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A NOTE ON $U_n \times U_m$ MODULAR INVARIANTS

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ABSTRACT. We consider the rings of invariants R^G, where R is the symmetric algebra of a tensor product between two vector spaces over the field F_p and $G = U_n \times U_m$. A polynomial algebra is constructed and these invariants provide Chern classes for the modular cohomology of U_{n+m}.

1**. Introduction.** Quillen calculated the cohomology of GL_n in the non-modular case. The modular case, important mainly because of its universal role in modular group representation, is still under investigation.

In [4] Milgram and Priddy generalizing a method of L. Dickson [1] constructed a family of invariants of $\mathrm{GL}_n \times \mathrm{GL}_m$ and provided elements in the cohomology of the Unipotent group $U_{n+m}(F_p)$. Then using the transfer, they gave explicit classes in $H^*\big(\mathrm{GL}_{n+m}(F_p); F_p\big)$.

Using a slightly different approach from [4], we study the case of the Unipotent group and its relation with GL_{n+m}. In Section 2, notation and an embedding of $\mathrm{GL}_n \times \mathrm{GL}_m$ in GL_{n+m} is given. An action on the symmetric algebra of the tensor product of two vector spaces is induced from the action of the corresponding Weyl group in cohomology. Invariant polynomials of $U_n \times U_m$ are studied in Section 3 where a polynomial subalgebra of the ring of invariants is constructed. Application in the modular cohomology of the Unipotent group is discussed at the end.

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2**. Notation.** Let V and W be n and m dimensional vector spaces over F_p where p is a prime. If bases are fixed, $\langle x_1, \ldots, x_n\rangle$ and $\langle y_1, \ldots, y_m\rangle$ for V and W^* respectively, then $\langle x_{i,j} = x_i \otimes y_j \mid 1 \le i \le n, 1 \le j \le m\rangle$ is a base for $V \otimes W^*$.

Since $V \otimes W^* \equiv \mathrm{Hom}(W, V)$, we may identify $V \otimes W^*$ and $M_{n,m}$, the additive group of $n \times m$ matrices over F_p. We consider $x_{i,j}$ as the $n \times m$ matrix with 1 in the (i, j)-th position and zeros elsewhere.

Let U_n be the subgroup of upper triangular matrices with one along the main diagonal. Then $U_n \times U_m \le \mathrm{GL}_n \times \mathrm{GL}_m \le \mathrm{GL}_{nm}$.

Let $M_{n,m} \cong \begin{pmatrix} I_n & M_{n,m} \\ 0 & I_m \end{pmatrix} \le U_{n+m} \le \mathrm{GL}_{n+m}$. Then the Weyl group of $M_{n,m}$ in U_{n+m} is $U_n \times U_m$ and in GL_{n+m} is $\mathrm{GL}_n \times \mathrm{GL}_m$. An action is induced on $M_{n,m}$ by conjugation (see [4]) $(g_1, g_2)h(g_1^{-1}, g_2^{-1}) = g_1 h g_2^{-1}$.

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The action of $\mathrm{GL}_n \times \mathrm{GL}_m$ on $M_{n,m}$ above is given by $(g,h)M := gMh^{-1}$.

Since $M_{n,m} \equiv F_p^{nm}$ as vector spaces, GL_{nm} acts on $M_{n,m}$ naturally:

$$Ax_{i,j} = \sum_{t=1}^{n}\sum_{s=1}^{m} a_{(i-1)m+j,(t-1)m+s}x_{t,s} \tag{1}$$

Here $A = (a_{(i-1)m+j,(t-1)m+s}) \in \mathrm{GL}_{nm}$ and $x_{i,j}$ is a base element for $M_{n,m}$. Hence $\mathrm{GL}_n \times \mathrm{GL}_m \leq \mathrm{GL}_{nm}$ and we consider this inclusion as follows: Let $g = (a_{ij})$, $h = (b_{ij})$, and $M = (m_{ij})$, then

$$gMh = \Big(c_{ij} = \sum_{t=1}^{n}\sum_{s=1}^{m} a_{it}m_{ts}b_{sj}\Big). \tag{2}$$

Explicitly: $(g,h) := (d_{ij}) \in \mathrm{GL}_{nm}$ where $d_{ij} = a_{[\frac{i}{m}][\frac{j}{m}]'}b_{(i \bmod m)'(j \bmod m)'}$. Here $[\frac{i}{m}]' = \begin{cases} [\frac{i}{m}]+1, & i \not\equiv 0 \bmod m \\ [\frac{i}{m}], & i = 0 \bmod m \end{cases}$ and $(i \bmod m)' = \begin{cases} i \bmod m, & i \not\equiv 0 \bmod m \\ m, & i = 0 \bmod m \end{cases}$.

