Groups of infinite rank in which every subgroup is either normal or contranormal are characterised in terms of their subgroups of infinite rank.


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1. Introduction

A group $G$ is said to have finite (Prüfer) rank $r$ if every finitely generated subgroup of $G$ can be generated by at most $r$ elements, and $r$ is the least positive integer with this property; if such an $r$ does not exist, we will say that the group $G$ has infinite rank. The investigation of the influence on a (generalised) soluble group of the behaviour of its subgroups of infinite rank has been developed in a series of recent papers (see, for instance, [2–5, 7, 8]). The aim of this paper is to provide some new contributions to this topic, by considering groups $G$ in which every subgroup of infinite rank is either normal or contranormal. A subgroup $H$ of $G$ is said to be contranormal in $G$ if it is not contained in a proper normal subgroup of $G$, that is, if $H^G = G$ (see, for instance, [13]). Groups satisfying this property will be called $\mathcal{AN}_\infty$-groups, in analogy with the symbol $\mathcal{AN}$ used to denote the class of groups in which every nonnormal subgroup is contranormal. The structure of $\mathcal{AN}$-groups has been studied in [14].

We will work within the universe of strongly locally graded groups, a class of generalised soluble groups that can be defined as follows. Recall that a group $G$ is locally graded if every finitely generated nontrivial subgroup of $G$ contains a proper subgroup of finite index. Let $\mathfrak{G}$ be the class of all periodic locally graded groups, and let $\mathfrak{D}$ be the closure of $\mathfrak{G}$ by the operators $\tilde{P}, \tilde{P}, R, L$ (we use the first chapter of the monograph [12] as a general reference for definitions and properties of closure operations on group classes). It is easy to prove that any $\mathfrak{D}$-group is locally graded, any

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locally (soluble-by-finite) group is a $\mathcal{D}$-group and the class $\mathcal{D}$ is closed with respect to forming subgroups. Moreover, Černikov proved that every $\mathcal{D}$-group of finite rank contains a locally soluble subgroup of finite index. Obviously, all residually finite groups belong to $\mathcal{D}$, and hence the consideration of any free nonabelian group shows that the class $\mathcal{D}$ is not closed with respect to homomorphic images. For this reason, it is better in some cases to replace $\mathcal{D}$-groups by strongly locally graded groups, that is, groups in which every section belongs to $\mathcal{D}$. The class of strongly locally graded groups has been introduced in [5]. Most of our notation is standard and can be found in [11].

2. $\mathcal{AN}_\infty$-groups

As in many problems concerning groups of infinite rank, the existence of a proper normal subgroup of infinite rank plays a crucial role. Recall that a group $G$ is said to be a Dedekind group if all its subgroups are normal.

**Lemma 2.1.** Let $G$ be a strongly locally graded $\mathcal{AN}_\infty$-group and let $N$ be a proper normal subgroup of infinite rank of $G$. Then every subgroup of $N$ is normal in $G$.

**Proof.** Every subgroup of infinite rank of $N$ is normal in $G$ so, in particular, $N$ is a Dedekind group (see [8, Theorem C]). Let $L$ be a subgroup of finite rank of $N$. Since $N$ is nilpotent, it contains a direct product $A_1 \times A_2$ such that both the subgroups $A_1$ and $A_2$ have infinite rank and $L \cap (A_1 \times A_2) = \{1\}$ (see [10]). Clearly the subgroups $A_1$ and $A_2$ are normal in $G$. Hence the subgroups of infinite rank $LA_1$ and $LA_2$ are normal in $G$, and $L = LA_1 \cap LA_2$ is normal in $G$. $\square$

Our next lemma shows, in particular, that any strongly locally graded group of infinite rank whose proper normal subgroups have finite rank must admit a simple homomorphic image of infinite rank.

**Lemma 2.2.** Let $G$ be a strongly locally graded group. Then every proper normal subgroup of $G$ has finite rank if and only if the subgroup generated by all proper normal subgroups of $G$ has finite rank.

