Can. J. Math., Vol. XXXVIII, No. 1, 1986, pp. 87-108

ALGEBRAS OF BOUNDED ANALYTIC FUNCTIONS CONTAINING THE DISK ALGEBRA

KEIJI IZUCHI AND YUKO IZUCHI

1. Introduction. Let D be the open unit disk and let ∂D be its boundary. We denote by C the algebra of continuous functions on ∂D, and by L^∞ the algebra of essentially bounded measurable functions with respect to the normalized Lebesgue measure m on ∂D. Let H^∞ be the algebra of bounded analytic functions on D. Identifying with their boundary functions, we regard H^∞ as a closed subalgebra of L^∞. Let $A = H^\infty \cap C$, which is called the disk algebra. The algebras A and H^∞ have been studied extensively [**5**, **6**, **7**]. In these fifteen years, norm closed subalgebras between H^∞ and L^∞, called Douglas algebras, have received considerable attention in connection with Toeplitz operators [**12**]. A norm closed subalgebra between A and H^∞ is called an *analytic subalgebra*. In [**2**], Dawson studied analytic subalgebras and he remarked that there are many different types of analytic subalgebras. One problem is to study which analytic subalgebras are backward shift invariant. Here, a subset E of H^∞ is called backward shift invariant if

$$f^* = (f(z) - f(0))/z \in E \quad \text{for every } f \text{ in } E.$$

Backward shift invariant subspaces of the Hardy space H^p are studied in [**3**, **4**]. The other problems come from Sarason's theorem [**11**, Theorem 2]. It is well known that if B is a Douglas algebra with $H^\infty \subsetneq B$, then B contains C. Sarason studied its generalization. For an analytic subalgebra $\mathscr{A}$, let ϕ_0 be the evaluation homomorphism of $\mathscr{A}$ at the origin;

$$\phi_0 : \mathscr{A} \ni f \rightarrow f(0).$$

SARASON'S THEOREM. *Let $\mathscr{A}$ be an analytic subalgebra and $\mathscr{C}$ be a C^*-subalgebra of L^∞ with $\mathscr{A} \subset \mathscr{C}$. If*

(a) *$\mathscr{A}$ is backward shift invariant, and*

(b) *ϕ_0 of $\mathscr{A}$ has a unique norm-preserving Hahn-Banach extension to $\mathscr{C}$, then every closed subalgebra B with $\mathscr{A} \subset B \subset \mathscr{C}$ and $B \not\subset H^\infty$ contains C.*

Sarason's theorem provides us the following problems. For an analytic subalgebra $\mathscr{A}$ and a C^*-subalgebra $\mathscr{C}$:

Received March 19, 1984 and in revised form March 4, 1985.

(1) When does ϕ_0 of $\mathscr{A}$ have a unique norm-preserving Hahn-Banach extension to $\mathscr{C}$?

(2) When does every closed subalgebra B with $\mathscr{A} \subset B \subset \mathscr{C}$ and $B \not\subset H^\infty$ contain C?

It is difficult to answer these problems completely. In this paper, we shall study a special type of algebras $qH^\infty + A$, where q is an inner function, which is studied by Stegenga [**13**]. For an inner function q, we put

$$\text{supp } q = \{\lambda \in \partial D; \text{ there is a sequence } \{z_n\} \text{ in } D \text{ such that } z_n \to \lambda \text{ and } q(z_n) \to 0\}.$$

Then supp q is a closed subset of ∂D. Stegenga proved that $qH^\infty + A$ is norm closed if and only if $m(\text{supp } q) = 0$ or 1. Our results are continuations of Stegenga's works [**13**] concerning the questions mentioned above. The following is a main theorem proved in Section 3 (actually we shall prove it under more general situations).

THEOREM. *Let $\mathscr{B}$ be the norm closure of $qH^\infty + A$. Then $\mathscr{B}$ is an analytic subalgebra and the following assertions are equivalent.*

(i) *$\mathscr{B}$ is backward shift invariant.*

(ii) $m(\text{supp } q) < 1$.

(iii) *ϕ_0 of $\mathscr{B}$ has a unique norm-preserving Hahn-Banach extension to L^∞.*

(iv) *If B is a closed subalgebra with $\mathscr{B} \subset B \subset L^\infty$ and $B \not\subset H^\infty$, then B contains C.*

Some partial assertions in this theorem have been already proved in [**2**, **8**, **10**]. In Section 2, we shall study backward shift invariant analytic subalgebras. Proposition 2.1 answers Dawson's question in [**2**, p. 94]. In Section 4, we shall give more precise properties of $\mathscr{B}$. We shall describe the maximal ideal space of $\mathscr{B}$ and study representing measures for ϕ_0 of $\mathscr{B}$. Our final result is that condition (a) in Sarason's theorem can not be removed. In [**11**], Sarason remarked that condition (b) can not be removed.

We give some notations and definitions. For $f \in L^\infty$, let $\|f\|$ denote the essential sup-norm of f. For a subset E of L^∞, $[E]$ denotes the norm closed subalgebra generated by E. For a closed subalgebra B between A and L^∞, we denote by $M(B)$ the maximal ideal space of B. Identifying a function in B with its Gelfand transform, we regard B as a subalgebra of $C(M(B))$, the space of continuous functions on $M(B)$. For $\lambda \in \partial D$, put

$$M_\lambda(B) = \{x \in M(B); z(x) = \lambda\},$$

which we call the fiber at λ. We note that

$$M(H^\infty)\backslash D = M(H^\infty + C) = \{x \in M(H^\infty); |z(x)| = 1\}.$$

The map

$$\pi_0 : M(H^\infty + C) \ni x \to z(x) \in \partial D$$

is called the fiber projection. For a subset E of $H^\infty + C$, put

$$Z(E) = \{x \in M(H^\infty + C); f(x) = 0 \text{ for every } f \in E\}.$$

A function q in H^∞ with $|q| = 1$ a.e. on ∂D is called inner. If q is inner, then

$$\text{supp } q = \pi_0(Z(q)).$$

Let H^1 be the classical Hardy space. We denote by H^1_0 the space of functions in H^1 which vanish at the origin. For regular Borel measures μ and λ, $\mu \ll \lambda$ means that μ is absolutely continuous with respect to λ, $\mu \perp \lambda$ means that μ and λ are mutually singular, and $\|\mu\|$ means the total variation norm of μ.

2. Backward shift invariant analytic subalgebras. It is well known that $H^\infty + C$ is a closed subalgebra of L^∞ [**12**]. In [**10**], Nishizawa pointed out that $\mathscr{A} + C$ is a closed subspace for every analytic subalgebra $\mathscr{A}$. Using this fact, it is easy to give some characterizations of backward shift invariant analytic subalgebras as follows [**2**, **10**].

LEMMA 2.1. *Let $\mathscr{A}$ be an analytic subalgebra, then the following conditions are equivalent.*

(1) *$\mathscr{A}$ is backward shift invariant.*

(2) *For each a in D, $(f(z) - f(a))/(z - a) \in \mathscr{A}$ for every $f \in \mathscr{A}$.*

(3) *For each a in D, $(z - a)\mathscr{A} + A = \mathscr{A}$.*

(4) *$\mathscr{A} + C$ is a closed subalgebra.*

(5) *$\mathscr{A} = H^\infty \cap [\mathscr{A} + C]$.*

But these characterizations are not sufficient to check whether a given $\mathscr{A}$ is backward shift invariant or not. In [**14**], Wolff proved that if $\mathscr{A}$ contains QA then $\mathscr{A}$ is backward shift invariant, where

$$QC = (H^\infty + C) \cap \overline{(H^\infty + C)} \quad \text{and} \quad QA = H^\infty \cap QC.$$

The following lemma is also available to check whether a given $\mathscr{A}$ is backward shift invariant or not.

