Canad. Math. Bull. Vol. **56** (3), 2013 pp. 534–543
http://dx.doi.org/10.4153/CMB-2011-184-7

A Cohomological Property of π-invariant Elements

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Abstract. Let A be a Banach algebra and let $\pi\colon A \to \mathscr{L}(H)$ be a continuous representation of A on a separable Hilbert space H with $\dim H = \mathfrak{m}$. Let π_{ij} be the coordinate functions of π with respect to an orthonormal basis and suppose that for each $1 \le j \le \mathfrak{m}$, $C_j = \sum_{i=1}^{\mathfrak{m}} \|\pi_{ij}\|_{A^*} < \infty$ and $\sup_j C_j < \infty$. Under these conditions, we call an element $\overline{\Phi} \in l^\infty(\mathfrak{m}, A^{**})$ left π-invariant if $a \cdot \overline{\Phi} = {}^t\pi(a)\overline{\Phi}$ for all $a \in A$. In this paper we prove a link between the existence of left π-invariant elements and the vanishing of certain Hochschild cohomology groups of A. Our results extend an earlier result by Lau on F-algebras and recent results of Kaniuth, Lau, Pym, and and the second author in the special case where $\pi\colon A \to \mathbf{C}$ is a non-zero character on A.

1 Introduction

Let A be a Banach algebra and let A^{**} be its double dual Banach algebra equipped with the left Arens product $\square$ (*cf.* Arens [1] or Dales [3]). For continuous finite-dimensional representations $\pi\colon A \to M_n(\mathbf{C})$, the (left) π-invariant elements of $l^\infty(n, A^{**})$ were recently studied by the authors in connection with the characterization of finite-dimensional left ideals in the dual of left introverted subspaces of A^* (Filali–Monfared [5]). In this paper we extend the concept of π-invariance to continuous representations on Hilbert spaces, and we prove an interesting link between the existence of π-invariant elements and the vanishing of certain Hochschild cohomology groups of A. In our proofs we use a modified version of a technique first employed by Lau in his study of F-algebras [7]. We remark that our results generalize an earlier result by Lau on F-algebras [7, Theorem 4.1] and recent results of Kaniuth–Lau–Pym [6, Theorem 1.1], and of the second author [9, Theorem 2.3], which were obtained for the special case where $\pi\colon A \to \mathbf{C}$ is a non-zero character on A.

Throughout this paper, we assume that A is a Banach algebra and H is a separable Hilbert space and $\dim H = \mathfrak{m}$ ($1 \le \mathfrak{m} \le \aleph_0$). We shall assume that H is equipped with an orthonormal basis $(e_i)_{1 \le i \le \mathfrak{m}}$, and, unless otherwise stated, $\mathscr{L}(H)$, the space of all continuous linear operators on H, is equipped with its weak operator topology (here and elsewhere in the paper, if $\mathfrak{m} = \aleph_0$, then in an equality such as $1 \le i \le \mathfrak{m}$, we shall always assume that $i < \mathfrak{m}$). If $\pi\colon A \to \mathscr{L}(H)$ is a continuous representation, for each $1 \le i, j \le \mathfrak{m}$, we define the coordinate function $\pi_{ij} \in A^*$ by $\pi_{ij}(a) = (\pi(a)e_j|e_i)$, $(a \in A)$. We denote the canonical extension of π to A^{**} by $\widetilde{\pi}$, so that $\widetilde{\pi}$ is a w^*-continuous representation of A^{**} on H and for every $\Phi \in A^{**}$, $(\widetilde{\pi}(\Phi)e_j|e_i) = \langle \Phi, \pi_{ij}\rangle$ (*cf.* Filali–Monfared [5]). The projection map on the (i, j)-th coordinate is defined by $\mathrm{pr}_{ij}\colon \mathscr{L}(H) \to \mathbf{C}$, $T \mapsto (Te_j|e_i)$.

Received by the editors March 27, 2011; revised June 7, 2011.
Published electronically February 3, 2012.
The second author was partially supported by NSERC
AMS subject classification: **46H15**, 46H25, 13N15.
Keywords: Banach algebras, π-invariance, derivations, representations.

We recall that if E is a Banach left [right] A-module, then its dual space E^* is a Banach right [left] A-module in a canonical way:

$$\langle \lambda \cdot a, x\rangle = \langle \lambda, a \cdot x\rangle, \quad \big[\langle a \cdot \lambda, x\rangle = \langle \lambda, x \cdot a\rangle\big] \quad (a \in A, x \in E, \lambda \in E^*).$$

In particular, it follows that both A^* and A^{**} have canonical Banach A-bimodule structures induced from the multiplication of A.

