

A RATIO LIMIT THEOREM FOR CONTRACTION PROJECTIONS AND APPLICATIONS

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1. The similarities between martingale convergence theory and pointwise ergodic theory are now well known [5, 7, 9, 10]. In [5] the similarity between the proofs of the Hopf–Dunford–Schwartz individual ergodic theorem and the martingale convergence theorem is systematically exploited to produce very general “maximal ergodic” inequalities for certain sequences of contractions on L^1 -spaces. A different approach by Rota [10] and Rao [9] leads to a unified convergence theory for martingales and Abel limits. Bishop [1] has produced “upcrossing” inequalities which yield both the Chacon–Ornstein theorem [4] and the martingale convergence theorem.

In the present note a “ratio limit” theorem, or analogue of the Chacon–Ornstein result is proved for certain sequences of positive contractions on L^1 . The proof is based upon Hopf’s proof [4] of the Chacon–Ornstein theorem and is in the spirit of the analogy exploited in [5]. The set where convergence obtains is characterised in the case when the contractions are also projections. This is true when we are dealing with decreasing martingales, so that the present theorem can be regarded as a “ratio” martingale convergence theorem.

In the last section the ratio limit theorem is applied to sequences of contractive projections satisfying a uniform integrability condition, to prove an extension of the Doob martingale convergence theorem. The treatment here was suggested by a paper of Ito [6] on the individual ergodic theorem, though the method of proof here differs substantially from his methods.

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2. A “maximal ergodic” lemma. Let $(\Omega, \mathcal{B}, \mu)$ be a complete σ -finite measure space and let $L^1 = L^1(\Omega, \beta, \mu)$ denote the space of (equivalence classes of) μ -integrable real-valued functions on Ω . A contraction T is a linear operator $T: L^1 \rightarrow L^1$ with $\|T\|_1 \leq 1$. A sequence of contractions $\{T_n\}_{n \geq 0}$ on L^1 is called *decreasing* if $T_{n+1}T_n = T_{n+1}$ for all $n \geq 0$. By induction, this implies that $T_m T_n = T_m$ for all $m > n$. In particular, if $f \in L^1$ and $T_n f = E(f | \mathcal{B}_n)$, where the σ -subalgebras \mathcal{B}_n of \mathcal{B} are decreasing with n , $\{T_n\}$ is a decreasing sequence of contractive projections on L^1 . For decreasing sequences of contractions on L^1 , we prove an inequality that is analogous to the maximal ergodic theorem [4, 5] both in statement and proof. (Throughout, $B' = \Omega \setminus B$, for any $B \subseteq \Omega$.)

LEMMA 1. Let $\{T_n\}_{n \geq 0}$ be a decreasing sequence of positive contractions on L^1 , and, for $f \in L^1$, let $A_f = \{\omega: \limsup_{n \rightarrow \infty} (T_n f)(\omega) > 0\}$. Let $B \subset A_f$ be measurable. Then

$$\int_B f d\mu + \int_{B'} f^+ d\mu \geq 0.$$

Proof. As $h_n^- \downarrow 0$ a.e. in A_f , choose n_0 such that $\int h_n^- d\mu < \varepsilon$ and let $h_\varepsilon = h_{n_0}$; (ii) follows from equation (2), since

$$\int (f - h_n) d\mu = \sum_0^{n-1} \int (1 - T_i) h_i^+ d\mu \geq 0$$

and (iv) follows from equation (3).

3. The ratio limit theorem. Analogously to the proof of the Chacon–Ornstein theorem in [4], the following theorem now follows from Lemma 1.

THEOREM 1. *If $\{T_n\}_{n \geq 0}$ is a decreasing sequence of positive contractions in L^1 , then, for all $f \in L^1$, $p \in L^1_+$, $\lim_{n \rightarrow \infty} [(T_n f)(\omega)/(T_n p)(\omega)]$ exists and is finite a.e. (μ) in*

$$B_p = \{\omega : \liminf_{n \rightarrow \infty} (T_n p)(\omega) > 0\}.$$

Proof. Note that we may assume $f \geq 0$.

