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RUSSIAN DOLLS

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1. **Introduction.** In an earlier number of this Bulletin, P. Erdös [1] posed the following problem. "For each line ℓ of the plane, A_ℓ is a segment of ℓ. Show that the set $\bigcup_\ell A_\ell$ contains the sides of a triangle." One objective of this paper is to prove a strengthened version of this result in N-dimensions. As usual $\aleph_0$ denotes the cardinality of the natural numbers and c, the cardinality of the real numbers.

THEOREM 1. *For each $(N-1)$-flat π of Euclidean N-space $(N \geq 2)$, let $A_\pi \subset \pi$ be an $(N-1)$-dimensional open set. Then $X = \bigcup_\pi A_\pi$ contains the boundary of an N-simplex. In fact X contains the boundaries of c dissimilar N-simplices. Moreover these c boundaries may be chosen so that each contains a dense nest of $\aleph_0$ homothetic images of itself, all lying in X.*

Theorem 1 *is trivial if* X has non-void interior. However an example due to D. Hammond Smith shows that this need not be the case.

In [2], I attempted a solution to the original problem of Erdös, stated the first part of Theorem 1 and mentioned the example. Unfortunately this "solution" contains a fallacy: the fixed angles θ_1, θ_2 and θ_3 need not be represented in smaller and smaller neighbourhoods of Q. In developing a counterexample to show that this approach could not be repaired, I discovered an interesting companion to Theorem 1.

THEOREM 2. *In Euclidean N-space $(N \geq 2)$ it is possible to choose c different directions and, perpendicular to each of these directions, c different $(N-1)$-flats so that an $(N-1)$-ball of radius* 1 *may be lodged in each of the chosen $(N-1)$-flats in such a way that no two of these $(N-1)$-balls intersect.*

To contrast Theorem 2 with Theorem 1 note that if we take all the different directions and, perpendicular to each direction, all the different $(N-1)$-flats then we are taking every $(N-1)$-flat and Theorem 1 guarantees intersections galore, even without a uniform bound on the size of the $(N-1)$-balls.

2. **D. Hammond Smith's example**. The example which we will construct has void interior, as was mentioned in §1, and in addition it is measurable, but with arbitrarily small N-measure. If the set $X = \bigcup_\pi A_\pi$ is measurable, then it must have positive N-measure. Just fix a direction $\vec{d}$ and note that for any $(N-1)$-flat π perpendicular to $\vec{d}$, $X \cap \pi \supset A_\pi$ and therefore $X \cap \pi$ has positive

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$(N-1)$-dimensional measure. To make X measurable but with arbitrarily small N-measure we take it to be an open tubular neighbourhood of the coordinate axes which tapers sufficiently quickly as we move away from the origin. Since every $(N-1)$-flat π meets at least one of the coordinate axes, we may set $A_\pi = \pi \cap X$ and then $\bigcup_\pi A_\pi = X$.

We modify X to X' so that X' has the same N-measure as X but void interior. Let Q denote the rational numbers. Then $X_1 = X' - Q^N$ has void interior. If the $(N-1)$-flat π is such that $\dim_Q(\pi \cap Q^N) \le N-2$ then $A^1_\pi = (\pi \cap X) - (\overline{\pi \cap Q^N})$ is an $(N-1)$-dimensional open set. There are only countably many $(N-1)$-flats π such that $\dim_Q(\pi \cap Q^N) = N-1$. For the nth of these we let A'_π be any $(N-1)$-dimensional ball in $\pi_n - X$ and we set $X_2 = \bigcup_{n=1}^\infty A'_{\pi_n}$. It follows that $X' = \bigcup_\pi A'_\pi = X_1 \cup X_2$ and hence X' has the same N-measure as X but void interior.

3. **Proof of Theorem** 1. The proof depends on three lemmas. If $\vec{d}$ is a direction in N-space we write $\pi \in \vec{d}$ if π is an $(N-1)$-flat in the pencil with normal direction $\vec{d}$.

Lemma 1. *For each $(N-1)$-flat π of Euclidean N-space $(N \ge 2)$, let $A_\pi \subset \pi$ be an $(N-1)$-dimensional open set. Then for each direction $\vec{d}$ there is an N-ball $B = B(\vec{d})$ with the property that the $(N-1)$-balls $\{\pi \cap B : \pi \in \vec{d}$ and $\pi \cap B = A_\pi \cap B\}$ are dense in B.*

Proof. Let V be the $(N-1)$-dimensional subspace of $\mathbb{R}^N$ which is perpendicular to $\vec{d}$. Then the points of $\mathbb{R}^N$ have a unique representation $(u, \vec{v})$ where $u \in \mathbb{R}$ measures distance in direction $\vec{d}$ and $\vec{v} \in V$.

Let $\{\vec{v}_p\}_{p=1}^\infty$ be a sequence which is dense in V. For each pair of positive integers p and q let $U_{pq} = \{u : \exists \pi \in \vec{d}$ such that A_π contains an $(N-1)$-ball of radius $\ge 1/q$ with centre $(u, \vec{v})$ satisfying $\|\vec{v} - \vec{v}_p\| \le 1/2q\}$.

Because the $\vec{v}_p$ are dense in V, each number u belongs to some U_{pq} and it follows that $\mathbb{R} = \bigcup_{pq} U_{pq}$. Since this union is countable, the Baire Category Theorem implies that there is a pair of integers p_0, q_0 such that the closed set $\bar{U}_{p_0q_0}$ contains an interval, $[u_0 - \delta, u_0 + \delta]$.

Let $B = B(\vec{d})$ be the N-ball with centre $(u_0, \vec{v}_{p_0})$ and radius $r = \min\{1/2q_0, \delta\}$. It is clear that B has the required property.