2 π-invariance and Derivations

We use the following lemma repeatedly in the rest of this paper.

Lemma 2.1 *If $\pi\colon A \to \mathscr{L}(H)$ is a continuous representation, then for all $1 \leq i, j \leq \mathfrak{m}$, and all $a \in A$, we have*

$$a \cdot \pi_{ij} = w\text{-}\sum_{k=1}^{\mathfrak{m}} \pi_{ik}\pi_{kj}(a), \quad \pi_{ij} \cdot a = w\text{-}\sum_{k=1}^{\mathfrak{m}} \pi_{ik}(a)\pi_{kj}, \tag{2.1}$$

where $w\text{-}\sum$ means the convergence is in the weak topology $\sigma(A^*, A^{**})$.

Proof Let $\Phi \in A^{**}$, then

$$\begin{aligned}\langle \Phi, a \cdot \pi_{ij}\rangle = \langle \Phi \cdot a, \pi_{ij}\rangle &= \big(\widetilde{\pi}(\Phi \cdot a)e_j|e_i\big) = \big(\widetilde{\pi}(\Phi)\pi(a)e_j|e_i\big)\\ &= \Big(\widetilde{\pi}(\Phi)\Big(\sum_{k=1}^{\mathfrak{m}}(\pi(a)e_j|e_k)e_k\Big)\,|e_i\Big)\\ &= \sum_{k=1}^{\mathfrak{m}}\big(\pi(a)e_j|e_k\big)\big(\widetilde{\pi}(\Phi)e_k|e_i\big)\\ &= \sum_{k=1}^{\mathfrak{m}} \pi_{kj}(a)\langle \Phi, \pi_{ik}\rangle.\end{aligned}$$

The second statement is proved similarly. ∎

For our purpose of introducing well-defined Banach A-module operations using a continuous representation $\pi\colon A \to \mathscr{L}(H)$, we need to assume the following finiteness conditions on π:

$$C_j = \sum_{i=1}^{\mathfrak{m}} \|\pi_{ij}\|_{A^*} < \infty, \quad C = \sup_{1\leq j\leq \mathfrak{m}} C_j < \infty. \tag{2.2}$$

Of course continuous finite-dimensional representations automatically satisfy the conditions in (2.2). However, one can easily find examples of infinite-dimensional representations satisfying these conditions. For example, if $(\pi_n)_{n=1}^{\infty}$ is a sequence of finite-dimensional representations of A such that for some $M, N > 0$ we have $\|\pi_n\| \leq M$ and $\dim(\pi_n) \leq N$ for all n, then the direct sum representation $\bigoplus_{n=1}^{\infty} \pi_n$

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satisfies the conditions in (2.2) (with $C \le MN$). If π satisfies the conditions in (2.2) and if E is an arbitrary Banach right A-module, then we can turn the Banach space $l^1(\mathfrak{m}, E)$ into a Banach A-bimodule with the following operations in which $a \in A$ and $\overline{x} \in l^1(\mathfrak{m}, E)$:

$$(2.3) \qquad (a \cdot \overline{x})(i) = \big(\pi(a)\overline{x}\big)(i) = \sum_{j=1}^{\mathfrak{m}} \pi_{ij}(a)\overline{x}(j) \qquad (1 \le i \le \mathfrak{m}),$$

$$(2.4) \qquad (\overline{x} \cdot a)(i) = \overline{x}(i) \cdot a \qquad (1 \le i \le \mathfrak{m}).$$

It follows from (2.2) that the module operations in (2.3) and (2.4) are well defined on $l^1(\mathfrak{m}, E)$ and the convergence in (2.3) is absolute convergence. The space $l^1(\mathfrak{m}, E)^* = l^\infty(\mathfrak{m}, E^*)$ inherits the dual Banach A-bimodule structure given by

$$(2.5) \qquad (a \cdot \overline{\varphi})(i) = a \cdot \overline{\varphi}(i) \qquad (1 \le i \le \mathfrak{m}),$$

$$(2.6) \qquad (\overline{\varphi} \cdot a)(i) = \big({}^t\pi(a)\,\overline{\varphi}\big)(i) = \sum_{j=1}^{\mathfrak{m}} \pi_{ji}(a)\overline{\varphi}(j) \qquad (1 \le i \le \mathfrak{m}),$$

where $\overline{\varphi} \in l^\infty(\mathfrak{m}, E^*)$, ${}^t\pi(a)$ is the transpose of the infinite matrix $\pi(a) = (\pi_{ij}(a))$, and the convergence in (2.6) is absolute convergence.