We first apply Lemma 1 to $(f - ap)$ for $a > 0$ to find $\int_E (f - ap) d\mu + \int_E (f - ap)^+ d\mu \geq 0$, where

$$E = B_p \cap \{\omega : p(\omega) > 0\} \cap \left\{ \omega : \limsup_{n \rightarrow \infty} \frac{(T_n f)(\omega)}{(T_n p)(\omega)} = +\infty \right\}.$$

Hence $\int (f - ap)^+ d\mu \geq \int_E (f - ap)^- d\mu$. Letting $a \rightarrow \infty$, we obtain $\mu(E) = 0$. So

$$\limsup_{n \rightarrow \infty} [(T_n f)(\omega)/(T_n p)(\omega)] < \infty.$$

To show that, for any pair $a, b \in \mathcal{R}$, $a < b$, the set

$$A = B_p \cap \{\omega : p(\omega) > 0\} \cap \left\{ \omega : \lim_{n \rightarrow \infty} \frac{(T_n f)(\omega)}{(T_n p)(\omega)} < a < b < \overline{\lim}_{n \rightarrow \infty} \frac{(T_n f)(\omega)}{(T_n p)(\omega)} \right\}$$

is μ -null, we apply the inductive procedure of [4, Lemma 5], using the function h whose existence is guaranteed by Corollary 1. To see that $T_n h/T_n p$ has the same limit behaviour as $T_n f/T_n p$, we note that $T_n(h - bp) = T_n(f - bp)$ for all $n > n_0(\varepsilon)$. Similarly, we find f' as in [4] and again $T_n f' = T_n h = T_n f$ for sufficiently large n . Now $\mu(A) = 0$ follows exactly as in [4]. This procedure, when applied to all pairs of rationals a, b with $a < b$ gives convergence a.e. (μ) to a finite limit on the set $B_p \cap \{\omega : p(\omega) > 0\}$. But, for any $n \geq 0$, let $f_1 = T_n f$, $p_1 = T_n p$; then $T_m f_1/T_m p_1 = T_m f/T_m p$ whenever $m > n$ and $T_m p > 0$. So $\lim_{m \rightarrow \infty} (T_m f_1/T_m p_1) = \lim_{m \rightarrow \infty} (T_m f/T_m p)$ exists and is finite a.e. (μ) on the set $B_p \cap \{\omega : (T_n p)(\omega) > 0\}$. As $T_m p > 0$ infinitely often on B_p , we have proved the convergence on all of B_p .

4. Contractive projections. We proceed to identify the set B_p in the case when $\{T_n\}_{n \geq 0}$ is a decreasing sequence of pairwise commuting contractive projections. The set of all contractive projections on L^1 is partially ordered by the relation \leq defined by $P \leq Q$ if and only if $PQ = P = QP$ and a decreasing sequence of projections is decreasing in this ordering.

Define the *support* of a measurable function f by $S(f) = \{\omega : f(\omega) \neq 0\}$. Then we have:

LEMMA 2. Let $f, g \in L^1_+$ and let T be a positive linear operator on L^1 . Then $S(f) \subseteq S(g) \Rightarrow S(Tf) \subseteq S(Tg)$.

Proof. Put $f_n = f \wedge ng$. Then $Tf_n \leq nTg$, as T is positive and linear. Hence $S(Tf_n) \subseteq S(Tg)$. By [8, Lemma 3.9], $S(f) \subseteq S(g)$ if and only if $f_n \uparrow f$. Hence $Tf_n \uparrow Tf$ and so $S(Tf) = \bigcup_n S(Tf_n) \subseteq S(Tg)$.