LEMMA 2.2. *Let $a \in D$. Suppose that $\psi \in H^\infty$ is continuous on an open subarc V of ∂D and $\psi(a) = 0$. Then $\psi/(z - a)$ belongs to the uniform closure of $\psi A + A$.*

Proof. Let W be an open subarc of ∂D whose closure is contained in V. For a given $\epsilon > 0$, find $f \in A$ such that

$$\left| f(z) - \frac{1}{z - a} \right| < \epsilon/\|\psi\| \quad \text{for } z \in \partial D \setminus W.$$

Set

$$h = \psi\left(f - \frac{1}{z - a}\right).$$

Then $h \in H^\infty$, $|h(z)| < \epsilon$ for almost all z in $\partial D \setminus W$, and h is continuous on V. Hence if g is a Cesaro mean of h with sufficiently high index, we have $g \in A$ and

$$\left\|\psi\left(f - \frac{1}{z - a}\right) - g\right\| < \epsilon.$$

LEMMA 2.3. *Let $\mathscr{A}$ be a backward shift invariant analytic subalgebra. Suppose that $\psi \in H^\infty$ is continuous on an open subarc of ∂D. Then $[\mathscr{A}, \psi]$ and $[\psi\mathscr{A} + A]$ are backward shift invariant.*

Proof. $\psi(z) - \psi(0)$ satisfies the assumptions of Lemma 2.2 (for $a = 0$). Hence $\psi^* \in [A, \psi]$. Note that

$$(fg)^* = f^*g + f(0)g^* \quad \text{and} \quad \|f^*\| \leqq 2\|f\| \quad \text{for } f, g \in H^\infty.$$

Then it is easy to see that $[A, \psi]$ and $[\psi\mathscr{A} + A]$ are backward shift invariant.

The following lemma shows the existence of non backward shift invariant analytic subalgebras.

LEMMA 2.4. *Let E be a closed subspace of H^∞ with*

$$m(\pi_0(Z(E))) = 1.$$

Then $\mathscr{A} = E + A$ is a closed non backward shift invariant subspace. Moreover, if E is an algebra and $AE \subset E$, then $\mathscr{A}$ becomes a non backward shift invariant analytic subalgebra.

Proof. Let $f_n + g_n \in \mathscr{A}(f_n \in E, g_n \in A, n = 1, 2, \dots)$ be a Cauchy sequence. Then $\{g_n\}_{n=1}^\infty$ is a Cauchy sequence on $Z(E)$. Since

$$m(\pi_0(Z(E))) = 1,$$

$\{g_n\}_{n=1}^\infty$ is a Cauchy sequence on ∂D. Hence $\{f_n\}_{n=1}^\infty$ is also a Cauchy sequence. Thus $\mathscr{A}$ is closed. To see that $\mathscr{A}$ is not backward shift invariant, suppose that $\mathscr{A}$ is backward shift invariant. Let $f + g \in \mathscr{A}$ ($f \in E$, $g \in A$) with $f \neq 0$. Put

$$f(z) = \sum_{n=k}^{\infty} a_n z^n, \quad a_k \neq 0.$$

Then

$$f^{*(k)}(0) = a_k \quad \text{and} \quad f^{*(k+1)} = -a_k\bar{z} \quad \text{on } Z(E),$$

https://doi.org/10.4153/CJM-1986-005-9 Published online by Cambridge University Press

where $f^{*(k)} = f^{*(k-1)}$. Since $\mathscr{A}$ is backward shift invariant, we can represent

$$f^{*(k+1)} = f_1 + g_1,$$

where $f_1 \in E$ and $g_1 \in A$. Hence $g_1 = -a_k\bar{z}$ on $Z(E)$. Since

$$m(\pi_0(Z(E))) = 1,$$

$g_1(z) = -a_k\bar{z}$ and $\bar{z} \in A$. But this is a contradiction, so $\mathscr{A}$ is not backward shift invariant.

By our lemmas, we get easily the following proposition.

PROPOSITION 2.1. *Let q be an inner function.*

(1) *If E is a backward shift invariant subspace of H^∞, then $[qE, A]$ is backward shift invariant if and only if $m(\text{supp } q) < 1$.*

(2) *If $\mathscr{A}$ is a backward shift invariant analytic subalgebra, then $[q\mathscr{A} + A]$ is backward shift invariant if and only if $m(\text{supp } q) < 1$.*

(3) *$[A, q]$ is backward shift invariant if and only if $m(\text{supp } q) < 1$.*

Remark 2.1. The above results are proved in [**2**, **8**, **9**, **10**] when $m(\text{supp } q) = 0$ or 1.

3. Proof of the main theorem. We will prove a more generalized version of our theorem given in the introduction. To state the new version, assume that $\mathscr{A}$, $\mathscr{C}$ and q satisfy the following conditions throughout the rest of this paper.

(a) $\mathscr{A}$ is a backward shift invariant analytic subalgebra.

(b) $\mathscr{C}$ is a C^*-algebra with $\mathscr{A} \subset \mathscr{C} \subset L^\infty$.

(c) The evaluation functional ϕ_0 of $\mathscr{A}$ has a unique norm-preserving Hahn-Banach extension to $\mathscr{C}$.

(d) q is an inner function in $\mathscr{A}$.

We give some examples of $\mathscr{A}$ and $\mathscr{C}$ satisfying (a), (b) and (c).

Example 3.1. (1) $\mathscr{A} = H^\infty$ and $\mathscr{C} = L^\infty$.

(2) For an inner function ψ, put $\mathscr{C}$ the C^*-algebra generated by $[A, \psi]$ and put $\mathscr{A} = H^\infty \cap \mathscr{C}$. It is easy to check that $\mathscr{A}$ is backward shift invariant. By [**9**, Theorem 3], $\mathscr{A}$ becomes a Dirichlet subalgebra of $\mathscr{C}$. This implies (c).

(3) Let B be a Douglas algebra and let $\mathscr{C}$ be the C^*-algebra generated by invertible inner functions in B. Put $\mathscr{A} = H^\infty \cap \mathscr{C}$, then $\mathscr{A}$ is a logmodular subalgebra of $\mathscr{C}$ [**1**]. It is easy to see (a) and (c).

Under the above assumptions (a)-(d) for $\mathscr{A}$, $\mathscr{C}$ and q, we will investigate the algebra $q\mathscr{A} + A$. In [**13**], Stegenga proved that $qH^\infty + A$ is closed if and only if $m(\text{supp } q) = 0$ or 1. We shall show that the same assertion is true for $q\mathscr{A} + A$. The proof is the same as the one in [**13**].

LEMMA 3.1 ([**13**, Lemma 3.5]). *If* ψ *is an inner function with* $m(\text{supp }\psi) = 0$, *then*

$$\|g + \psi A \cap C\| \leqq \|g + \psi H^\infty\| + \|g\|_{\text{supp}\psi}$$

for every $g \in C$, *where*

$$\|g\|_{\text{supp}\psi} = \sup\{ |g(\lambda)|; \lambda \in \text{supp }\psi\}.$$

Proof. This is a slight generalization of Stegenga's lemma. We can prove the above estimate by the same way as the one in [**13**].