Definition 2.2 Let A be a Banach algebra and $\pi\colon A \to \mathscr{L}(H)$ be a continuous representation satisfying the conditions in (2.2). Suppose that A^* is equipped with its natural Banach right A-module action so that $l^\infty(\mathfrak{m}, A^{**})$ is a Banach left A-module with the canonical operation $(a \cdot \overline{\Phi})(i) = a \cdot \overline{\Phi}(i)$ $(1 \le i \le \mathfrak{m})$. We call an element $\overline{\Phi} \in l^\infty(\mathfrak{m}, A^{**})$ left π-invariant if for every $a \in A$, we have $a \cdot \overline{\Phi} = {}^t\pi(a)\overline{\Phi}$, or equivalently,

$$(a \cdot \overline{\Phi})(i) = \sum_{k=1}^{\mathfrak{m}} \pi_{ki}(a)\overline{\Phi}(k) \qquad (1 \le i \le \mathfrak{m}),$$

where the series is absolutely convergent. A right π-invariant element can be defined analogously (see the discussion prior to Theorem 2.10).

Definition 2.3 Let $\pi\colon A \to \mathscr{L}(H)$ be a continuous representation of A on a separable Hilbert space H. Then π is said to satisfy the strong Hahn–Banach separation property on the column $1 \le j \le \mathfrak{m}$, if there exists $\epsilon = \epsilon(j) > 0$ such that for every $1 \le i \le \mathfrak{m}$, $d(\pi_{ij}, E_{ij}) \ge \epsilon$, where

$$E_{ij} = \overline{\mathrm{lin}}^{\|\cdot\|}\{\pi_{kj}\colon k \ne i\} \subset A^*$$

and $d(\pi_{ij}, E_{ij})$ is the distance between π_{ij} and the subspace E_{ij}. The strong Hahn–Banach separation property on the rows of π is defined similarly.

Lemma 2.4 *Let $\pi\colon A \to \mathscr{L}(H)$ be a continuous, topologically irreducible representation.*

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(i) *For each $1 \le i \le \mathfrak{m}$, the set $\{\pi_{ij} : 1 \le j \le \mathfrak{m}\}$ is linearly independent in A^*.*
(ii) *For each $1 \le j \le \mathfrak{m}$, the set $\{\pi_{ij} : 1 \le i \le \mathfrak{m}\}$ is linearly independent in A^*.*

Proof (i) Suppose that $\sum_{j=1}^n \alpha_j \pi_{ik_j} = 0$, where $\alpha_j \in \mathbf{C}$ and $1 \le k_j \le \mathfrak{m}$, for $j = 1, \ldots, n$. Let $x \in H$ be defined by $x = \sum_{j=1}^n \alpha_j e_{k_j}$. Then for every $a \in A$,

$$\big(\pi(a)x|e_i\big) = \sum_{j=1}^n \alpha_j \big(\pi(a)e_{k_j}|e_i\big) = \sum_{j=1}^n \alpha_j \pi_{ik_j}(a) = 0.$$

It follows that x is not a cyclic vector for π, and hence irreducibility of π implies that $x = 0$. Therefore $\alpha_1 = \cdots = \alpha_n = 0$, which is what we needed to show.

(ii) Suppose that $\sum_{i=1}^n \alpha_i \pi_{k_i j} = 0$, where $\alpha_i \in \mathbf{C}$ and $1 \le k_i \le \mathfrak{m}$, for $i = 1, \ldots, n$. Let $x \in H$ be defined by $x = \sum_{i=1}^n \overline{\alpha_i} e_{k_i}$. Then for every $a \in A$,

$$\big(\pi(a)e_j|x\big) = \sum_{i=1}^n \alpha_i \big(\pi(a)e_j|e_{k_i}\big) = \sum_{i=1}^n \alpha_i \pi_{k_i j}(a) = 0.$$

Since π is irreducible, $\{\pi(a)e_j : a \in A\}$ is dense in H, and hence $x = 0$. Therefore $\alpha_1 = \cdots = \alpha_n = 0$, proving that $\{\pi_{ij} : 1 \le i \le \mathfrak{m}\}$ is linearly independent in A^*. ■

The following is an immediate corollary of the above lemma. We recall that by a result of Johnson, algebraically irreducible representations are automatically continuous (*cf.* Bonsall–Duncan [2, Theorem III.25.7]).

Corollary 2.5 *All finite-dimensional irreducible representations of a Banach algebra satisfy the strong Hahn–Banach separation property on each of its columns and rows.*

The following theorem is the main result of this paper.