Lemma 2. *Let B_i with centre P_i and radius r_i $(i \in I)$ be a collection of c N-balls in Euclidean N-space. Then there exists an N-ball B^* which lies in the interior of c of the given N-balls.*

Proof. For each positive integer m let $I_m = \{i \in I : r_i \ge 1/m\}$. Since $I = \bigcup_m I_m$ is a countable union while card $I = c$ it follows that one of the sets, I_{m_0} satisfies card $I_{m_0} = c$.

The set of c points $\{P_i : i \in I_{m_0}\}$ has a c-accumulation point Q. This means that every neighbourhood of Q contains c of these points.

Let B^* be the N-ball with centre Q and radius $1/2m_0$. Then B^* lies in the interior of the c balls $\{B_i: i \in I_{m_0}$ and dist $(P_i, Q) < 1/2m_0\}$

LEMMA 3. *The c directions corresponding to positions on the moment curve $\vec{d}(t) = (t, t^2, \ldots, t^N)$, $0 < t \leq 1$, have the property that any N of them are linearly independent. It follows that any $N+1$ of them can be normals to the faces of an N-simplex.*

Proof. The independence of $\vec{d}(t_1)$, $\vec{d}(t_2), \ldots, \vec{d}(t_N)$ when $t_1 < t_2 < \cdots < t_N$ is immediate from the non-vanishing of the Vandermonde determinant $V(t_1, t_2, \ldots, t_N)$.

Now the lemmas may be applied in succession to prove Theorem 1. Let $\vec{d}(t)$ be a direction from the moment curve and apply Lemma 1 to obtain an N-ball $B_t = B(\vec{d}(t))$. Apply Lemma 2 to the c N-balls B_t, $0 < t \leq 1$ to obtain an N-ball B^*. There are c directions of the form $\vec{d}(t)$ such that B^* is densely stratified by sets $\pi \cap B^* = A_\pi \cap B^*$ with $\pi \in \vec{d}(t)$. Since these directions come from the moment curve, Lemma 3 assures us that any $(N+1)$ of them can serve as normals to the faces of an N-simplex.

Let $\vec{d}_1$, $\vec{d}_2, \ldots, \vec{d}_{N+1}$ be any $N+1$ of our c special directions. Let $\pi_i \in \vec{d}_i$ $(i = 1, 2, \ldots, N+1)$ determine an N-simplex S which contains the centre Q of B^* and lies entirely inside of B^*. Then for each i $(i = 1, 2, \ldots, N+1)$ the dense set of $(N-1)$-flats $\pi \in \vec{d}_i$, which lie between Q and π_i and satisfy $\pi \cap B^* = A_\pi \cap B^*$, may be used to construct a dense nest of N-simplex boundaries homothetic to ∂S and lying in X.

4. **Proof of Theorem** 2. In dimension $N=2$, Theorem 2 reduces to the assertion that it is possible to choose c line segments of length 2 in each of c directions with no two line segments intersecting. Theorem 2 is actually equivalent to this special case because a 2-dimensional configuration may be extended into an orthogonal $(N-2)$-space without creating intersections.

We may try to build a suitable 2-dimensional configuration by considering line segments which join the point $(t - f(t), -1)$ to the point $(t + f(t), 1)$ where $f: [0, 1] \to \mathbb{R}$ is a suitable function. These line segments are of length ≥ 2 and they will be non-intersecting provided f satisfies the Lipschitz condition $|f(t_1) - f(t_2)| \leq |t_1 - t_2|$. To ensure that c directions occur a total of c times each we require that c values should be attained by f a total of c times each. The proof of Theorem 2 is completed by Lemma 4.

LEMMA 4. *There is a Lipschitz function f: $[0, 1] \to \mathbb{R}$ which assumes c different values a total of c times each.*

Proof. Let F be the set of $t \in [0, 1]$ which have an expansion $t = .t_1 t_2 t_3 \cdots$ in the scale of 3 which does not use the digit 2. F is a closed set and after we have defined f on F we will extend f to $[0, 1]$ by making it linear on the open

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intervals of $[0, 1]\backslash F$. For convenience we define $1 \in F$ and associate with it the expansion $.000\cdots$ of its fractional part.

Each t in F gives us a sequence of 0's and 1's and we begin by defining the auxiliary function $g(t) = .t_1\, t_3\, t_5 \cdots$ where $.t_1\, t_3\, t_5 \cdots$ is interpreted as a number in the scale of 10, i.e. an ordinary decimal. If we know t to $2n-1$ places in the scale of 3 then we know $g(t)$ to n places in the scale of 10. This leads to the inequality $|\Delta g| < 10^{-n}$ if $3^{-2(n+1)} < |\Delta t| \le 3^{-2n}$. For n sufficiently large ($n > 2[\log_3 \frac{10}{9}]^{-1}$) and therefore $|\Delta t|$ sufficiently small ($|\Delta t| \le 3^{-4[\log_3(10/9)]^1}$), we have $n/2(n+1) \log_3 10 > 1$ and we may rewrite the inequality for $|\Delta g|$ as

$$|\Delta g| < 10^{-n} = 3^{-n \log_3 10} = 3^{-2(n+1)\cdot[n/2(n+1)]\cdot \log_3 10} < |\Delta t|.$$

This proves that $\Delta g/\Delta t$ is bounded. It follows that there exists a number k with $0 < k < 1$ such that $f = kg$ satisfies the Lipschitz condition, $|\Delta f| < |\Delta t|$, on F.

The c values which f assumes on F are each assumed c different times because of the freedom of t in its digits $t_2, t_4, t_6, \cdots$. Moreover, the Lipschitz property of f is preserved when it is extended by linear interpolation from F to the rest of $[0, 1]$.

REFERENCES

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