LEMMA 3.2 [**13**, Lemma 2.3]. *Suppose that* X *and* Y *are closed subspaces of a Banach space* Z. *Then* $X + Y$ *is closed if and only if there exists a positive constant* K *with*

$$\|y + X \cap Y\| \leqq K\|y + X\| \quad \textit{for all } y \textit{ in } Y.$$

PROPOSITION 3.1. *Let* $\mathcal{S}$ *be an analytic subalgebra and* ψ *be an inner function. Then* $\psi\mathcal{S} + A$ *is closed if and only if* $m(\text{supp }\psi) = 0$ *or* 1.

Proof. Case 1. Suppose that $m(\text{supp }\psi) = 0$. By Lemma 3.1,

$$\|g + \psi\mathcal{S} \cap C\| \leqq \|g + \psi A \cap C\| \leqq \|g + \psi\mathcal{S}\| + \|g\|_{\text{supp}\psi}$$

for every $g \in C$. Since $\|g + \psi\mathcal{S}\| \geqq \|g\|_{\text{supp}\psi}$,

$$\|g + \psi\mathcal{S} \cap C\| \leqq 2\|g + \psi\mathcal{S}\| \quad \text{for every } g \in C.$$

By Lemma 3.2, $\psi\mathcal{S} + C$ is closed. Hence

$$\psi\mathcal{S} + A = H^\infty \cap (\psi\mathcal{S} + C)$$

is closed.

Case 2. Suppose that $m(\text{supp }\psi) = 1$. Since

$$\|\psi h + g\| \geqq \|g\| \quad \text{for every } h \in \mathcal{S} \text{ and } g \in A,$$

it is easy to see that $\psi\mathcal{S} + A$ is closed.

Case 3. Suppose that $0 < m(\text{supp }\psi) < 1$. Note that

$$\psi\mathcal{S} \cap A = \{0\}.$$

Let F be a closed subarc of ∂D such that

$$F \cap \text{supp }\psi = \phi.$$

Since ψ is continuous on F and $m(F) < 1$, there exists a function h_n in A such that

$$|(\psi - h_n)(\lambda)| < 1/n$$

for every $\lambda \in F$ $(n = 1, 2, \dots)$. Fix a point a in F. Then $|\psi(a)| = 1$. Choose a function g in A such that $g(a) = 1$ and $|g(\lambda)| < 1$ for every $\lambda \in \partial D\backslash\{a\}$. Then for each n, there exists a positive integer k_n such that

$$\|(\psi - h_n)g^{k_n}\| < 1/n.$$

Hence

$$\|h_n g^{k_n} + \psi\mathscr{S}\| < 1/n.$$

But

$$\|h_n g^{k_n}\| \geqq |h_n(a)| \geqq |\psi(a)| - 1/n = 1 - 1/n.$$

By Lemma 3.2, $\psi\mathscr{S} + A$ is not closed.

COROLLARY 3.1. *$q\mathscr{A} + A$ is an analytic subalgebra if and only if $m(\text{supp } q) = 0$ or 1.*

From now on, let put $X = M(\mathscr{C})$, the maximal ideal space of $\mathscr{C}$, and put $\mathscr{B}$ the closure of $q\mathscr{A} + A$. Then $\mathscr{B}$ is an analytic subalgebra. Now we can state our theorem.

THEOREM 3.1. *The following assertions are equivalent.*

(i) *$\mathscr{B}$ is backward shift invariant.*

(ii) $m(\text{supp } q) < 1$.

(iii) *The evaluation functional ϕ_0 of $\mathscr{B}$ has a unique norm-preserving Hahn-Banach extension to $\mathscr{C}$.*

(iv) *If B is a closed subalgebra with $\mathscr{B} \subset B \subset \mathscr{C}$ and $B \not\subset H^\infty$, then B contains C.*

By Proposition 2.1(2), (i) and (ii) are equivalent. To see (iii), we regard $\mathscr{B}$ as a closed subalgebra of $C(X)$. Then the study of norm-preserving Hahn-Banach extensions of ϕ_0 of $\mathscr{B}$ to $\mathscr{C}$ is the same as the study of representing measures for ϕ_0 on X. We start out to study some properties of measures on X.

For $\mu \in L^1(m)$, there exists a unique measure $\hat{\mu}$ on X such that

$$\int_X f d\hat{\mu} = \int_{\partial D} f d\mu \quad \text{for every } f \in \mathscr{C}.$$

The map $\mu \to \hat{\mu}$ is one-to-one and norm-preserving from $L^1(m)$ onto $L^1(\hat{m})$. For a given $\mu \in L^1(m)$, $\hat{\mu}$ is determined uniquely by the following conditions;

$$\hat{\mu} \in L^1(\hat{m}) \quad \text{and}$$

$$\int_X f d\hat{\mu} = \int_{\partial D} f d\mu \quad \text{for every } f \in C.$$

For a measure μ on X, there is a unique measure $\pi(\mu)$ on ∂D such that

$$\int_{\partial D} f d\pi(\mu) = \int_X f d\mu \quad \text{for every } f \in C.$$

The measure $\pi(\mu)$ is the image of μ by the fiber projection π from X onto ∂D; $\pi(x) = z(x)$ for $x \in X$. If $\mu \in L^1(m)$, then $\pi(\hat{\mu}) = \mu$. If two measures μ_1 and μ_2 in $L^1(\hat{m})$ satisfy $\pi(\mu_1) = \pi(\mu_2)$, then $\mu_1 = \mu_2$. From now on, we identify μ with $\hat{\mu}$ for every $\mu \in L^1(m)$, but we will use $\hat{m}$ to avoid the confusion with m.

For a measure μ on X, put

$$\mu = \mu_a + \mu_s, \quad \mu_a \ll \hat{m} \quad \text{and} \quad \mu_s \perp \hat{m}.$$

By (c), $\hat{m}$ is the unique representing measure on X for ϕ_0 of $\mathscr{A}$. Set

$$\mathscr{A}_0 = \{f \in \mathscr{A}; f(0) = 0\} \quad \text{and} \quad A_0 = \{f \in A; f(0) = 0\}.$$

For a subset E of $C(X)$ and for a measure μ on X, we write $\mu \perp E$ if

$$\int_X f d\mu = 0 \quad \text{for every } f \in E.$$

LEMMA 3.3. *If a measure μ on X satisfies $\mu \perp \mathscr{A}$ (or $\mathscr{A}_0$), then we get the following assertions.*

(1) $\mu_a, \mu_s \perp \mathscr{A}$ *(or* $\mathscr{A}_0$*).*

(2) *If we put* $d\pi(\mu_a) = fdm$, *then* $f \in H^1_0$ *(or* H^1*).*

(3) $\mu_s \perp \mathscr{A} + C$.

Proof. We shall prove the case $\mu \perp \mathscr{A}$. By the same way, it is easy to get (1)-(3) for the case $\mu \perp \mathscr{A}_0$.

(1) follows from the abstract F. and M. Riesz theorem [**5**, p. 44].

(2) Since $\mu_a \perp \mathscr{A}$, $\pi(\mu_a) \perp A$ and $f \in H^1_0$.

(3) Put

$$a = \int_X \bar{z} d\mu_s.$$

Since $\mathscr{A}$ is backward shift invariant, for $f \in \mathscr{A}$,

$$\begin{aligned}\int_X f d(\bar{z}\mu_s - a\hat{m}) &= \int_X f\bar{z} d\mu_s - \int_X \bar{z} d\mu_s \int_X f d\hat{m} \\ &= \int_X \bar{z}(f - f(0))d\mu_s \\ &= \int_X f^* d\mu_s = 0.\end{aligned}$$

By (1), $\bar{z}\mu_s \perp \mathscr{A}$ and $\mu_s \perp \bar{z}\mathscr{A}$. Repeating these arguments,

$$\mu_s \perp \bar{z}^n\mathscr{A} \quad \text{for every } n.$$

Thus we get (3).