Theorem 2.6 *Let A be a Banach algebra and let $\pi : A \to \mathscr{L}(H)$ be a continuous representation such that π satisfies the conditions in (2.2) as well as the strong Hahn–Banach separation property on the column j, for some $1 \le j \le \mathfrak{m}$. Suppose that for every Banach right A-module E for which $l^1(\mathfrak{m}, E)$ is equipped with the bimodule structure defined in (2.3)–(2.4), all continuous derivations $d : A \to l^1(\mathfrak{m}, E)^*$ are inner. In that case, there exists a left π-invariant element $\overline{\Phi}_j \in l^\infty(\mathfrak{m}, A^{**})$ such that*

$$\langle \overline{\Phi}_j(i), \pi_{kj} \rangle = \delta_{ik} \quad (1 \le i, k \le \mathfrak{m}).$$

Proof Let us define

$$F = \overline{\mathrm{lin}}^{w}\{\pi_{ij} : 1 \le i \le \mathfrak{m}\} = \overline{\mathrm{lin}}^{\|\cdot\|}\{\pi_{ij} : 1 \le i \le \mathfrak{m}\} \subset A^*.$$

The equality of the two closures in the weak and norm topologies of A^* follows from the Mazur's theorem (*cf.* Dunford–Schwartz [4, Theorem V.3.13]). We assume that A^* has its natural Banach right A-module structure, and we turn $l^1(\mathfrak{m}, A^*)$ and its dual $l^\infty(\mathfrak{m}, A^{**})$ into Banach A-bimodules with the operations defined in

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(2.3)–(2.4) and (2.5)–(2.6), respectively. It follows from (2.1) that F is a Banach right A-submodule of A^*. If we let $E = A^*/F$ be the quotient Banach right A-module, then the space $l^1(\mathfrak{m}, E) \cong l^1(\mathfrak{m}, A^*)/l^1(\mathfrak{m}, F)$ inherits Banach A-bimodule operations from $l^1(\mathfrak{m}, A^*)$, and hence $l^1(\mathfrak{m}, E)$ and $l^1(\mathfrak{m}, E)^* = l^\infty(\mathfrak{m}, E^*)$ can also be equipped with the Banach bimodule multiplications defined in (2.3)–(2.4) and (2.5)–(2.6), respectively.

By our assumption, π satisfies the strong Hahn–Banach separation property on the column j, and hence by the Hahn–Banach theorem, for each $1 \leq i \leq \mathfrak{m}$, we can choose $\Psi_{ij} \in A^{**}$ such that

$$\langle \Psi_{ij}, \pi_{kj} \rangle = \delta_{ik}, \quad \|\Psi_{ij}\| = d(\pi_{ij}, E_{ij})^{-1} \leq 1/\epsilon(j), \tag{2.7}$$

where $E_{ij} = \overline{\mathrm{lin}}^{\|\cdot\|}\{\pi_{kj} \colon k \neq i\} \subset A^*$. Let $\overline{\Psi}_j \in l^\infty(\mathfrak{m}, A^{**})$ be defined by $\overline{\Psi}_j(i) = \Psi_{ij}$, $1 \leq i \leq \mathfrak{m}$. We now show that the image of the inner derivation

$$\delta_{\overline{\Psi}_j} \colon A \longrightarrow l^\infty(\mathfrak{m}, A^{**}), \quad a \mapsto a \cdot \overline{\Psi}_j - \overline{\Psi}_j \cdot a = a \cdot \overline{\Psi}_j - {}^t\pi(a)\overline{\Psi}_j,$$

is a subset of $l^1(\mathfrak{m}, E)^* = l^1(\mathfrak{m}, F)^\circ$ (the polar of $l^1(\mathfrak{m}, F)$ in $l^\infty(\mathfrak{m}, A^{**})$). Let $\overline{\lambda} \in l^1(\mathfrak{m}, F)$ be such that for each $1 \leq l \leq \mathfrak{m}$,

$$\overline{\lambda}(l) = \sum_{i \in F_l} \alpha_{il}\pi_{ij} \quad (\alpha_{il} \in \mathbf{C}),$$

where F_l is a finite subset of $\mathbf{N}$. Such elements are norm dense in $l^1(\mathfrak{m}, F)$, therefore it suffices to show that for each $a \in A$, $\delta_{\overline{\Psi}_j}(a)$ annihilates such an element. For each $a \in A$ and $1 \leq l \leq \mathfrak{m}$, using (2.1), we have:

$$\begin{aligned} \overline{\lambda}(l) \cdot a = \sum_{i \in F_l} \alpha_{il}\pi_{ij} \cdot a &= \sum_{i \in F_l} \alpha_{il} \Big(w\text{-}\sum_{k=1}^{\mathfrak{m}} \pi_{ik}(a)\pi_{kj} \Big) \\ &= w\text{-}\sum_{k=1}^{\mathfrak{m}} \Big(\sum_{i \in F_l} \alpha_{il}\pi_{ik}(a) \Big) \pi_{kj}. \end{aligned} \tag{2.8}$$