**Proof.** Suppose that $G$ has infinite rank but all its proper normal subgroups have finite rank. Clearly $G$ is perfect and so it is not locally nilpotent, by [1, Lemma 2.3]. Hence $G$ contains a proper normal subgroup $N$ such that $G/N$ is a simple group of infinite rank (see [5, Lemma 2.4]). Therefore $N$ has finite rank. Let $H$ be any proper normal subgroup of $G$. Since $H$ has finite rank, $HN$ also has finite rank and so it is a proper subgroup of $G$. Then $HN = N$ and it follows that $H \leq N$ so that $N$ is the subgroup generated by all proper normal subgroups of $G$. $\square$

The following result will be often used in our proofs.

**Lemma 2.3.** Let $G$ be a group containing an abelian subgroup $A$ of infinite rank and let $H$ be a subgroup of $G$ such that $H^G$ has finite rank. Then there exists a subgroup $B$ of $A$ such that $B$ has infinite rank and $H^G B$ is a proper subgroup of $G$. 

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Proof. Since $H^G$ is a proper subgroup of $G$, we can take an element $x \in G \setminus H^G$. Then $A$ contains a direct product $B \times C$ such that the subgroups $B$ and $C$ both have infinite rank and $BC \cap H^G \langle x \rangle = \{1\}$. Now

$$H^G B \cap H^G \langle x \rangle = H^G (B \cap H^G \langle x \rangle) = H^G,$$

so $x \notin H^G B$, and hence $H^G B$ is a proper subgroup of $G$. □

**Proposition 2.4.** Let $G$ be a strongly locally graded $\mathcal{AN}_\infty$-group. If $G$ contains a proper normal subgroup of infinite rank, then $G$ is an $\mathcal{AN}$-group.

**Proof.** Let $N$ be a proper normal subgroup of infinite rank of $G$. By Lemma 2.1, every subgroup of $N$ is normal in $G$ and so $N$ is a Dedekind group. Let $H$ be any subgroup of finite rank of $G$ which is not contranormal, so that $H^G$ is a proper normal subgroup of $G$. If $H^G$ has infinite rank, then every subgroup of $H^G$ is normal in $G$ (by Lemma 2.1) and so $H$ is normal in $G$. Suppose now that $H^G$ has finite rank. Since $N$ is a Dedekind group, it contains an abelian subgroup $A$ of infinite rank. By Lemma 2.3, there exists $B \leq A$ of infinite rank such that $H^G B$ is a proper normal subgroup of $G$. Therefore $H$ is normal in $G$ (by Lemma 2.1) and $G$ is an $\mathcal{AN}$-group. □

It is now easy to prove the main result of this section.

**Theorem 2.5.** Let $G$ be a locally soluble $\mathcal{AN}_\infty$-group. Then $G$ is an $\mathcal{AN}$-group.

**Proof.** Since $G$ is locally soluble, $G$ contains a proper normal subgroup of infinite rank. Therefore $G$ is an $\mathcal{AN}$-group, by Proposition 2.4. □

### 3. $\mathcal{SC}_\infty$-groups

In this section we will consider groups $G$ in which every subgroup of infinite rank is either subnormal or contranormal. Groups satisfying this property will be called $\mathcal{SC}_\infty$-groups, in analogy with the symbol $\mathcal{SC}$ used to denote the class of groups in which every nonsubnormal subgroup is contranormal. This class is a natural extension of the class of $\mathcal{AN}$-groups, where the normality is replaced by subnormality. The structure of $\mathcal{SC}$-groups has been studied in [6]. We need the following elementary property.

**Lemma 3.1.** Let $G$ be a locally (soluble-by-finite) $\mathcal{SC}_\infty$-group and let $K$ be a proper subnormal subgroup of infinite rank of $G$. Then every subgroup of infinite rank of $K$ is subnormal in $G$.

In particular, it follows that every proper subnormal subgroup of infinite rank of a $\mathcal{SC}_\infty$-group is soluble (see [9, Theorem 2]).