LEMMA 3.4. *If μ is a probability measure on X and $\mu \perp q\mathscr{A}_0 + A_0$, then we have the following assertions.*

(1) $\pi(\mu) = m$.

(2) $\pi(\mu_s)$ *is concentrated on* supp q.

(3) *If we put* $d\pi(\mu_a) = fdm$, *then* $qf \in H^1$.

Proof. (1) Since $\mu \perp A_0$, $\pi(\mu) \perp A_0$. Since $\pi(\mu) \geqq 0$ and $\|\pi(\mu)\| = 1$, $\pi(\mu) = m$.

(2) First we note that $q\mu \perp \mathscr{A}_0$. By Lemma 3.3 (3),

$$q\mu_s \perp \mathscr{A} + C \quad \text{and} \quad \mu_s \perp q(\mathscr{A} + C).$$

Then $\mu_s \perp qC$, and thus

$$\pi(\mu_s) \perp qC \cap C.$$

Consequently, $\pi(\mu_s)$ is concentrated on supp q.

(3) By Lemma 3.3 (2),

$$d\pi(q\mu_a) = qfdm \quad \text{and} \quad qf \in H^1.$$

The following lemma shows that in Theorem 3.1, (i) implies (iii).

LEMMA 3.5. *Let $\mathscr{A}_1$ be a backward shift invariant analytic subalgebra such that $\mathscr{B} \subset \mathscr{A}_1 \subset \mathscr{C}$. Then ϕ_0 of $\mathscr{A}_1$ has a unique norm-preserving Hahn-Banach extension to $\mathscr{C}$.*

Proof. Let μ be a representing measure on X for ϕ_0 of $\mathscr{A}_1$. To show our assertion, it is sufficient to prove $\mu = \hat{m}$. To see this, put

$$\mu = \mu_a + \mu_s,\ \mu_a \ll \hat{m} \quad \text{and} \quad \mu_s \perp \hat{m}.$$

Since $\mu \perp q\mathscr{A}_0 + A_0$, it is clear that

(1) $\mu_s \perp q(\mathscr{A} + C)$

in the proof of Lemma 3.4 (2). Put $d\pi(\mu_a) = fdm$. By Lemma 3.4 (3),

(2) $qf \in H^1$.

Set

$$q(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Since $q \in \mathscr{A}_1$ and $\mathscr{A}_1$ is backward shift invariant,

$$q^{*(n)} = \left(q(z) - \sum_{k=0}^{n-1} a_k z^k\right)\Big/ z^n \in \mathscr{A}_1 \quad (n = 0, 1, 2, \dots).$$

Hence for each n,

$$\begin{aligned} a_n &= \int_X q^{*(n)} d\mu \\ &= \int_X q/z^n d\mu - \int_X \sum_{k=0}^{n-1} a_k z^{k-n} d\mu \\ &= \int_X q/z^n d\mu_a + \int_X q/z^n d\mu_s \end{aligned}$$

(since $\mu \geqq 0$ and $\mu \perp A_0$)

$$= \int_{\partial D} qf/z^n dm$$

by (1).

The above equations and (2) give us that $qf = q$. Since q is inner, $f = 1$ a.e. dm and $\mu = \hat{m}$.

As a corollary, we get the following.

COROLLARY 3.2. *Conditions* (ii) *and* (iv) *in Theorem* 3.1 *are equivalent.*

Proof. (ii) $\Rightarrow$ (iv) Suppose that $m(\text{supp } q) < 1$. Then $\mathscr{B}$ is backward shift invariant by Proposition 2.1(2). By Lemma 3.5 and Sarason's theorem, we get (iv).

(iv) $\Rightarrow$ (ii) Suppose that $m(\text{supp } q) = 1$. Put

$$B = q(\mathscr{A} + C) + A.$$

By the same way as in the proof of Lemma 2.4, B is a closed subalgebra with $\mathscr{B} \subset B \subset \mathscr{C}$. Since $q\bar{z}^n \notin H^\infty$ for some n, $B \not\subset H^\infty$. Since $q[\mathscr{A} + C] \cap C = \{0\}$, $C \not\subset B$. But this contradicts (iv).

To complete the proof of Theorem 3.1, we need to prove (iii) $\Rightarrow$ (ii). The following lemma is its special case.

LEMMA 3.6. *If* $m(\text{supp } q) = 1$ *and* $q(0) = 0$, *then* ϕ_0 *of* $\mathscr{B}$ *does not have a unique norm-preserving Hahn-Banach extension to* $\mathscr{C}$.

Proof. There exists a positive integer n such that

$$q/z^n \in H^\infty \quad \text{and} \quad (q/z^n)(0) \neq 0.$$

Put $\psi = q/z^n$, then $\psi \in \mathscr{A}$. Also put

$$\alpha : \psi\mathscr{A} + A \ni \psi h + f \rightarrow f(0).$$

Then α is a non-zero complex homomorphism of $\psi\mathscr{A} + A$. Let μ be a representing measure for α on X. Since

$$\int_X (\psi + f) d\hat{m} = \psi(0) + f(0) \quad \text{for } f \in A,$$

we obtain $\mu \neq \hat{m}$. Note that for $h \in \mathscr{A}$ and $f \in A$,

$$\int_X (qh + f) d\mu = \int_X (\psi z^n h + f) d\mu = f(0) = (qh + f)(0).$$

Hence μ is a representing measure for ϕ_0 on $\mathscr{B}$.

To remove the assumption $q(0) = 0$ in Lemma 3.6, we study the structure of representing measures for a complex homomorphism α_0 of $\mathscr{B} = q\mathscr{A} + A$ with $m(\text{supp } q) = 1$ defined as follows;

$$\alpha_0 : q\mathscr{A} + A \ni qh + f \rightarrow f(0).$$

If $q(0) = 0$, then $\alpha_0 = \phi_0$. By the proof of Lemma 3.6, there exists a representing measure μ for α_0 such that $\mu \neq \hat{m}$. Fix such a measure μ and put $\mu = \mu_a + \mu_s$. Set

$$d\pi(\mu_a) = fdm \quad (f \in L^1(m)).$$

Then $f \neq 1$ and $0 \leqq f \leqq 1$. Since $\mu \perp q\mathscr{A} + A_0$, $\pi(\mu) = m$ by the same way as in the proof of Lemma 3.4. By Lemma 3.3,

$$qf \in H_0^1 \quad \text{and} \quad \mu_s \perp q(\mathscr{A} + C).$$

The following lemma shows that the converse of the above fact is affirmative. This is the key to prove our theorem.

LEMMA 3.7. *Let* $g \in L^1(m)$ *such that* $0 \leqq g \leqq 1$ a.e. dm *and* $qg \in H_0^1$. *Then there exists a representing measure* λ *on* X *for* α_0 *such that* $d\pi(\lambda_a) = gdm$.