From the identities

$$(\overline{\lambda} \cdot a)(l) = \overline{\lambda}(l) \cdot a \quad \text{and} \quad (a \cdot \overline{\lambda})(l) = \sum_{j=1}^{\mathfrak{m}} \pi_{lj}(a)\overline{\lambda}(j),$$

we find that

$$\begin{aligned} \big\langle \delta_{\overline{\Psi}_j}(a), \overline{\lambda} \big\rangle &= \langle a \cdot \overline{\Psi}_j - \overline{\Psi}_j \cdot a, \overline{\lambda} \rangle = \langle \overline{\Psi}_j, \overline{\lambda} \cdot a \rangle - \langle \overline{\Psi}_j, a \cdot \overline{\lambda} \rangle \\ &= \sum_{l=1}^{\mathfrak{m}} \big\langle \overline{\Psi}_j(l), \overline{\lambda}(l) \cdot a \big\rangle - \sum_{i=1}^{\mathfrak{m}} \Big\langle \overline{\Psi}_j(i), \sum_{l=1}^{\mathfrak{m}} \pi_{il}(a)\overline{\lambda}(l) \Big\rangle. \end{aligned}$$

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Using (2.8), we have

$$\langle \delta_{\overline{\Psi}_j}(a), \overline{\lambda} \rangle = \sum_{l=1}^{\mathfrak{m}} \Big\langle \overline{\Psi}_j(l), w\text{-}\sum_{k=1}^{\mathfrak{m}} \Big(\sum_{i\in F_l} \alpha_{il}\pi_{ik}(a) \Big) \pi_{kj} \Big\rangle - \sum_{i=1}^{\mathfrak{m}} \sum_{l=1}^{\mathfrak{m}} \pi_{il}(a) \big\langle \overline{\Psi}_j(i), \overline{\lambda}(l) \big\rangle$$
$$= \sum_{l=1}^{\mathfrak{m}} \sum_{k=1}^{\mathfrak{m}} \sum_{i\in F_l} \alpha_{il}\pi_{ik}(a) \big\langle \overline{\Psi}_j(l), \pi_{kj} \big\rangle - \sum_{i=1}^{\mathfrak{m}} \sum_{l=1}^{\mathfrak{m}} \sum_{k\in F_l} \alpha_{kl}\pi_{il}(a) \big\langle \overline{\Psi}_j(i), \pi_{kj} \big\rangle.$$

Finally, by (2.7), we find that

$$\big\langle \delta_{\overline{\Psi}_j}(a), \overline{\lambda} \big\rangle = \sum_{l=1}^{\mathfrak{m}} \sum_{i\in F_l} \alpha_{il}\pi_{il}(a) - \sum_{i\in F_l} \sum_{l=1}^{\mathfrak{m}} \alpha_{il}\pi_{il}(a) = 0.$$

Thus $\delta_{\overline{\Psi}_j}(a) \in l^1(\mathfrak{m}, F)^\circ = l^1(\mathfrak{m}, E)^*$ for all $a \in A$. Therefore by assumption, the continuous derivation $\delta_{\overline{\Psi}_j}\colon A \to l^1(\mathfrak{m}, E)^*$ must be inner; that is, there exists $\overline{\Psi}_j' \in l^1(\mathfrak{m}, E)^*$ such that $\delta_{\overline{\Psi}_j} = \delta_{\overline{\Psi}_j'}$ (in the special case that $F = A^*$, we have $l^1(\mathfrak{m}, F)^\circ = l^1(\mathfrak{m}, E)^* = \{0\}$ and $\overline{\Psi}_j' = 0$). Thus, for every $a \in A$, we have $\delta_{\overline{\Psi}_j}(a) = \delta_{\overline{\Psi}_j'}(a)$, which is equivalent to

$$a \cdot (\overline{\Psi}_j - \overline{\Psi}_j') = {}^t\pi(a)(\overline{\Psi}_j - \overline{\Psi}_j') \qquad (a \in A).$$

If we define $\overline{\Phi}_j = \overline{\Psi}_j - \overline{\Psi}_j'$, then for every $a \in A$, we have

$$a \cdot \overline{\Phi}_j = {}^t\pi(a)\overline{\Phi}_j \quad \text{and} \quad \langle \overline{\Phi}_j(i), \pi_{kj} \rangle = \langle \overline{\Psi}_j(i), \pi_{kj} \rangle = \delta_{ik},$$

completing the proof of the theorem. ∎

We can prove the following partial converse to Theorem 2.6.

