Jordan–Chevalley Decomposition in Lie Algebras

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Abstract. We prove that if $s$ is a solvable Lie algebra of matrices over a field of characteristic 0 and $A \in s$, then the semisimple and nilpotent summands of the Jordan–Chevalley decomposition of $A$ belong to $s$ if and only if there exist $S, N \in s$, $S$ is semisimple, $N$ is nilpotent (not necessarily $[S, N] = 0$) such that $A = S + N$.

1 Introduction

All Lie algebras and representations considered in this paper are finite dimensional over a field $F$ of characteristic 0. An important question concerning a given representation $\pi: g \rightarrow gl(V)$ of a Lie algebra $g$ is (cf. [B2, Ch. VII, §5])

(∗) Does $\pi(g)$ contain the semisimple and nilpotent parts of the Jordan–Chevalley decomposition (JCD) in $gl(V)$ of $\pi(x)$ for a given $x \in g$?

For semisimple Lie algebras, this is true for any representation and this classic result is a cornerstone of the representation theory of semisimple Lie algebras (see [Hu, §6.4 and Ch. VI] or [FH, §9.3 and Ch. 14]). We are interested in the classification of indecomposable representations of certain families of non semisimple Lie algebras (see [CS2, CS3]), and an extension of the classical result to more general Lie algebras will prove useful in this endeavour. In a different direction, the recent article [Ki], studies the existence of a Jordan–Chevalley–Seligman decomposition in prime characteristic.

The question (∗) led us to study the existence and uniqueness of abstract JCDs in arbitrary Lie algebras [CS]. Recall that an element $x$ of a Lie algebra $g$ is said to have an abstract JCD if there exist unique $s, n \in g$ such that $x = s + n, [s, n] = 0$ and given any finite dimensional representation $\pi: g \rightarrow gl(V)$ the JCD of $\pi(x)$ in $gl(V)$ is $\pi(x) = \pi(s) + \pi(n)$. The Lie algebra $g$ itself is said to have an abstract JCD if every one of its elements does. The main results of [CS] are Theorems 1 and 2, and they respectively state that a Lie algebra has an abstract JCD if and only if it is perfect, and an element of a Lie algebra $g$ has an abstract JCD if and only if it belongs to $[g, g]$. These theorems, though related to question (∗), do not provide a satisfactory answer to it.

The purpose of this note is two-fold: on one hand we prove Theorem 1.1 below, which directly addresses question (∗) and allows us to derive from it [CS, Theorems 1 and 2]. On the other hand, we recently realized that there is a gap in the original

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proof of [CS, Theorems 1 and 2], since [CS, Lemma 2.1] is not true. Therefore, we leave [CS, Theorems 1 and 2] in good standing by giving a correct proof derived from Theorem 1.1.

**Theorem 1.1**  Let $s$ be a solvable Lie algebra of matrices, let $A \in s$, and assume that $A = S + N$ with $S, N \in s$, $S$ semisimple, $N$ nilpotent (we are not assuming $[S, N] = 0$). Then the semisimple and nilpotent summands of the JCD of $A$ belong to $s$.

This theorem is a consequence of the following result.

**Theorem 1.2**  Let $\mathbb{F}$ be algebraically closed. Given a square matrix $A = S + N$ with $S$ semisimple and $N$ nilpotent, let $\{S_n\}$ and $\{N_n\}$ be sequences defined inductively by

$$S_0 = S \quad \text{and} \quad N_0 = N,$$

and, if $[S_n, N_m] \neq 0$, let $(N_n)_{\lambda_n}$ be a non-zero eigenmatrix of $\text{ad}(S_n)$ corresponding to a non-zero eigenvalue $\lambda_n$ appearing in the $\text{ad}(S_n)$-decomposition of $N_n$, and let

$$S_{n+1} = S_n + (N_n)_{\lambda_n} \quad \text{and} \quad N_{n+1} = N_n - (N_n)_{\lambda_n}.$$

(The sequences depend on the choice of the non-zero eigenvalues.)

If $\{S, N\}$ generates a solvable Lie algebra $s$, then (independently of the choice of the eigenvalues)

(i) $S_n$ is semisimple, $N_n$ is nilpotent, and $S_n, N_n \in s$ for all $n$,

(ii) there is $n_0$ such that $[S_{n_0}, N_{n_0}] = 0$.