Proof. Since

$$dm = d\pi(\mu) = d\pi(\mu_a) + d\pi(\mu_s) = fdm + d\pi(\mu_s),$$

we have

$$d\pi(\mu_s) = (1 - f)dm.$$

Since $qf \in H_0^1$ as mentioned above, $q(1 - f) \in H^1$. Thus

$$1 - f \neq 0 \text{ a.e. } dm.$$

Set

$$h = (1 - g)/(1 - f).$$

Then h is a non-negative Borel measurable function on ∂D, and $h \in L^1(\pi(\mu_s))$; in fact,

$$\int_{\partial D} |h| d\pi(\mu_s) = \int_{\partial D} (1 - g)dm < \infty.$$

Here there is a sequence of non-negative functions $\{f_n\}_{n=1}^{\infty}$ in C such that

$$f_n \rightarrow h (n \rightarrow \infty) \quad \text{in } L^1(\pi(\mu_s)).$$

Since $\|\pi(\nu)\| = \|\nu\|$ for a non-negative measure ν of X,

$$\begin{aligned} \|f_n\mu_s - f_k\mu_s\| &\leqq \| |f_n - f_k|\mu_s\| = \|\pi(|f_n - f_k|\mu_s)\| \\ &= \| |f_n - f_k|\pi(\mu_s)\| \rightarrow 0 \quad (n, k \rightarrow \infty). \end{aligned}$$

Thus $f_n\mu_s$ converges to a non-negative measure λ_s on X with $\lambda_s \ll \mu_s$. Since $\|\pi(\nu)\| \leqq \|\nu\|$ for a measure ν on X,

$$\|\pi(\lambda_s) - (1 - g)m\| = \|\pi(\lambda_s) - h(1 - f)m\|$$

$$\begin{aligned}\|\pi(\lambda_s) - (1 - g)m\| &= \|\pi(\lambda_s) - h\pi(\mu_s)\| \\ &= \lim_{n\to\infty} \|\pi(\lambda_s) - f_n\pi(\mu_s)\| \\ &= \lim_{n\to\infty} \|\pi(\lambda_s - f_n\mu_s)\| \\ &\leqq \overline{\lim_{n\to\infty}} \|\lambda_s - f_n\mu_s\| = 0.\end{aligned}$$

This leads us to

$$d\pi(\lambda_s) = (1 - g)dm.$$

If we put $d\lambda = gdm + d\lambda_s$, then

$$d\pi(\lambda) = gdm + d\pi(\lambda_s) = dm.$$

Thus λ is a probability measure on X. Since $qg \in H_0^1$,

$$gdm \perp qA.$$

Since $\mu_s \perp q(\mathscr{A} + C)$,

$$f_n\mu_s \perp q(\mathscr{A} + C).$$

Since $f_n\mu_s \to \lambda_s$,

$$\lambda_s \perp q(\mathscr{A} + C).$$

These facts show

$$\lambda = gdm + d\lambda_s \perp q\mathscr{A}.$$

Consequently

$$\int_X (qh + f)d\lambda = f(0) \quad \text{for } qh + f \in q\mathscr{A} + A.$$

This completes the proof.

Proof of Theorem 3.1. (i) $\Leftrightarrow$ (ii) follows from Proposition 2.1 (2).
(ii) $\Leftrightarrow$ (iv) is already proved in Corollary 3.2.
(i) $\Rightarrow$ (iii) follows from Lemma 3.5.
We shall prove (iii) $\Rightarrow$ (ii). To see this, suppose that

$$m(\text{supp } q) = 1.$$

By Lemma 3.6, it is sufficient to see that if $q(0) \neq 0$, then ϕ_0 of $\mathscr{B}$ does not have a unique representing measure on X.

Claim. There exists $g \in L^1(m)$ such that $0 < g < 1$ a.e. dm, $qg \in H^1$ and $\int_{\partial D} qgdm = q(0)$.

First, on the assumption of our claim, we shall complete the proof. By our claim and Lemma 3.7, there exists a representing measure λ on X for α_0 of $qz\mathscr{A} + A$ such that $d\pi(\lambda_a) = gdm$. Then

$$\lambda \perp qz\mathscr{A} \quad \text{and} \quad qd\lambda \perp \mathscr{A}_0.$$

Hence for every $qf + h \in \mathscr{B} = q\mathscr{A} + A$, we get

$$\int_X (qf + h)d\lambda = \int_X q(f - f(0))d\lambda + f(0)\int_X qd\lambda + \int_X hd\lambda$$

$$= \int_X qzf^*d\lambda + f(0)\int_X qd\lambda + h(0)$$

$$= f(0)\int_X qd\lambda_a + f(0)\int_X qd\lambda_s + h(0)$$

$$= f(0)\int_{\partial D} qgdm + h(0)$$

(by Lemma 3.3 (3))

$$= f(0)q(0) + h(0)$$

(by our claim)

$$= \phi_0(qf + h).$$

Therefore λ is a representing measure for ϕ_0 of $q\mathscr{A} + A$. Since $0 < g < 1$ and $d\pi(\lambda_a) = gdm$, $\lambda \neq \hat{m}$. This completes the proof of Theorem 3.1.

We shall prove our claim. Put $a = q(0)$. We may assume that $0 < a < 1$. Set

$$F = (1 + a^2)/a - (q + \bar{q}).$$

Then $F \in L^\infty$ and $F \neq 0$. Since $(1 + a^2)/a - 2 > 0$ and $-2 \leqq q + \bar{q} \leqq 2$, $0 < F$ a.e. dm. Take a small positive number c such that $0 < 1 - cF < 1$ a.e. dm, and put $G = 1 - cF$. Since

$$qF = (1 + a^2)q/a - (q^2 + 1) \in H^\infty,$$

$qG \in H^\infty$. Since

$$(qF)(0) = (1 + a^2) - (a^2 + 1) = 0,$$

we get

$$\int_{\partial D} qGdm = \int_{\partial D} qdm - c\int_{\partial D} qFdm = q(0).$$

This completes the proof of our claim.

4. Additional properties of $qH^{\infty} + A$ type algebras. In this section, we use the same notations as the ones in Section 3. Here we discuss

(1) the maximal ideal space of $\mathscr{B}$,

(2) the corona theorem for $\mathscr{B}$,

(3) singular representing measures for ϕ_0 of $\mathscr{B}$, and

(4) Sarason's theorem when one of the conditions is dropped.

Let $\mathscr{S}$ be an analytic subalgebra. For each z in D, put

$$\phi_z : \mathscr{S} \ni f \to f(z).$$

Then ϕ_z is a complex homomorphism of $\mathscr{S}$. We may identify D with $\{\phi_z; z \in D\}$. Then $D \subset M(\mathscr{S})$. We say that the corona theorem holds for $\mathscr{S}$ if D is dense in $M(\mathscr{S})$. For each λ in ∂D, put

$$M_\lambda(\mathscr{S}) = \{x \in M(\mathscr{S}); z(x) = \lambda\}.$$

In [**2**, p. 43], Dawson remarked that if $\mathscr{S}$ is backward shift invariant then

$$M(\mathscr{S}) = D \cup \{M_\lambda(\mathscr{S}); \lambda \in \partial D\}.$$

By Lemma 2.1, it is easy to see the above fact and that

$$M(\mathscr{S} + C) = M(\mathscr{S})\backslash D.$$

If $\mathscr{S}$ is not backward shift invariant, it is difficult to describe $M(\mathscr{S})$, but it is easy to see

$$M([\mathscr{S} + C]) = \{x \in M(\mathscr{S}); |z(x)| = 1\}.$$

The first result in this section is to describe $M(\mathscr{B})$. Since $\mathscr{B} \subset \mathscr{A}$, there is a continuous restriction map Γ from $M(\mathscr{A})$ to $M(\mathscr{B})$. If $m(\text{supp } q) = 1$, for $z \in D$ we put

$$\alpha_z : \mathscr{B} = q\mathscr{A} + A \ni qh + f \to f(z).$$

Then α_z is a complex homomorphism of $\mathscr{B}$.

THEOREM 4.1. (1) $\Gamma(M(\mathscr{A})) = D \cup \{x \in M(\mathscr{B}); |z(x)| = 1\}$.