Theorem 2.7 *Let A be a Banach algebra and let $\pi\colon A \to \mathscr{L}(H)$ be a continuous representation satisfying the conditions in (2.2). Suppose that for each $1 \le j \le \mathfrak{m}$, there exists a left π-invariant element $\overline{\Phi}_j \in l^\infty(\mathfrak{m}, A^{**})$ such that*

(i) $\sup_j \|\overline{\Phi}_j\|_\infty < \infty$;

(ii) $\langle \overline{\Phi}_j(i), \pi_{kj} \rangle = \delta_{ik} \quad (1 \le i, k \le \mathfrak{m})$.

If E is a Banach right A-module and $l^1(\mathfrak{m}, E)$ is equipped with the Banach A-bimodule structure defined in (2.3)–(2.4), then every continuous derivation

$$d = (d_i)_{1\le i\le \mathfrak{m}}\colon A \longrightarrow l^1(\mathfrak{m}, E)^*$$

is inner, provided that

$$d_i^{**}(\overline{\Phi}_i(i)) = d_j^{**}(\overline{\Phi}_j(i)) \quad (1 \le i, j \le \mathfrak{m}), \tag{2.9}$$

where $d_j^{**}\colon A^{**} \to E^{***}$ is the double adjoint of d_j.

Proof Let us define $\overline{F} \in l^{\infty}(\mathfrak{m}, E^*)$ by

$$\overline{F}(j) = d_j^{**}(\overline{\Phi}_j(j))|_E \in E^* \quad (1 \le j \le \mathfrak{m}).$$

We claim that $d = \delta_{-\overline{F}}$. To show this, let $a \in A$ and $\overline{x} \in l^1(\mathfrak{m}, E)$, then

$$\begin{aligned}
(2.10)\quad \langle \delta_{\overline{F}}(a), \overline{x} \rangle &= \langle a \cdot \overline{F} - \overline{F} \cdot a, \overline{x} \rangle = \langle \overline{F}, \overline{x} \cdot a - \pi(a)\overline{x} \rangle \\
&= \sum_{i=1}^{\mathfrak{m}} \Big\langle d_i^{**}(\overline{\Phi}_i(i)), \overline{x}(i) \cdot a \Big\rangle - \sum_{i=1}^{\mathfrak{m}} \Big\langle d_i^{**}(\overline{\Phi}_i(i)), \sum_{j=1}^{\mathfrak{m}} \pi_{ij}(a)\overline{x}(j) \Big\rangle \\
&= \sum_{i=1}^{\mathfrak{m}} \Big\langle \overline{\Phi}_i(i), d_i^{*}(\overline{x}(i) \cdot a) \Big\rangle - \sum_{i=1}^{\mathfrak{m}} \sum_{j=1}^{\mathfrak{m}} \pi_{ij}(a) \Big\langle d_i^{**}(\overline{\Phi}_i(i)), \overline{x}(j) \Big\rangle .
\end{aligned}$$

However,

$$(2.11)\qquad d_i^*(x \cdot a) = d_i^*(x) \cdot a - \sum_{j=1}^{\mathfrak{m}} \langle d_j(a), x \rangle \pi_{ji}, \qquad (a \in A,\ x \in E).$$

In fact, for all $b \in A$, we can write:

$$\begin{aligned}
\langle d_i^*(x \cdot a), b \rangle &= \Big\langle x, \mathrm{pr}_i(a \cdot d(b)) \Big\rangle = \Big\langle x, \mathrm{pr}_i(d(ab) - {}^t\pi(b)d(a)) \Big\rangle \\
&= \Big\langle x, d_i(ab) \Big\rangle - \Big\langle x, \sum_{j=1}^{\mathfrak{m}} \pi_{ji}(b) d_j(a) \Big\rangle \\
&= \Big\langle d_i^*(x) \cdot a, b \Big\rangle - \Big\langle \sum_{j=1}^{\mathfrak{m}} \Big\langle d_j(a), x \Big\rangle \pi_{ji}, b \Big\rangle \\
&= \Big\langle d_i^*(x) \cdot a - \sum_{j=1}^{\mathfrak{m}} \Big\langle d_j(a), x \Big\rangle \pi_{ji}, b \Big\rangle ,
\end{aligned}$$

which proves (2.11). Now, if in (2.10), we substitute the value of $d_i^*(\overline{x}(i) \cdot a)$ from (2.11), and subsequently use condition (ii) in our theorem, we obtain