In particular, $A = S_{n_0} + N_{n_0}$ is the Jordan–Chevalley decomposition of $A$ with both components $S_{n_0}, N_{n_0} \in s$. Moreover, if $\pi : s \to \mathfrak{gl}(V)$ is a representation such that $\pi(S)$ is semisimple and $\pi(N)$ is nilpotent, then $\pi(A) = \pi(S_{n_0}) + \pi(N_{n_0})$ is the Jordan–Chevalley decomposition of $\pi(A)$.

## 2 Jordan–Chevalley Decomposition of Upper Triangular Matrices

This section is devoted to proving Theorem 1.2, and thus we assume that $\mathbb{F}$ is algebraically closed. Let $t$ denote the Lie algebra of upper triangular $n \times n$ matrices over $\mathbb{F}$, let $t' = [t, t]$, and let $s$ be a Lie subalgebra of $t$.

**Lemma 2.1**  Let $S, X, N \in s$ and assume that $\text{ad}_s(S)(N) = \lambda N$, with $\lambda \in \mathbb{F}$, and $\text{ad}_s(S)(X) = \mu X$, with $0 \neq \mu \in \mathbb{F}$ (in particular, $X \in t'$). Then

$$\exp\left( -\mu^{-1} \text{ad}_s(X) \right)(N) = \sum_{j=0}^{n-1} \frac{(-\mu)^{-j}}{j!} \text{ad}_s(X)^j(N)$$

is an eigenmatrix of $\text{ad}_s(S + X)$ of eigenvalue $\lambda$, and it belongs to $s$. In particular, $S$ is semisimple if and only if $S + X$ is semisimple.
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Proof Since $X \in t'$, we see that $\mu^{-1} \text{ad}_S(X)$ is a nilpotent derivation of $s$, so $\exp(\mu^{-1} \text{ad}_S(X)) \in \text{Aut}(s)$. In particular, $\exp(\mu^{-1} \text{ad}_S(X))(N) \in s$ and

\[
\left[ \exp \left( -\mu^{-1} \text{ad}_S(X) \right)(S), \exp \left( -\mu^{-1} \text{ad}_S(X) \right)(N) \right] = \exp \left( -\mu^{-1} \text{ad}_S(X) \right)([S, N]) = \lambda \exp \left( -\mu^{-1} \text{ad}_S(X) \right)(N).
\]

But $[S, X] = 0$ yields $\exp(\mu^{-1} \text{ad}_S(X))(S) = S + X$, so $\exp(-\mu^{-1} \text{ad}_S(X))(N)$ is an eigenmatrix of $\text{ad}_S(S + X)$ of eigenvalue $\lambda$. Consequently, if $\text{ad}_t(S)$ is semisimple then $\exp(\mu^{-1} \text{ad}_S(X))$ transforms a basis of eigenmatrices of $\text{ad}_t(S)$ into a basis of eigenmatrices of $\text{ad}_S(S + X)$.

To complete the proof it is sufficient to show that a matrix $A \in t$ is semisimple if and only if $\text{ad}_t(A)$ is semisimple. The ‘only if’ part is clear. Conversely, if $\text{ad}_t(A)$ is semisimple and $A = A_1 + A_n$ is the JCD of $A$, then $A_1, A_n \in t$ (both are polynomials in $A$), and it follows that $\text{ad}_t(A) = \text{ad}_t(A_1) + \text{ad}_t(A_n)$ is the JCD of $\text{ad}_t(A)$. By uniqueness, $\text{ad}_t(A_1) = 0$, and this implies that $A_n = 0$, since $A_n \in t'$ and the centralizer of $t$ in $t'$ is 0.

Let $S \in s$ be semisimple. Let $\Lambda$ be the set of eigenvalues of $\text{ad}_S(S)$, and for each $\lambda \in \Lambda$, let $s_\lambda \subseteq s$ be the corresponding eigenspace. Given $N \in s$, let

\[
N = \sum_{\lambda \in \Lambda} N_\lambda,
\]

where each $N_\lambda \in s_\lambda$. We refer to the above as the $\text{ad}_S(S)$-decomposition of $N$.

For $k = 0, \ldots, n - 1$, let $t_k$ be the subspace of $t$ consisting of those matrices whose non-zero entries lay only on the diagonal $(i, j)$ such that $j - i = k$. Given $N \in t$, let $d_k(N) \in t_k$ be defined so that $N = \sum_{k=0}^{n-1} d_k(N)$. We now introduce a function that will help to measure how close two matrices are to commuting with each other.

