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H-SIMPLE H-MODULE ALGEBRAS

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ABSTRACT. Let A be an H-simple commutative H-module algebra, with $A^H = k$ and $\dim_k H \leq \dim_k A < \infty$. We show that this implies that $A \# H$ is isomorphic to $M_n(k)$, a central simple algebra. We apply this to characterize certain group graded algebras, algebras acted upon by a group as automorphisms, or by a nilpotent Lie algebra as derivations.

Let k be a field, H a Hopf algebra over k and A an H-module algebra over k. The smash product of A by H, $A \# H$, is the generalization of the classical crossed product $A * G$, where G is a group of automorphisms of A. If A is a Galois extension of k, then it is well known that $A * G$ is simple [1]. This result was extended by Sweedler to smash products of field extensions A, by certain "Galois" Hopf algebras [6, Thm. 10.1.1]. In this note we extend Sweedler's result from fields to H-simple commutative algebras, using a similar method of proof.

We apply the theorem to characterize finite dimensional commutative algebras A acted upon by either:

(a) $H = kG$, G acting as automorphisms on A.

(b) $H = (kG)^*$, that is, A is G-graded.

(c) $H = u(L)$, the restricted enveloping algebra of a nilpotent Lie algebra L, acting as derivations on A.

where $\dim_k H \leq \dim_k A$, and $A^H = k$.

Let us recall that

$$A^H = \{a \in A \mid h \cdot a = \epsilon(h)a, \text{ all } h \in H\},$$

and that A is H-simple (H-semiprime) if A contains no nonzero H-stable ideals (H-stable nilpotent ideals).

THEOREM. *Let H be a finite-dimensional Hopf algebra (not necessarily with an antipode), and A, a commutative n-dimensional, H-simple, H-module algebra such that $A^H = k$ and $\dim_k H \leq n$, then*:

(a) $\dim_k H = \dim_k A = n$

(b) $A \# H$ is a central simple algebra (isomorphic to $M_n(k)$).

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PROOF. Consider $A \otimes_k A$ and $A \# H$ as A-modules via left multiplication. Furthermore, A is a left $A \# H$ module by: $(a \# h)\cdot b = a(h\cdot b)$. Define f: $A \otimes A \to \mathrm{Hom}_A(A \# H, A)$ by:

$$(f(a \otimes b))(c \# h) = ac(h\cdot b) = a((c \# h)\cdot b).$$

Since A is commutative, $f(a \otimes b)$ is indeed an A-module map. Now,

$$\dim_k (\mathrm{Hom}_{,A} \times (A \# H, A)) = \dim_k H \dim_k A \leq n^2,$$

and $\dim_k(A \otimes A) = n^2$. Thus, if we prove that f is injective, it follows that $\dim_k H = n$ (thus proving part (a)), and that f is surjective.

Knowing that f is surjective implies that A is a faithful $A \# H$-module. To see this, let $0 \neq x = \Sigma c_i \# h_i \in A \# H$, where $\{h_i\}$ is a k-basis for H; and $c_m \neq 0$ for some m. Let g be the projection map to the m-th coefficient, then $g \in \mathrm{Hom}_A(A \# H, A)$, and $g(x) \neq 0$. Since f is surjective, $g = f(\Sigma_j a_j \otimes b_j)$, for some $\Sigma a_j \otimes b_j \in A \otimes A$, and,

$$0 \neq g(x) = f\Big(\sum_j a_j \otimes b_j\Big)\Big(\sum_i c_i \# h_i\Big).$$

Thus, for some t

$$f(a_t \otimes b_t)\Big(\sum_i c_i \# h_i\Big) \neq 0.$$

but this implies that $a_t(x\cdot b_t) = a_t(\Sigma c_i \# h_i)\cdot b_t) \neq 0$. We have shown that if $x \neq 0$ then $x\cdot A \neq 0$, that is, A is a faithful $A \# H$-module. This in turn implies that $A \# H$ is isomorphic to a subalgebra of $\mathrm{End}_k(A)$. But now, since $\dim_k A \# H = \dim_k \mathrm{End}_k A = n^2$, we deduce that $A \# H \cong \mathrm{End}_k(A) \cong M_n(k)$, proving part (b).