$$\begin{aligned}
(2.12)\quad \Big\langle \delta_{\overline{F}}(a), \overline{x} \Big\rangle &= \sum_{i=1}^{\mathfrak{m}} \Big\langle \overline{\Phi}_i(i), d_i^*(\overline{x}(i)) \cdot a \Big\rangle - \sum_{i=1}^{\mathfrak{m}} \sum_{j=1}^{\mathfrak{m}} \Big\langle d_j(a), \overline{x}(i) \Big\rangle \Big\langle \overline{\Phi}_i(i), \pi_{ji} \Big\rangle \\
&\quad - \sum_{i=1}^{\mathfrak{m}} \sum_{j=1}^{\mathfrak{m}} \pi_{ij}(a) \Big\langle d_i^{**}(\overline{\Phi}_i(i)), \overline{x}(j) \Big\rangle \\
&= \sum_{i=1}^{\mathfrak{m}} \Big\langle \mathrm{pr}_i(a \cdot \overline{\Phi}_i), d_i^*(\overline{x}(i)) \Big\rangle - \Big\langle d(a), \overline{x} \Big\rangle \\
&\quad - \sum_{i=1}^{\mathfrak{m}} \sum_{j=1}^{\mathfrak{m}} \pi_{ij}(a) \Big\langle d_i^{**}(\overline{\Phi}_i(i)), \overline{x}(j) \Big\rangle .
\end{aligned}$$

Since by the definition of left π-invariance we have $a \cdot \overline{\Phi}_i = {}^t\pi(a)\overline{\Phi}_i$, we can also write

$$(2.13)\qquad \sum_{i=1}^{\mathfrak{m}} \big\langle \mathrm{pr}_i(a \cdot \overline{\Phi}_i), d_i^*(\overline{x}(i))\big\rangle = \sum_{i=1}^{\mathfrak{m}} \Big\langle \sum_{j=1}^{\mathfrak{m}} \pi_{ji}(a)\overline{\Phi}_i(j), d_i^*(\overline{x}(i))\Big\rangle$$
$$= \sum_{i=1}^{\mathfrak{m}} \sum_{j=1}^{\mathfrak{m}} \pi_{ji}(a)\big\langle d_i^{**}(\overline{\Phi}_i(j)), \overline{x}(i)\big\rangle$$
$$= \sum_{j=1}^{\mathfrak{m}} \sum_{i=1}^{\mathfrak{m}} \pi_{ij}(a)\big\langle d_j^{**}(\overline{\Phi}_j(i)), \overline{x}(j)\big\rangle .$$

Therefore, using (2.13), we may rewrite (2.12) as

$$(2.14)\quad \langle \delta_{\overline{F}}(a), \overline{x}\rangle = \sum_{j=1}^{\mathfrak{m}} \sum_{i=1}^{\mathfrak{m}} \pi_{ij}(a)\big\langle d_j^{**}(\overline{\Phi}_j(i)), \overline{x}(j)\big\rangle - \big\langle d(a), \overline{x}\big\rangle$$
$$- \sum_{i=1}^{\mathfrak{m}} \sum_{j=1}^{\mathfrak{m}} \pi_{ij}(a)\big\langle d_i^{**}(\overline{\Phi}_i(i)), \overline{x}(j)\big\rangle .$$

By (2.9), the first and the third terms in the right-hand side of (2.14) cancel each other, and thus we obtain

$$\langle \delta_{\overline{F}}(a), \overline{x}\rangle = -\langle d(a), \overline{x}\rangle \qquad \big(a \in A,\ \overline{x} \in l^1(\mathfrak{m}, E)\big),$$

which implies that $d = \delta_{-\overline{F}}$, completing the proof of the theorem. ■

Remark 2.8 We have been unable to remove condition (2.9) in the above theorem. Of course, this condition holds automatically when $n = 1$ and π is a non-zero character on A, but in general it seems to impose a strong restriction on d. It would be interesting to know whether (2.9) can be removed or substituted by a weaker assumption.

Remark 2.9 As was kindly pointed out to us by the referee, it would be interesting to study the relation of right π-invariant elements of $l^\infty(n, A^{**})$ and fixed point properties similar to those in Lau and Zhang [8].

In the following, we shall formulate Theorems 2.6 and 2.7 for right π-invariant elements of $l^\infty(\mathfrak{m}, A^{**})$. The proofs that are similar to those given above are omitted. We remark that using the same methods given in Filali–Monfared [5], right π-invariant elements of $l^\infty(n, A^{**})$ can be used to characterize finite-dimensional *right* ideals in A^{**} equipped with the right Arens product $\diamond$. More generally, if X is a faithful, right introverted subspace of A^* for which X^* is equipped with the induced right Arens product $\diamond$, then [5, Lemma 2.2] remains true without any change, and the analogues of [5, Lemma 2.4, Theorems 2.7 and 2.8] can be readily formulated and proved for right π-invariants and finite-dimensional right ideals in X^*.