**Theorem 3.2.** Let $G$ be a torsion-free locally (soluble-by-finite) $\mathcal{SC}_\infty$-group. If $G$ contains a proper normal subgroup of infinite rank, then $G$ is an $\mathcal{SC}$-group.
Proof. Let $N$ be a proper normal subgroup of $G$ of infinite rank. Then $N$ is soluble, by Lemma 3.1. Let $H$ be any subgroup of $G$ of finite rank such that $H$ is not contranormal in $G$. Then $H^G$ is a proper normal subgroup of $G$. Clearly, there exists a proper subnormal subgroup $K$ of $G$, of infinite rank, which contains $H$. In fact, if $H^G$ has infinite rank, we can put $K = H^G$; if $H^G$ has finite rank, since $N$ contains an abelian subgroup $A$ of infinite rank, by Lemma 2.3 there exists $B \leq A$ of infinite rank such that $H^GB$ is a proper subnormal subgroup of $G$ and in this case we can chose $K = H^GB$. By Lemma 3.1, all subgroups of infinite rank of $K$ are subnormal in $G$ and hence $K$ is nilpotent (by [9, Theorem 3]), so that $H$ is subnormal in $G$. □

Recall that the periodic radical of a group $G$ is the largest periodic normal subgroup of $G$. Moreover, $G$ is a Baer group if all its cyclic subgroups are subnormal. The following lemma will be used to prove the last theorem of the paper.

Lemma 3.3. Let $G$ be a locally (soluble-by-finite) $SC_\infty$-group containing a proper normal subgroup $N$ of infinite rank. If the periodic radical of $G$ has infinite rank, then every subgroup of $N$ is subnormal in $G$.

Proof. By Lemma 3.1, every subgroup of infinite rank of $N$ is subnormal in $G$. So $N$ is soluble and, in particular, a Baer group (see [9, Theorem 2]). Let $H$ be any subgroup of finite rank of $N$. We can suppose that the largest periodic subgroup $K$ of $N$ has finite rank (otherwise $H$ is subnormal in $G$, by [9, Theorem 5]). Denote by $T$ the periodic radical of $G$ and consider the subgroup $NT$. If $NT$ is a proper normal subgroup of $G$, then all subgroups of infinite rank of $NT$ are subnormal in $G$ and, since $T$ has infinite rank, $H$ is subnormal in $NT$ (by [9, Theorem 5]), and so it is subnormal in $G$.

Suppose that $G = NT$. Clearly, $K$ is a periodic normal subgroup of $G$ and hence it is contained in $T$. On the other hand, $T \cap N$ is contained in $K$, so $T \cap N = K$. Hence

$$\frac{N}{T \cap N} \cong \frac{NT}{T} = \frac{G}{T}$$

is a torsion-free group and so $T$ is the set of all elements of finite order of $G$.

Now $G/T$ has infinite rank and all its subgroups of infinite rank are subnormal, so (by [9, Theorem 3]) it is nilpotent. Hence $HT$ is a proper subnormal subgroup of $G$. By Lemma 3.1, every subgroup of infinite rank of $HT$ is subnormal, but $T$ has infinite rank and so $H$ is subnormal in $HT$, by [9, Theorem 5]. Therefore $H$ is subnormal in $G$. □

Theorem 3.4. Let $G$ be a locally (soluble-by-finite) $SC_\infty$-group containing a proper normal subgroup of infinite rank. If the periodic radical of $G$ has infinite rank, then $G$ is an $SC$-group.

Proof. Let $H$ be any subgroup of $G$ of finite rank which is not contranormal in $G$. Then $H^G$ is a proper normal subgroup of $G$. If $H^G$ has infinite rank, then $H$ is subnormal in $G$ by Lemma 3.3. Suppose now that $H^G$ has finite rank. If $N$ is a proper normal
subgroup of $G$ of infinite rank, then $N$ is soluble (by Lemma 3.1), and so it contains an abelian subgroup $A$ of infinite rank. By Lemma 2.3, there exists $B \leq A$ of infinite rank such that $H^G B$ is a proper subgroup of $G$. Therefore $H^G B$ is subnormal in $G$ and, by Lemma 3.1, all its subgroups of infinite rank are subnormal in $G$, so that $H$ is subnormal in $G$, by Lemma 3.3. This completes the proof of the theorem. □

The hypotheses of Theorems 3.2 and 3.4 cannot be weakened. Kurdachenko and Smith have proved the existence of a metabelian locally nilpotent group of infinite rank such that the largest periodic subgroup has finite rank and all subgroups of infinite rank are subnormal, but there exists a nonsubnormal subgroup of finite rank (see [9, Theorem 4]). Obviously, this subgroup cannot even be contranormal.

References


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