**Definition 2.2** Let $S, N \in s$, with $S$ semisimple, and let $N = \sum_{\lambda \in \Lambda} N_\lambda$ be the decomposition of $N$ as a sum of eigenmatrices of $\text{ad}_S(S)$. For $k = 0, \ldots, n - 1$, let

\[
C_{S, k}(N) = \{ \lambda \in \Lambda : \lambda \neq 0 \text{ and } d_k(N_\lambda) \neq 0 \},
\]

let $\epsilon_{S, k}(N)$ be the number of elements in $C_{S, k}(N)$ ($\epsilon_{S, 0}(N) = 0$, since $\lambda \neq 0$ implies that $N_\lambda \in t'$), and let

\[
y_S(N) = (\epsilon_{S, 1}(N), \ldots, \epsilon_{S, n-1}(N)) \in \mathbb{Z}_{\geq 0}^{n-1}.
\]

It is clear that $\epsilon_{S, k}(N) \leq \dim s$ for all $k$ and $[S, N] = 0$ if and only if $y_S(N) = (0, \ldots, 0)$.

**Lemma 2.3** Let $S, X, N \in s$ with $S$ semisimple and $\text{ad}_S(S)(X) = \mu X$, with $0 \neq \mu \in F$. Let $k_0 \geq 1$ be the lowest $k$ such that $d_k(X) \neq 0$ ($\mu \neq 0$ implies $X \in t'$ and hence $k_0 \geq 1$). Then $C_{S+X, k}(N) = C_{S, k}(N)$ for all $k \leq k_0$.

**Proof** We first point out that it follows from Lemma 2.1 that $S + X$ is semisimple, and thus it makes sense to consider $C_{S+X, k}(N)$. 

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Let

\[ N = \sum_{\lambda \in \Lambda} N_{\lambda}, \quad N_{\lambda} \in \mathfrak{s}, \]

be the \( ad(S) \)-decomposition of \( N \). Let

\[ \tilde{N}_{\lambda,0} = \exp \left( -\mu^{-1} ad_e(X) \right) (N_{\lambda}), \]

and, for \( j \geq 1 \), let \( \tilde{N}_{\lambda,j} = \frac{\mu^{-j}}{j!} ad_e(X)^j (\tilde{N}_{\lambda,0}). \)

It follows from Lemma 2.1 that \( \tilde{N}_{\lambda,j} \) is an eigenmatrix of \( ad_e(S + X) \) of eigenvalue \( \lambda + j\mu \). Since

\[ N_{\lambda} = \exp \left( -\mu^{-1} ad_e(X) \right) (\tilde{N}_{\lambda,0}) = \tilde{N}_{\lambda,0} + \tilde{N}_{\lambda,1} + \tilde{N}_{\lambda,2} + \cdots, \]

it follows that

\[ N = \sum_{\lambda \in \Lambda} \sum_{j \geq 0} \tilde{N}_{\lambda,j} = \sum_{\lambda \in \Lambda} \tilde{N}_{\lambda,0} + \sum_{\lambda \in \Lambda} \sum_{j \geq 1} \tilde{N}_{\lambda,j} \]

and this leads to the decomposition of \( N \) as a sum of eigenmatrices of \( ad_e(S + X) \) (after adding up those \( \tilde{N}_{\lambda,j} \) with the same eigenvalue).

Let \( k \leq k_0 \) (recall that \( k_0 \) is the lowest \( k \) such that \( d_k(X) \neq 0 \)). Since \( k_0 \geq 1 \), it follows that

\[ d_k(\tilde{N}_{\lambda,j}) = \begin{cases} d_k(N_{\lambda}) & \text{if } j = 0, \\ 0 & \text{if } j \geq 1. \end{cases} \]

This implies \( C_{S+X,k}(N) = C_{S,k}(N) \). \( \blacksquare \)

**Lemma 2.4** Let \( S, N \in \mathfrak{s}, \) with \( S \) semisimple, and let \( N = \sum_{\lambda \in \Lambda} N_{\lambda} \) be the \( ad(S) \)-decomposition of \( N \). Assume that there is \( \lambda_0 \in \Lambda \) with \( \lambda_0 \neq 0 \) such that \( N_{\lambda_0} \in \mathfrak{s}_{\lambda_0} \) is non-zero. Then

\[ y_{S+N_{\lambda_0}}(N - N_{\lambda_0}) < y_{S}(N) \]

in the lexicographical order. (The pair \( (S + N_{\lambda_0}, N - N_{\lambda_0}) \) is closer to commuting than the pair \( (S, N) \).)