(2) *If the corona theorem holds for $\mathscr{A}$ and $m(\text{supp } q) < 1$, then the corona theorem holds for $\mathscr{B}$.*

(3) *If $m(\text{supp } q) = 1$, then*

$$M(\mathscr{B}) = \Gamma(M(\mathscr{A})) \cup \{\alpha_z; z \in D \text{ with } q(z) \neq 0\}$$

and the corona theorem does not hold for $\mathscr{B}$.

Proof. (1) From the definition of Γ, it is easy to see that

$$\Gamma(D) = D \quad \text{and} \quad \Gamma(M_\lambda(\mathscr{A})) \subset M_\lambda(\mathscr{B}) \quad \text{for } \lambda \in \partial D.$$

Since $M(\mathscr{A}) = D \cup \{x \in M(\mathscr{A}); |z(x)| = 1\}$,

$$\Gamma(M(\mathscr{A})) \subset D \cup \{x \in M(\mathscr{B}); |z(x)| = 1\}.$$

To see the converse inclusion, let $x \in M(\mathscr{B})$ with $z(x) = \lambda$ and $|\lambda| = 1$.

Case 1. Suppose that $q(x) \neq 0$. Put

$$\gamma : \mathscr{A} \ni f \to (qf)(x)/q(x).$$

Then γ is a non-zero complex homomorphism of $\mathscr{A}$. In fact, for $f, g \in \mathscr{A}$

$$\begin{aligned}\gamma(f)\gamma(g) &= (qf)(x)(qg)(x)/q(x)^2 = (qqfg)(x)/q(x)^2 \\ &= q(x)(qfg)(x)/q(x)^2 = \gamma(fg).\end{aligned}$$

Hence there is a point x' in $M(\mathscr{A})$ such that

$$\gamma(f) = f(x') \quad \text{for } f \in \mathscr{A}.$$

Since $h(x') = \gamma(h) = h(x)$ for $h \in \mathscr{B}$, we get

$$x \in \Gamma(M(\mathscr{A})).$$

Case 2. Suppose that $q(x) = 0$. Since

$$(qf)^2(x) = q(x)(qf^2)(x) = 0 \quad \text{for } f \in \mathscr{A},$$

$$\{h \in \mathscr{B};\ h(x) = 0\} \supset q\mathscr{A}.$$

Since $q(x) = 0$, q is not constant on $M_\lambda(\mathscr{B})$. Hence q is not constant on $M_\lambda(\mathscr{A})$. Consequently there is a point x' in $M_\lambda(\mathscr{A})$ such that $q(x') = 0$. Since

$$(qf + g)(x') = g(x') = g(\lambda) = g(x) = (qf + g)(x)$$

for $f \in \mathscr{A}$ and $g \in A$, we get

$$x = \Gamma(x') \in \Gamma(M(\mathscr{A})).$$

Thus we get (1). Note that Γ is one-to-one on

$$\{x \in M(\mathscr{A});\ |z(x)| = 1 \text{ and } q(x) \neq 0\},$$

and

$$\Gamma\{x \in M(\mathscr{A});\ |z(x)| = 1 \text{ and } q(x) = 0\}$$

is a one point set by our proof.

(2) Suppose that the corona theorem holds for $\mathscr{A}$. Then by (1), D is dense in

$$D \cup \{x \in M(\mathscr{B});\ |z(x)| = 1\}.$$

If $m(\operatorname{supp} q) < 1$, then $\mathscr{B}$ is backward shift invariant by Theorem 3.1. So

$$M(\mathscr{B}) = D \cup \{x \in M(\mathscr{B});\ |z(x)| = 1\}.$$

Thus we get (2).

(3) We have

$$M(\mathscr{B}) = \cup\{M_\lambda(\mathscr{B});\ \lambda \in \partial D\} \cup \{x \in M(\mathscr{B});\ |z(x)| < 1\}.$$

We shall prove that

$$\{x \in M(\mathscr{B}); |z(x)| < 1\} = D \cup \{\alpha_z;\ z \in D \text{ with } q(z) \neq 0\}.$$

It is obvious that

$$D \cup \{\alpha_z;\ z \in D \text{ with } q(z) \neq 0\} \subset \{x \in M(\mathscr{B});\ |z(x)| < 1\}.$$

To see the converse inclusion, take $x \in M(\mathscr{B})$ with $|z(x)| < 1$. If $(qh + f)(x) = f(x)$ for every $qh + f \in \mathscr{B}$, we have $x = \alpha_{z(x)}$, because $f(x) = f(z(x))$. Moreover if $q(z(x)) = 0$, then

$$x = \alpha_{z(x)} = \phi_{z(x)}.$$

Next, suppose that

$$(qh_0 + f_0)(x) \neq f_0(x) \quad \text{for some } qh_0 + f_0 \in \mathscr{B},$$

that is, $(qh_0)(x) \neq 0$ for some $h_0 \in \mathscr{A}$. We shall prove $x = \phi_{z(x)}$. We have

$$(qh)(x) = (q(h - h(z(x))))(x) + h(z(x))q(x)$$

for each $h \in \mathscr{A}$. Since $\mathscr{A}$ is backward shift invariant, we may represent

$$h - h(z(x)) = (z - (z(x))h',$$

where $h' \in \mathscr{A}$, by Lemma 2.1. Then

$$(q(h - h(z(x))))(x) = (qh')(x)(z - z(x))(x) = 0.$$

Thus

$$(qh)(x) = h(z(x))q(x).$$

In particular

$$q(x)^2 = q(z(x))q(x) \text{ and } q(x)\{q(x) - q(z(x))\} = 0.$$

Since $(qh_0)(x) \neq 0$,

$$h_0(z(x))q(x) \neq 0 \quad \text{and} \quad q(x) \neq 0.$$

Then $q(x) = q(z(x))$ and

$$(qh + f)(x) = q(z(x))h(z(x)) + f(z(x)) = (qh + f)(z(x)).$$

Thus we get $x = \phi_{z(x)}$, and

$$M(\mathscr{B}) = \Gamma(M(\mathscr{A})) \cup \{\alpha_z;\ z \in D \text{ with } q(z) \neq 0\}.$$

Since $\Gamma(M(\mathscr{A}))$ is a compact subset of $M(\mathscr{B})$, the corona theorem does not hold for $\mathscr{B}$ by (1).

If $m(\text{supp } q) = 1$, then $q(\mathscr{A} + C) + A$ is a closed subalgebra by the same way as in the proof of Proposition 2.1.

THEOREM 4.2. *Suppose that $m(\text{supp } q) = 1$. Then*

(1) *α_0 of $\mathscr{B}$ has a singular representing measure on X, where singular means with respect to $\hat{m}$.*

(2) *ϕ_0 of $\mathscr{B}$ has a singular representing measure on X if and only if $q(0) = 0$.*

(3) *If $q(0) = 0$, then the set of singular representing measures for ϕ_0 of $\mathscr{B}$ coincides with the set of representing measures for the complex homomorphism*

$$\beta_0:\phi q(\mathscr{A} + C) + A \ni qh + f \to f(0).$$

Proof. (1) We take $g = 0$ as the one in Lemma 3.7. Then there is a representing measure λ on X for α_0 such that $d\pi(\lambda_a) = 0$. Thus λ is singular.