Let A be a Banach algebra and let $\pi\colon A \to \mathscr{L}(H)$ be a continuous representation satisfying the conditions

$$C_i = \sum_{j=1}^{\mathfrak{m}} \|\pi_{ij}\|_{A^*} < \infty, \quad C = \sup_{1\le i\le \mathfrak{m}} C_i < \infty. \tag{2.15}$$

If E is an arbitrary Banach left A-module, then we can turn $l^1(\mathfrak{m}, E)$ into a Banach A-bimodule with the following operations:

$$(a\cdot \overline{x})(i) = a\cdot \overline{x}(i), \quad (\overline{x}\cdot a)(i) = \big({}^t\pi(a)\,\overline{x}\big)(i) = \sum_{k=1}^{\mathfrak{m}} \pi_{ki}(a)\overline{x}(k), \tag{2.16}$$

where $1 \le i \le \mathfrak{m}$. The dual space $l^\infty(\mathfrak{m}, E^*) = l^1(\mathfrak{m}, E)^*$ inherits the canonical Banach A-bimodule structure, given by:

$$(a\cdot \overline{\varphi})(i) = (\pi(a)\overline{\varphi})(i) = \sum_{k=1}^{\mathfrak{m}} \pi_{ik}(a)\overline{\varphi}(k), \quad (\overline{\varphi}\cdot a)(i) = \overline{\varphi}(i)\cdot a,$$

for every $1 \le i \le \mathfrak{m}$.

Suppose that π satisfies the conditions in (2.15) and that A^* is equipped with its natural Banach left A-module action so that $l^\infty(\mathfrak{m}, A^{**})$ is a Banach right A-module with the canonical action $(\overline{\Phi}\cdot a)(i) = \overline{\Phi}(i)\cdot a$ $(1 \le i \le \mathfrak{m})$. We call an element $\overline{\Phi} \in l^\infty(\mathfrak{m}, A^{**})$ right π-invariant if for every $a \in A$, we have $\overline{\Phi}\cdot a = \pi(a)\overline{\Phi}$.

Theorem 2.10 *Let A be a Banach algebra and let $\pi\colon A \to \mathscr{L}(H)$ be a continuous representation such that π satisfies the conditions in (2.15) and the strong Hahn–Banach separation property on the row i, for some $1 \le i \le \mathfrak{m}$. Suppose that for every Banach left A-module E for which $l^1(\mathfrak{m}, E)$ is equipped with the bimodule structure defined in (2.16), all continuous derivations $d\colon A \to l^1(\mathfrak{m}, E)^*$ are inner. In that case, there exists a right π-invariant element $\overline{\Phi}_i \in l^\infty(\mathfrak{m}, A^{**})$ such that*

$$\langle \overline{\Phi}_i(j), \pi_{ik}\rangle = \delta_{jk} \quad (1 \le j, k \le \mathfrak{m}).$$

Theorem 2.11 *Let A be a Banach algebra and let $\pi\colon A \to \mathscr{L}(H)$ be a continuous representation satisfying the conditions in (2.15). Suppose that for each $1 \le i \le \mathfrak{m}$, there exists a right π-invariant element $\overline{\Phi}_i \in l^\infty(\mathfrak{m}, A^{**})$ such that*

(i) $\sup_i \|\overline{\Phi}_i\|_\infty < \infty$;
(ii) $\langle \overline{\Phi}_i(j), \pi_{ik}\rangle = \delta_{jk} \quad (1 \le j, k \le \mathfrak{m})$.

If E is a Banach left A-module and $l^1(\mathfrak{m}, E)$ is equipped with the Banach A-bimodule structure defined in (2.16), then every continuous derivation $d = (d_i)_{1\le i\le \mathfrak{m}}\colon A \to l^1(\mathfrak{m}, E)^*$ is inner, provided that

$$d_i^{**}\big(\overline{\Phi}_i(i)\big) = d_j^{**}\big(\overline{\Phi}_j(i)\big) \quad (1 \le i, j \le \mathfrak{m}),$$

where $d_j^{**}\colon A^{**} \to E^{***}$ is the double adjoint of d_j.

Acknowledgment The authors would like to thank the referee for his/her careful reading of the paper and for useful suggestions.

https://doi.org/10.4153/CMB-2011-184-7 Published online by Cambridge University Press

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