**Proof** Let \( k_0 \) be the lowest \( k \) such that \( d_k(N_{\lambda_0}) \neq 0 \) \( (k_0 \geq 1, \text{ since } N_{\lambda_0} \in \mathfrak{t}') \). It is clear that

\[ c_{S,k}(N - N_{\lambda_0}) = \begin{cases} c_{S,k}(N) & \text{if } k < k_0, \\ c_{S,k}(N) - 1 & \text{if } k = k_0, \end{cases} \]

and thus \( y_{S}(N - N_{\lambda_0}) < y_{S}(N) \).

It follows from Lemma 2.3 that, for \( k \leq k_0 \),

\[ c_{S+N_{\lambda_0},k}(N - N_{\lambda_0}) = c_{S,k}(N - N_{\lambda_0}), \]

and this, combined with (2.1), implies \( y_{S+N_{\lambda_0}}(N - N_{\lambda_0}) < y_{S}(N) \) in the lexicographical order. \( \blacksquare \)

We are now in a position to prove Theorem 1.2.
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Proof of Theorem 1.2  Since \{S, N\} generates a solvable Lie algebra \( \mathfrak{s} \), and \( F \) is algebraically closed, it follows from Lie's Theorem that we can assume \( S, N \in \mathfrak{s} \subset \mathfrak{t} \), and since \( N \) is nilpotent, \( N \in \mathfrak{t}' \).

We will prove (i) by induction. Assume (i) is true for \( S_n \) and \( N_n \) and let us suppose that \( [S_n, N_n] \neq 0 \). Since \( \lambda_n \neq 0 \), we have \( (N_n)_{\lambda_n} \in \mathfrak{t}' \), and hence \( N_{n+1} \) is nilpotent. It follows from Lemma 2.1 that \( S_{n+1} \) is semisimple and \( S_{n+1}, N_{n+1} \in \mathfrak{s} \). This proves (i).

It follows from Lemma 2.4 that

\[
y_{S_{n+1}}(N_{n+1}) = y_{S_n + (N_n)_{\lambda_n}}(N_n - (N_n)_{\lambda_n}) < y_{S_n}(N_n)
\]

in the lexicographical order. This implies that there exists \( n_0 \) such that \( y_{S_{n_0}}(N_{n_0}) = 0 \) and hence \( [S_{n_0}, N_{n_0}] = 0 \). This proves (ii), and it is clear \( A = S_{n_0} + N_{n_0} \) is the Jordan–Chevalley decomposition of \( A \).

Finally, let \( \pi: \mathfrak{s} \to \mathfrak{gl}(V) \) be a representation such that \( \pi(S) = \pi(S_0) \) is semisimple and \( \pi(N) = \pi(N_0) \) is nilpotent. Since \( \pi \) is a representation, if \( N_n = \sum_{\lambda \in \Lambda_n} (N_n)_{\lambda} \) is the \( \text{ad}_n \)-decomposition of \( N_n \), then

\[
\pi(N_n) = \sum_{\lambda \in \Lambda_n} \pi((N_n)_{\lambda})
\]

is the \( \text{ad}_{\pi(n)}(\pi(S_n)) \)-decomposition of \( \pi(N_n) \). Therefore, assuming that \( \pi(S_n) \) is semisimple and \( \pi(N_n) \) is nilpotent, it follows, just as above, that \( \pi(S_{n+1}) \) is semisimple and \( \pi(N_{n+1}) \) is nilpotent. This implies that \( \pi(A) = \pi(S_{n_0}) + \pi(N_{n_0}) \) is the Jordan–Chevalley decomposition of \( \pi(A) \).

Proof of Theorem 1.1  Theorem 1.2 shows that Theorem 1.1 is true when \( \hat{F} \) is algebraically closed, since in this case, Lie's Theorem allows us to assume that \( \mathfrak{s} \subset \mathfrak{t} \).