(2) If $q(0) = 0$, then $\alpha_0 = \pi_0$. Hence the if part follows from (1). To see the inverse direction, suppose that ϕ_0 of $\mathscr{B}$ has a singular representing measure λ on X. Since $q\lambda \perp \mathscr{A}_0$, $\lambda \perp q(\mathscr{A} + C)$ by Lemma 3.3(3). Then

$$q(0) = \int_X q d\lambda = 0.$$

(3) Let λ be a singular representing measure on X for ϕ_0 of $\mathscr{B}$. Then $\lambda \perp q(\mathscr{A} + C)$, and λ becomes a representing measure for β_0. Let ν be a representing measure on X for β_0. ν is also a representing measure for ϕ_0. Since $\nu \perp q\mathscr{A}$,

$$\nu_a \perp q\mathscr{A} \quad \text{and} \quad \nu_s \perp q(\mathscr{A} + C)$$

by Lemma 3.3. Moreover since $\nu \perp q(\mathscr{A} + C)$,

$$\nu_a \perp q(\mathscr{A} + C) \quad \text{and} \quad qd\nu_a \perp C.$$

Hence $qd\nu_a = 0$. Since q is inner, $\nu_a = 0$. Thus ν is singular.

For the rest of this section, denote

$$\mathscr{A}_1 = H^\infty \cap \{q(\mathscr{A} + C) + A\} \quad \text{and}$$

$$\mathscr{A}_2 = H^\infty \cap \{q(\mathscr{A} + C) + C\}.$$

Here we shall study some properties of $\mathscr{A}_1$ and $\mathscr{A}_2$. If $m(\text{supp } q) = 1$, then $\mathscr{A}_1$ and $\mathscr{A}_2$ are analytic subalgebras with $\mathscr{B} \subset \mathscr{A}_1 \subset \mathscr{A}_2$. Since $q(\mathscr{A} + C) \cap A = \{0\}$, for each $z \in D$,

$$\gamma_z:\mathscr{A}_1 \ni q(h + g) + f \to f(z)$$

is a complex homomorphism of $\mathscr{A}_1$.

THEOREM 4.3. *Suppose that* $m(\text{supp } q) = 1$. *Then*

(1) $\mathscr{A}_1$ *is not a backward shift invariant analytic subalgebra.*

(2) $\mathscr{A}_2$ *is the smallest backward shift invariant analytic subalgebra containing* $\mathscr{B}$.

(3) *In the family of analytic subalgebras* $\mathscr{S}$ *between* $\mathscr{B}$ *and* $\mathscr{C}$ *with the property that* ϕ_0 *of* $\mathscr{S}$ *has a unique norm-preserving Hahn-Banach extension to* $\mathscr{C}$, $\mathscr{A}_1$ *is the smallest one.*

(4) $\mathscr{B} \subsetneq \mathscr{A}_1 \subsetneq \mathscr{A}_2$.

(5) *There are no other analytic subalgebras between* $\mathscr{A}_1$ *and* $\mathscr{A}_2$.

(6) $M(\mathscr{A}_1) = \{x \in M(\mathscr{B}); |z(x)| = 1\} \cup D \cup \{\gamma_z; z \in D\}$.

We need the following lemma proved essentially in [**13**, Lemma 3.4].

LEMMA 4.1. *Let* $\mathscr{S}$ *be a backward shift invariant analytic subalgebra and let* ψ *be a unimodular function in* L^∞. *If* $\psi\mathscr{S} \cap C \neq \{0\}$, *then* $\psi\mathscr{S} \cap C$ *is weak star dense in* ψH^∞.

Proof of Theorem 4.3. (1) We may represent

$$\mathscr{A}_1 = \{H^\infty \cap q(\mathscr{A} + C)\} + A.$$

Then $E = H^\infty \cap q(\mathscr{A} + C)$ satisfies

$$m(\pi_0(Z(E))) = 1.$$

By Lemma 2.4, $\mathscr{A}_1$ is a non-backward shift invariant analytic subalgebra.

(2) By the same way as in the proof of Lemma 2.4, $q(\mathscr{A} + C) + C$ is a closed subalgebra. Since

$$\mathscr{A}_2 \subset H^\infty \cap [\mathscr{A}_2 + C] \subset H^\infty \cap [q(\mathscr{A} + C) + C] = \mathscr{A}_2,$$

$\mathscr{A}_2$ is backward shift invariant by Lemma 2.1 (5). Let $\mathscr{A}_3$ be a backward shift invariant analytic subalgebra containing $\mathscr{B}$. By Lemma 2.1 (4),

$$q(\mathscr{A} + C) + C = [\mathscr{B} + C] \subset \mathscr{A}_3 + C.$$

Then $\mathscr{A}_2 \subset \mathscr{A}_3$.

(3) Let $\mathscr{A}_4$ be an analytic subalgebra between $\mathscr{B}$ and $\mathscr{C}$ such that ϕ_0 of $\mathscr{A}_4$ has a unique norm-preserving Hahn-Banach extension to $\mathscr{C}$. Let λ be a measure on X such that $\lambda \perp \mathscr{A}_4$, then

$$\lambda_a \perp A \quad \text{and} \quad \lambda_s \perp q(\mathscr{A} + C).$$

These imply that

$$\lambda \perp H^\infty \cap \{q(\mathscr{A} + C) + A\},$$

so that

$$H^\infty \cap \{q(\mathscr{A} + C) + A\} \subset \mathscr{A}_4.$$

We shall prove that ϕ_0 of $H^\infty \cap \{q(\mathscr{A} + C) + A\}$ has a unique representing measure. We may asusme $q(0) \neq 0$. Because, if $q = z^n q'$, where $q' \in \mathscr{A}$ and $q'(0) \neq 0$, then

$$q(\mathscr{A} + C) + A = q'(\mathscr{A} + C) + A.$$

Let μ be a representing measure on X for ϕ_0. Since μ is also a representing measure for ϕ_0 of $q\mathscr{A} + A$,

$$\mu_a \perp q\mathscr{A}_0 \quad \text{and} \quad \mu_s \perp q(\mathscr{A} + C)$$

by Lemma 3.3. Then

$$q(0) = \int_X q d\mu = \int_X q d\mu_a.$$

By Lemma 4.1, $z\bar{q}\mathscr{A} \cap C$ is weak star dense in $z\bar{q}H^\infty$. Hence $\mathscr{A}_0 \cap qC$ is weak star dense in H_0^∞. Then for each $h \in H_0^\infty$ there exists a net $\{f_\alpha\}_\alpha$ in C such that

$$f_\alpha q \in \mathscr{A}_0 \quad \text{and} \quad f_\alpha q \to h \quad \text{(weak star topology)}.$$

Since

$$0 = \phi_0(f_\alpha q) = \int_X f_\alpha q d\mu = \int_X f_\alpha q d\mu_a = \int_{\partial D} f_\alpha q d\pi(\mu_a)$$
$$\to \int_{\partial D} h d\pi(\mu_a),$$

we have

$$\int_{\partial D} h d\pi(\mu_a) = 0 \quad \text{for } h \in H_0^\infty.$$

Since $\pi(\mu_a)$ is a non-negative measure on ∂D, $\pi(\mu_a) = cm$ for some constant c with $0 \leqq c \leqq 1$. Since $q(0) \neq 0$ and

$$q(0) = \int_X q d\mu_a = c \int_X q d\hat{m} = cq(0),$$

we obtain $c = 1$. So $\mu_a = \hat{m}$.

(4) The first inequality follows from (3) and Theorem 3.1. The second one follows from (1) and (2).