We prove now that f is injective. Let $\Sigma a_i \otimes b_i \in \mathrm{Ker} f$, let $s \in A$ and $g \in H$, we show that $\Sigma a_i \otimes s(g\cdot b_i) \in \mathrm{Ker} f$. Let $c \# h \in A \# H$ then,

$$\begin{aligned}\Big[f\Big(\sum a_i \otimes s(g\cdot b_i)\Big)\Big](c \# h) &= \sum a_i ch\cdot(sg\cdot b_i) \\ &= \sum_i \sum_{(h)} a_i c(h_1\cdot s)(h_2 g\cdot b_i) \\ &= \sum_{(h)} \Big[f\Big(\sum_i a_i \otimes b_i\Big)\Big](ch_1\cdot s \# h_2 g) = 0.\end{aligned}$$

Hence $\Sigma a_i \otimes s(g\cdot b_i) \in \mathrm{Ker} f$. If $\mathrm{Ker} f \neq 0$, let $0 \neq \Sigma_{i=1}^r a_i \otimes b_i \in \mathrm{Ker} f$ of shortest length with $\{a_i\}$ linearly independent. Let

$$I = \Big\{y \in A \,\Big|\, a_1 \otimes y + \sum_{j=2}^r a_j \otimes m_j \in \mathrm{Ker} f\Big\}$$

By the above, I is a nonzero H-stable ideal of A, which now equals A, since A is H-simple. Thus, we may assume that $b_1 = 1$. Let $h \in H$, then by the above,

$\Sigma_{i=1}^{r} a_i \otimes (h - \epsilon(h))\cdot b_i \in \operatorname{Ker} f$. This element is of shorter length since $b_1 = 1$, and thus $(h - \epsilon(h))\cdot b_i = 0$. So, $(h - \epsilon(h))\cdot b_i = 0$ for each i and each $h \in H$. That is, $b_i \in A^H = k$ for all i. Thus $r = 1$, and $0 \neq a \otimes 1 \in \operatorname{Ker} f$. This is impossible however, for

$$0 = [f(a \otimes 1)](1 \# 1) = a \neq 0.$$

Hence $\operatorname{Ker} f = 0$, and we are done.

As an application we get:

COROLLARY 2. *Let G be a finite group acting as automorphisms on a $|G|$-torsion free, commutative, n-dimensional algebra with $A^G = k$, and $|G| \leq n$. Then the following are equivalent:*

(a) A is kG-semiprime
(b) A is kG-simple
(c) $|G| = n$ and $A \# kG \cong M_n(k)$
(d) A is semisimple
(e) A is a direct sum of m fields, $m \leq n$.

PROOF. (a) $\rightarrow$ (b) If $I \neq 0$ is a G-stable ideal then $I \cap A^G \neq 0$ by [3], but since $A^G = k$, I must equal A. (b) $\rightarrow$ (c) by the theorem. (c) $\rightarrow$ (d) the Jacobson radical of A, $J(A)$, is a G-stable ideal, and hence $J(A) \# kG$ is an ideal of the simple ring $A \# kG$. Thus $J(A) = 0$. (d) $\rightarrow$ (e) since A is commutative and finite-dimensional. (e) $\rightarrow$ (a) is obvious.

If G is a finite group then A is G-graded if and only if A is a $(kG)^*$-module algebra [4]. So saying that A is graded-simple (semiprime) means it is $(kG)^*$-simple (semiprime), that is, A has non-trivial graded ideals (graded nilpotent ideals). Here $A^H = A_1$.

COROLLARY 3. *Let G be a finite group, and A a commutative, n-dimensional, G-graded algebra with $A_I = k$ and $|G| \leq n$. Then the following are equivalent:*

(a) A is graded-semiprime
(b) A is graded-simple
(c) $|G| = n$ and $A \# (kG)^* \cong M_n(k)$
(d) $A = k*G$ (the classical crossed product), and G is abelian.

PROOF. (a) $\Rightarrow$ (b) by [5] and the fact that $A_1 = k$. (b) $\Rightarrow$ (c) by the theorem. (c) $\Rightarrow$ (d). Since $A \# (kG)^*$ is simple, A is $A \# (kG)^*$—faithful (for $Ann_{A \# (kG)^*}(A)$ is an ideal of $A \# (kG)^*$). Hence, for each $g \in G$, $p_g \cdot A \neq 0$. This means that $A_g \neq 0$ for each $g \in G$. Hence by [5, Prop. 1.11] $A = k*G$. (d) $\Rightarrow$ (a) is obvious.

REMARK. $A = k*G$ is semisimple if and only if A is $|G|$-torsion free. However, no such assumption is necessary for Corollary 3 to hold (unlike corollary 2).

If L is a Lie algebra of derivations of A then $A^L = \{a \in A \mid \ell \cdot a = 0, \text{ all } \ell \in L\}$, and the Hopf algebra acting on A is $U(L)$, the universal enveloping algebra of L. If $ch(k) = p$ and L is a restricted Lie algebra, then $H = u(L)$, the restricted enveloping algebra.

COROLLARY 4. *Let $ch(k) = p$, and let L be a nilpotent m-dimensional restricted Lie algebra acting as derivations on a commutative, n-dimensional algebra with $A^L = k$ and $p^m \leq n$. Then the following are equivalent:*

(a) A is L-semiprime,
(b) A is L-simple
(c) $p^m = n$ and $A \# u(L)$ is simple.

PROOF. (a) $\Rightarrow$ (b) by [2, Cor. 1.9]. The rest follows as before.

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