h^2 de \wedge {}^*de &\geq \sum_{j=1}^{\infty} \int_{U(a_j, \rho_j)} h^2 de \wedge {}^*de \\ &> \sum_{j=1}^{\infty} j^2 D_{U(a_j,\rho_j)}(e) \\ &\geq \sum_{j=1}^{\infty} j^2 \cdot 16^{-1} \cdot j^{-2} = \infty .\end{aligned}$$

Therefore, by (33), $D_R(eh) = \infty$. By the Singer criterion (6), this means that $h \notin T_D(PD(R))$, i.e. T_D is not surjective.

The proof of Theorem is herewith complete.

Added in Proof. The author feels it very fortunate that the referee of this paper was at least careful enough to keep the manuscript of this paper safely for almost three years in his drawer without losing it. In the meantime further developements based on this paper have been published by the present author in the following two papers:

1) *Extremizations and Dirichlet integrals on Riemann surfaces*, J. Math. Soc. Japan, **28** (1976), 581–603;

2) *Malformed subregions of Riemann surfaces*, J. Math. Soc. Japan, **29** (1977), 779–782.

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