To apply Lemma 2 to a decreasing sequence $\{P_n\}$ of contractive projections, let $h \in L^1$, $h > 0$ a.e. (μ) . By Lemma 2, if $g \in R(P_n)$, then $S(g) \subseteq S(P_n h)$. But $P_{n+1} h = P_n P_{n+1} h \in R(P_n)$; so $S(P_{n+1} h) \subseteq S(P_n h)$ and so $\{S(P_n h)\}$ is a decreasing sequence of sets. As $P_n \geq 0$, $S(P_n h) = \{\omega : (P_n h)(\omega) > 0\}$ and so Theorem 1 becomes

THEOREM 2. If $\{P_n\}$ is a decreasing sequence of positive contractive projections on L^1 and if $f \in L^1$, $h \in L^1$ and $h > 0$ a.e. (μ) , then $\lim_{n \rightarrow \infty} [(P_n f)(\omega)/(P_n h)(\omega)]$ exists and is finite a.e. (μ) in $\bigcap_{n=1}^{\infty} S(P_n h)$.

5. Application: Almost everywhere convergence for projections. In this section we follow the lead of Ito [6] and use the ratio theorem to prove the following theorem.

THEOREM 3. Let $(\Omega, \mathcal{B}, \mu)$ be a probability space and let $\{P_n\}$ be a decreasing sequence of pairwise commuting positive contractive projections on $L^1(\Omega, \mathcal{B}, \mu)$. If the sequence $\{P_n 1\}$, where 1 denotes the function taking constant value 1 μ -a.e., is uniformly integrable, then the sequence $\{P_n f\}$ converges μ -a.e. for any $f \in L^1$.

We need two lemmas. The first asserts the existence of a positive contraction P on L^1 , commuting with the P_n and which is their limit in the strong operator topology. The statement and proof are close to that in [4], but we give the proof for the sake of completeness.

LEMMA 3. There is a positive contraction P on L^1 such that $P_n P = P P_n$ for all n , and $P_n \rightarrow P$ in the strong operator topology.

Proof. As $\{P_n 1\}$ is a uniformly integrable sequence, the set $\{P_n 1\}$ is weakly sequentially compact, hence conditionally weakly compact, by the Eberlein-Šmulian theorem [3, p. 430]. As in [6, Lemmas 1 and 2], for any $B \in \mathcal{B}$, the set $\{P_n \chi_B\}$ is conditionally weakly compact, since $0 \leq \chi_B \leq 1$ implies $0 \leq P_n \chi_B \leq P_n 1$ for all n . So the set of all $f \in L^1$ such that $\{P_n f\}$ is conditionally weakly compact is total in L^1 . But this implies that it is all of L^1 , since a set of this kind is a closed linear manifold [3, Chapter II, 3.30].

Fix $f \in L^1$. Let $0(f)$ be the (strongly) closed convex hull of $\{P_n f\}$. $0(f)$ is weakly compact, and is mapped into itself by $\{P_n\}$; hence we may apply the Markov-Kakutani fixed point theorem [3, Chapter V, 10.6] to deduce the existence of a fixed point $g \in L^1$. Now for given $\varepsilon > 0$ there is a convex combination $\sum_{i=1}^k \lambda_i P_{n_i} f$ such that $\|g - \sum_{i=1}^k \lambda_i P_{n_i} f\| < \varepsilon$. Choose

$m > \max_{i=1, \dots, k} n_i$; then $g = P_m g$ and $P_m \left(\sum_{i=1}^k \lambda_i P_{n_i} f \right) = P_m f$. So

$$\|g - P_m f\| = \left\| P_m \left(g - \sum_{i=1}^k \lambda_i P_{n_i} f \right) \right\| \leq \left\| g - \sum_{i=1}^k \lambda_i P_{n_i} f \right\| < \varepsilon.$$

Therefore $P_m f \rightarrow g$ strongly. As the limit is unique, we may define $P : L^1 \rightarrow L^1$ by $Pf = g$ for each $f \in L^1$, with f and g related as above. The other statements follow at once.