(5) Let $\mathscr{A}_5$ be a closed subalgebra with $\mathscr{A}_1 \subsetneq \mathscr{A}_5 \subset \mathscr{A}_2$. Then there exists a function f in $\mathscr{A}_5$ such that

$$f = q(h + g_1) + g_2, \quad h \in \mathscr{A}, g_1, g_2 \in C \text{ and } g_2 \notin A.$$

Let μ be a measure on X with $\mu \perp \mathscr{A}_5$. By (3), ϕ_0 of $\mathscr{A}_5$ has a unique representing measure on X. Then $\mu_a \perp \mathscr{A}_5$ and $\mu_s \perp \mathscr{A}_5$. Consequently, $\mu_a \perp \mathscr{A}_2$. Since $[A, f] \subset \mathscr{A}_5$, we get $[A, f] \perp \mu_s$. Since $\mathscr{B} \perp \mu_s$,

$$q(\mathscr{A} + C) \perp \mu_s$$

by Lemma 3.3. Hence $[A, g_2] \perp \mu$. Since $g_2 \notin A$, $\mu_s \perp C$. These imply that

$$\mu_s \perp q(\mathscr{A} + C) + C \quad \text{and} \quad \mu_s \perp \mathscr{A}_2.$$

Thus $\mu \perp \mathscr{A}_2$ and $\mathscr{A}_5 = \mathscr{A}_2$.

(6) Since

$$\mathscr{A}_{1|M_\lambda(\mathscr{C})} = \mathscr{B}_{|M_\lambda(\mathscr{C})} \quad \text{for } \lambda \in \partial D,$$

it is clear that $M_\lambda(\mathscr{A}_1) = M_\lambda(\mathscr{B})$. We shall show that

$$\{x \in M(\mathscr{A}_1);\ |z(x)| < 1\} = D \cup \{\gamma_z;\ z \in D\}.$$

It is obvious that

$$D \cup \{\gamma_z;\ z \in D\} \subset \{x \in M(\mathscr{A}_1);\ |z(x)| < 1\}.$$

To see the converse inclusion, let $x \in M(\mathscr{A}_1)$ with $|z(x)| < 1$ and let x' be the restriction homomorphism of x onto $\mathscr{B}$. Then $x' \in M(\mathscr{B})$ and $z(x) = z(x')$. By Theorem 4.1, $x' = \phi_{z(x)}$ or $x' = \alpha_{z(x)}$. By the same way as in the proof of (3), we may assume that $q(z(x)) \neq 0$.

Case 1. Suppose that $x' = \phi_{z(x)}$. We shall prove that

$$x = \phi_{z(x)}.$$

Let $g \in C$ with $qg \in H^\infty$. By Lemma 2.1, $qg \in \mathscr{A}$. Then $q^2 g \in \mathscr{B}$, and

$$\begin{aligned} q(z(x))(qg)(x) &= q(x')(qg)(x) = q(x)(qg)(x) \\ &= (q^2 g)(x) = (q^2 g)(x') = (q^2 g)(z(x)) \\ &= q(z(x))(qg)(z(x)). \end{aligned}$$

Hence

$$(qg)(x) = (qg)(z(x))$$

and for $q(f + g) + h \in \mathscr{A}_1$,

$$\begin{aligned} \{q(f + g) + h\}(x) &= (qf)(x') + (qg)(x) + h(x') \\ &= \{q(f + g) + h\}(z(x)). \end{aligned}$$

Thus $x = \phi_{z(x)}$.

Case 2. Suppose that $x' = \alpha_{z(x)}$. We shall prove that

$$x = \gamma_{z(x)}.$$

Let μ be a representing measure for x on X. Then μ is also a representing measure for $x' = \alpha_{z(x)}$. Since $\mu \perp q\mathscr{A}$,

$$\mu_a \perp q\mathscr{A} \quad \text{and} \quad \mu_s \perp q(\mathscr{A} + C)$$

by Lemma 3.3. If we put

$$\phi(z) = (z - z(x))/(1 - \overline{z(x)}z),$$

then $\phi \in A$. For $g \in C$ with $qg \in H^\infty$,

$$\begin{aligned} 0 &= \phi(z(x))(qg)(x) = \phi(x)(qg)(x) = (\phi qg)(x) \\ &= \int_X \phi qg d\mu = \int_X \phi qg d\mu_a = \int_{\partial D} \phi qg d\pi(\mu_a). \end{aligned}$$

Since $H^\infty \cap qC$ is weak star dense in H^∞ by Lemma 4.1,

$$\int_{\partial D} \phi h d\pi(\mu_a) = 0 \quad \text{for } h \in H^\infty.$$

We may represent

$$q = q(z(x)) + \phi h_0 \quad \text{for some } h_0 \in H^\infty.$$

Then

$$0 = \int_X q d\mu = \int_X q d\mu = q(z(x))\|\mu_a\| + \int_X \phi h_0 d\mu_a$$
$$= q(z(x))\|\mu_a\|.$$

Hence $\|\mu_a\| = 0$, and $\mu = \mu_s$. Since $\mu = \mu_s \perp q(\mathscr{A} + C)$, we get

$$(q(f + g) + h)(x) = h(x) = h(z(x))$$

for $q(f + g) + h \in \mathscr{A}_1$. Thus $x = \gamma_{z(x)}$.

Remark 4.1. (1) In Sarason's theorem, we can't remove condition (a). Because, if $m(\text{supp } q) = 1$, then

$$H^\infty \cap \{q(\mathscr{A} + C) + A\} \subsetneq q(\mathscr{A} + C) + A,$$

$$C \not\subset q(\mathscr{A} + C) + A \quad \text{and } q(\mathscr{A} + C) + A \not\subset H^\infty.$$

While, ϕ_0 on $H^\infty \cap \{q(\mathscr{A} + C) + A\}$ has a unique norm-preserving Hahn-Banach extension to $\mathscr{C}$ by Theorem 4.3.

(2) If q is a singular inner function with $m(\text{supp } q) = 1$, then the restriction map from $M(\mathscr{A}_1)$ to $M(\mathscr{B})$ is a homeomorphism. This follows from Theorems 4.1 and 4.3.

The authors are grateful to the referee for shortening the original proof of Lemma 2.2.

References

1. S. Y. Chang and D. Marshall, *Some algebras of bounded analytic functions containing the disk algebra*, Lecture Notes in Math. *604*, 12-20 (Springer-Verlag, Berlin Heidelberg, 1977).
2. D. Dawson, *Subalgebras of* H^∞, Thesis, Univ. of Indiana (1975).
3. S. Fisher, *Algebras of bounded functions invariant under the restricted backward shift*, J. Funct. Anal. *12* (1973), 236-245.
4. ——— *Invariant subalgebras of the backward shift*, Amer. J. Math. *95* (1973), 537-552.
5. T. Gamelin, *Uniform algebras* (Prentice-Hall, Englewood Cliffs, N. J., 1969).
6. J. Garnett, *Bounded analytic functions* (Academic Press, New York, 1981).
7. K. Hoffman, *Banach spaces of analytic functions* (Prentice-Hall, Englewood Cliffs, N. J., 1962).
8. T. Muto and T. Nakazi, *Maximality theorems for some closed subalgebras between A and* H^∞, Arch. Math. *43* (1984), 173-178.
9. T. Nakazi, *Algebras generated by z and an inner function*, Arch. Math. *42* (1984), 545-548.

10. K. Nishizawa, *On closed subalgebras between A and H^{∞} II*, Tokyo J. Math. *5* (1982), 157-169.
11. D. Sarason, *Algebras of functions on the unit circle*, Bull. Amer. Math. Soc. *79* (1973), 286-299.
12. ——— *Function theory on the unit circle*, Lecture Note (Virginia Polytec. Inst. and State Univ., Virginia, 1978).
13. D. Stegenga, *Sums of invariant subspaces*, Pacific J. Math. *70* (1977), 567-584.
14. T. Wolff, *Two algebras of bounded functions*, Duke Math. J. *49* (1982), 321-328.

Kanagawa University,
Yokohama, Japan;
Kanagawa Dental College,
Yokosuka, Japan