In general, let \( \hat{F} \) be an algebraic closure of \( F \). Suppose \( A, S, N \in \mathfrak{s} \), where \( A = S + N, S \) is semisimple, and \( N \) is nilpotent. Let \( A = S' + N' \) be the JCD of \( A \) in \( \mathfrak{gl}(n, \hat{F}) \), as ensured in [HK, §7.5]. The minimal polynomial of \( S' \), say \( p \), is a product of distinct monic irreducible polynomials over \( \hat{F} \) [HK, §7.5]. Since \( \hat{F} \) has characteristic 0, it follows that \( p \) has distinct roots in \( \hat{F} \), whence \( S' \) is diagonalizable over \( \hat{F} \). Therefore, \( A = S' + N' \) is the JCD of \( A \) in \( \mathfrak{gl}(n, \hat{F}) \). Let \( \hat{\mathfrak{s}} \) be the \( \hat{F} \)-linear span of \( \mathfrak{s} \) in \( \mathfrak{gl}(n, \hat{F}) \). Then \( \hat{\mathfrak{s}} \) is a solvable subalgebra of \( \mathfrak{gl}(n, \hat{F}) \). As the theorem is true over \( \hat{F} \), we infer \( S', N' \in \hat{\mathfrak{s}} \). Thus, \( S', N' \in \mathfrak{gl}(n, \hat{F}) \cap \hat{\mathfrak{s}} = \mathfrak{s} \). This completes the proof of Theorem 1.1.

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Theorem 3.1  An element \( x \) of a Lie algebra \( \mathfrak{g} \) has an abstract JCD if and only if \( x \) belongs to the derived algebra \( [\mathfrak{g}, \mathfrak{g}] \), in which case the semisimple and nilpotent parts of \( x \) also belong to \( [\mathfrak{g}, \mathfrak{g}] \).

Necessity  This is clear, since any linear map from \( \mathfrak{g} \) to \( \mathfrak{gl}(V) \) such that \( \dim\pi(\mathfrak{g}) = 1 \), and \( \pi([\mathfrak{g}, \mathfrak{g}]) = 0 \) is a representation.

Sufficiency  By Ado’s theorem, we can assume that \( \mathfrak{g} \) is a Lie algebra of matrices. Fix a Levi decomposition \( \mathfrak{g} = \mathfrak{g}_s \rtimes \mathfrak{r} \) and let \( \mathfrak{n} = [\mathfrak{g}, \mathfrak{r}] \). We know that \( \mathfrak{n} \) is nilpotent (see [FH, Lemma C.20]). If \( x \in [\mathfrak{g}, \mathfrak{g}] \), then \( x = a + r \) for unique \( a \in \mathfrak{g}_s \) and \( r \in \mathfrak{n} \). If \( a = a_s + a_n \) is the JCD of the matrix \( a \), since \( \mathfrak{g}_s \) is semisimple, it follows that \( a_s, a_n \in \mathfrak{g}_s = [\mathfrak{g}_s, \mathfrak{g}_s] \) (see, for instance, [Hu, §6.4]). Let \( \mathfrak{s} = \mathcal{F}a_s \oplus \mathcal{F}a_n \oplus \mathfrak{n} \subset [\mathfrak{g}, \mathfrak{g}] \). Since \([\mathfrak{s}, \mathfrak{s}] \subset \mathfrak{n} \) and \( \mathfrak{n} \)
is nilpotent, we obtain that $s$ is a solvable Lie algebra. We now apply Theorem 1.1 to the Lie algebra $s$ with $S = a_s, N = a_n + r$. We obtain that if $x = S' + N'$ is the JCD of $x$, then $S', N' \in s \subset [g, g]$.

Finally, let $\pi : g \rightarrow \mathfrak{gl}(V)$ be a representation of $g$. Since $r \in n$, it follows that $\pi(r)$ is nilpotent (see [FH, Lemma C.19] or [B1, Ch.1, §5]). Since $g_s$ is semisimple, $\pi(S) = \pi(a_s)$ is semisimple and $\pi(a_n)$ is nilpotent. Since $s$ is solvable, it follows from Lie's Theorem that $\pi(N) = \pi(a_n + r)$ is nilpotent. It follows from Theorem 1.2 (applied over a field extension of $\mathbb{F}$) that $\pi(x) = \pi(S') + \pi(N')$ is the JCD of $\pi(x)$. ■

References


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