The following results can be found in [11]. Let Q be a positive contractive projection on L^1 and let $\Sigma = \{K \subset \Omega : K = S(f), f \geq 0, f \in \mathcal{R}(Q)\}$. Then

- (i) Σ is a σ -ring;
- (ii) if $f \in L^1$ has support contained in a member of Σ , then, for all $K \in \Sigma_0$, $Q(\chi_K f) = \chi_K Qf$;
- (iii) each $f \in \mathcal{R}(Q)$ is Σ -measurable.

Now let $\Sigma_n = \{K \subseteq \Omega : K = S(f), f \geq 0, f \in \mathcal{R}(P_n)\}$ and define $h_n = P_n 1$, $h = P 1$, $A_n = \Omega \setminus S(h_n)$ and $A = \Omega \setminus S(h)$. Then Lemma 2 implies that $\{A_n\}$ is an increasing sequence of sets, and $\bigcup_n A_n \subseteq A$.

LEMMA 4. $A = \bigcup_n A_n$.

Proof. We need to show that $A \subseteq \bigcup_n A_n$, or, equivalently, $\bigcap_n A'_n \subseteq A'$. Since h_n is Σ_n -measurable, there is an increasing sequence $\{g_{m,n}\}_m$ of Σ_n -simple functions converging to h_n . Hence $P_n(g_{m,n} \chi_{A'_n}) = g_{m,n} P_n \chi_{A'_n}$ and, by continuity, $P_n(h_n \chi_{A'_n}) = h_n P_n \chi_{A'_n}$. But $P_n h_n = P_n^2 1 = P_n 1 = h_n$; so $h_n = P_n h_n = P_n(h_n \chi_{A'_n}) = h_n P_n \chi_{A'_n}$. Hence, on A'_n , $P_n \chi_{A'_n} = 1$ a.e. (μ). But, on A_n , $0 \leq P_n \chi_{A'_n} \leq P_n 1 = h_n = 0$. Hence $0 \leq P_n \chi_{A'_n} = \chi_{A'_n} \leq h_n$ a.e. (μ). Therefore $h = \lim_n h_n \geq \chi_{\bigcap_n A'_n} \geq 0$, since the A'_n form a decreasing sequence. Hence, on A , $\chi_{\bigcap_n A'_n} = 0$ a.e. (μ), or $\chi_{\bigcap_n A'_n} \leq \chi_A$. This means that $\bigcap_n A'_n \subseteq A'$, i.e., that $A \subseteq \bigcup_n A_n$.

Proof of Theorem 3. We may assume that $f \geq 0$. By the ratio limit theorem, $\lim_n (P_n f / P_n h)$ exists a.e. (μ) on A' . But h is invariant under all the P_n ; so the limit is $(1/h) \lim_n P_n f$ on A' . Hence $\lim_n P_n f$ exists a.e. (μ) on A' . On A , $\lim_n (P_n f)(\omega) = 0$ a.e. (μ); we know that $\{\omega : (P_n f)(\omega) = 0\} \supseteq A_n$, by Lemma 2. The $\{A_n\}$ are increasing to A ; so $\lim_n (P_n f)(\omega) = 0$ a.e. (μ) on A .

REMARKS.

1. For functions with support contained in $S(h)$, P acts as conditional expectation operator with respect to Σ .
2. The assumption of commutativity is essential in Theorem 3, though the ratio theorem holds without it. In fact, Lemma 3 does not generalise. Consider $\Omega = [0, 1]$, \mathcal{B} = the Borel

field, $\mu =$ Lebesgue measure. Let E_n be the set of all $\omega \in \Omega$ with binary expansion $\omega = .x_1x_2\dots x_n\dots$ such that $x_n = 0$. Then let $P_n f = 2(\int_0^1 f d\mu)_{x_{E_n}}$. $\{P_n f\}$ clearly does not converge strongly for any $f \in L^1$, but, since $\{P_n 1\}$ converges weakly to 1, it is uniformly integrable. This example is due to Dona Strauss (oral communication).

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