ELLiptic Integrals in Terms of Legendre Polynomials

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1. Consider the elliptic integral of the first kind

\[ u = \text{sn}^{-1} x = \int_0^x \frac{dx}{\sqrt{(1-x^2)(1-k'^2x^2)}} \]

or, alternatively,

\[ u = F(\phi, k) = \int_0^\phi \frac{d\phi}{\sqrt{1-k^2\sin^2 \phi}} \]

where \( x = \sin \phi \). It is customary in the theory of elliptic integrals to let \( k = \sin \alpha \), \( k' = \cos \alpha \), so that \( k'^2 + k^2 = 1 \). For convenience we shall also introduce the parameter

\[ \lambda = k'^2 - k^2. \]

The substitution \( x = \sin \phi \) is equivalent to

\[ x = \frac{2v}{1+v^2} \]

where \( v = \tan \frac{\phi}{2} \). By means of this substitution we find that

\[ u = 2\int_0^v \frac{dv}{\sqrt{1+2\lambda v^2 + v^4}}. \]

We have

\[ (1 + 2\lambda v^2 + v^4)^{-\frac{1}{4}} = \sum_{n=0}^{\infty} P_n(-\lambda)v^{2n} = \sum_{n=0}^{\infty} (-1)^n P_n(\lambda)v^{2n}, \]

where \( P_n(\lambda) \) denotes the Legendre polynomial of order \( n \). This series converges uniformly with respect to \( v \) and \( \lambda \) when \( |\lambda| \leq a, |v| \leq b \), and \( a \) and \( b \) being positive constants such that \( 2ab^2 + b^4 \leq 1 - \delta (\delta > 0) \).

Hence it follows that

\[ u = 2\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} P_n(\lambda) v^{2n+1} \]

or

\[ u = 2\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} P_n(\lambda) \tan^{2n+1} \frac{\phi}{2}. \]

For \( \phi = \frac{\pi}{2} \) we find formally

\[ K = 2\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} P_n(\lambda). \]
a formula which we shall establish rigorously in (§ 2). If, in addition, \( \lambda = 1 \), the formula above gives

\[
\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n + 1} = 1 - \frac{1}{3} + \frac{1}{5} \ldots .
\]

Thus (3) is an extension of the Gregory-Leibnitz series.

2. Series expansions for \( K, K', E \) and \( E' \).

It is well known that the complete elliptic integral of the first kind \( K \) can be expressed in terms of Gauss's hypergeometric function as follows

\[
K = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}, 1 ; k^2\right). \quad \text{...........................................(4)}
\]

On the other hand, the Legendre function of degree \( n \) of the first kind is defined by means of the equation

\[
P_n(\lambda) = F\left(-n, n+1, 1 ; \frac{1-\lambda}{2}\right)
\]

which, for \( n = -\frac{1}{2} \), gives

\[
P_{-1/2}(\lambda) = F\left(\frac{1}{2}, \frac{1}{2}, 1 ; \frac{1-\lambda}{2}\right) \quad \text{...........................................(5)}
\]

Since \( \frac{1-\lambda}{2} = k^2 \), from (4) and (5) we obtain

\[
K = \frac{\pi}{2} P_{-1/2}(\lambda). \quad \text{...........................................(6)}
\]

Similarly,

\[
K' = \frac{\pi}{2} P_{-1/2}(-\lambda). \quad \text{...........................................(7)}
\]

Also, since

\[
E = \frac{\pi}{2} F\left(-\frac{1}{2}, \frac{1}{2}, 1 ; k^2\right)
\]

and

\[
E' = \frac{\pi}{2} F\left(-\frac{1}{2}, \frac{1}{2}, 1 ; k'^2\right)
\]

it can be shown that

\[
E = \frac{\pi}{4} \left[P_{-1/2}(\lambda) + P_{1/2}(\lambda)\right] \quad \text{...........................................(8)}
\]

and that

\[
E' = \frac{\pi}{4} \left[P_{-1/2}(-\lambda) + P_{1/2}(-\lambda)\right]. \quad \text{...........................................(9)}
\]

Now, if we expand the Legendre function \( P_{-1/2}(\lambda) \) in a series of Legendre polynomials we obtain

\[
P_{-1/2}(\lambda) = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n + 1} P_n(\lambda). \quad \text{...........................................(10)}
\]

Similarly, by expanding \( P_{1/2}(\lambda) \) in a series of Legendre polynomials we obtain

\[
P_{1/2}(\lambda) = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n + 1}{(2n+1)(2n+3)} P_n(\lambda). \quad \text{...........................................(11)}
\]
Whence,
\begin{equation}
K = 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} P_n(\lambda), \quad \text{.................................(12)}
\end{equation}
\begin{equation}
K' = 2 \sum_{n=0}^{\infty} \frac{1}{2n+1} P_n(\lambda), \quad \text{.................................(13)}
\end{equation}
\begin{equation}
E = 4 \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n-1)(2n+1)(2n+3)} P_n(\lambda), \quad \text{.................................(14)}
\end{equation}
\begin{equation}
E' = -4 \sum_{n=0}^{\infty} \frac{1}{(2n-1)(2n+1)(2n+3)} P_n(\lambda). \quad \text{.................................(15)}
\end{equation}

For \( \lambda = 1 \) it follows from (14) that
\[ \frac{\pi}{8} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n-1)(2n+1)(2n+3)}. \]

3. Series expansions for \( \text{cn}^{-1}x \) and for \( \varphi^{-1}(x) \).

By letting \( x = (1 - v^2)/(1 + v^2) \) it can be shown that
\[ \text{cn}^{-1}x = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(k'^2 + k^2x^2)}} = 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} P_n(\lambda) \left( \frac{1-x}{1+x} \right)^{n+1}. \]

Also, by letting \( x = e_1 + \gamma^2 v^2 \), where \( 3e_1 = 2\lambda\gamma^2, g_2 = 12e_1^2 - 4\gamma^4 \), it is easy to verify that
\[ \varphi^{-1}(x) = \int_0^x \frac{dz}{\sqrt{4x^2 - g_2 z - g_3}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \gamma^{2n} P_n(\lambda) (x - e_1)^{-(n+1)}. \]

4. Expansion for the elliptic integral of the second kind.

Since
\[ E(\phi, k) = \frac{k^2 \sin \phi \cos \phi}{\sqrt{1 - k^2 \sin^2 \phi}} + k'^2 \left[ k \frac{dF}{dk} + F \right] \]
we obtain, after introducing the parameter \( \lambda \), and making use of (2):
\[ E(\phi, \lambda) = \frac{(1-\lambda) \tan \phi}{\sqrt{4 + 2(1+\lambda) \tan^2 \phi}} + \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left[ 2(\lambda^2 - 1)P_n'(\lambda) + (\lambda + 1)P_n(\lambda) \right] \tan^{n+1} \frac{\phi}{2}, \]
or, alternatively,
\[ E(\phi, \lambda) = \frac{(1-\lambda) \tan \phi}{\sqrt{4 + 2(1+\lambda) \tan^2 \phi}} + \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left[ ((2n+1)\lambda + 1)P_n(\lambda) - 2nP_{n-1}(\lambda) \right] \tan^{n+1} \frac{\phi}{2}, \]
since
\[ (\lambda^2 - 1)P_n'(\lambda) = n\lambda P_n(\lambda) - nP_{n-1}(\lambda). \]

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