Namely, $(g,h) = \begin{pmatrix} a_{11}h & \cdots & a_{1n}h \\ & \vdots & \\ a_{n1}h & \cdots & a_{nn}h \end{pmatrix}$.

Let $S(M_{n,m})$ be the symmetric algebra on $M_{n,m}$, then the action above is extended to $S(M_{n,m})$ via the diagonal action. We are interested in studying the invariant rings: $S(M_{n,m})^{U_n \times U_m}$ and $S(M_{n,m})^{\mathrm{GL}_n \times \mathrm{GL}_m}$.

We note here that since $M_{n,m} \equiv F_p^{nm}$ and under this isomorphism $x_{i,j}$ is identified with the dual $(e_{(i-1)m+j})^* := z_{(i-1)m+j}$ of the basis element $e_{(i-1)m+j}$, the rings of invariants above and the isomorphic ones $F_p[z_1,\ldots,z_{nm}]^{U_n \times U_m}$ and $F_p[z_1,\ldots,z_{nm}]^{\mathrm{GL}_n \times \mathrm{GL}_m}$ will be interchanged for the shake of convenience. The algebras above are graded, where $|z_1| = 2$, if p is odd and $|z_1| = 1$, otherwise.

Instead of studying $S(M^*_{n,m})$, we work only with $S(M_{n,m})$, since $M^*_{n,m} \cong M_{m,n}$ and the last isomorphism is τ-equivariant. Here $\tau\colon \mathrm{GL}_n \times \mathrm{GL}_m \xrightarrow{\cong} \mathrm{GL}_m \times \mathrm{GL}_n$, (see [4]).

3. **Invariants of $U_n \times U_m$.** Let $f_n(x) = \prod_{u \in (V^n)^*}(x-u)$, then $f_n(x)$ is a GL_n-invariant and $f_n(x) = \sum_{i=0}^{n}(-1)^{n-i}x^{p^i}D_{n,i}$, where $D_{n,i}$ are the generators of the Dickson polynomial algebra: $F_p[z_1,\ldots,z_n]^{\mathrm{GL}_n}$. The generators of the upper triangular invariants are given by evaluating the polynomial above, $V_i := f_i(z_i) = \prod_{u \in (V^{i-1})^*}(z_i - u)$, for $1 \leq i \leq n$ [2], [6].

Next, we construct a family of polynomial invariants for $U_n \times U_m$ and discuss their relation with the invariants for U_{nm}.

The $\{V_1,\ldots,V_m\}$ contains invariant polynomials for $U_n \times U_m$. Instead of V_{m+1}, we take the following: let the subscript of z_1 in V_1 be replaced by $m+1$ and call it $V_{1;1}$. Let $V_{(1;1,0)} = \prod_{a \in F_p}(z_{m+1} + az_1)$. We define $V^{(n,m)}_{m+1} := V_{1;1}V_{(1;1,0)}$. Let $V_{2;1}$ be V_2 where all subscripts have been shifted by m and $V_{(2;1,0)} = \prod_{b \in F_p^*}\prod_{a \in F_p}(z_{m+2} + bz_2 + az_{m+1} + abz_1)$. We define $V^{(n,m)}_{m+2} := V_{2;1}V_{(2;1,0)}$. For the general case:

DEFINITION 1. Let $V_{i;k} = \prod_{a_t \in F_p}(z_{km+i} + \sum_{1 \leq t \leq i-1} a_t z_{km+t})$, *i.e.* all subscripts of V_i have been shifted by km.

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DEFINITION 2. Let

$$V_{(i;k_t,\ldots,k_1)} = \prod_{b_j\in F_p^*, b_t=1, a_s\in F_p} \Big(\sum_{1\le j\le t} b_j z_{k_j m+i} + \sum_{1\le s\le i-1}\sum_{1\le j\le t} b_j a_s z_{k_j m+s}\Big),$$

i.e., the corresponding $V_{i;k_t}$ are added componentwise.

Finally:

DEFINITION 3. Let $V^{(n,m)}_{km+i} = \prod_{1\le t\le k+1}\prod_{0\le k_1<\cdots<k_t=k} V_{(i;k_t,\ldots,k_1)}$, for $0 \le k \le n-1$.
1. The degree of $V^{(n,m)}_{km+i}$ is given by $|V^{(n,m)}_{km+i}| = |V_{k+1}|\,|V_i|$ and $|V_i| = 2p^{i-1}(2^{i-1}$, if $p = 2)$.

PROPOSITION 4. *The $V^{(n,m)}_{km+i}$'s are $U_n \times U_m$ invariants.*

PROOF. First, we note that the following polynomial is $U_n \times U_m$ invariant:

Let W_t^{i-1} be the vector space generated by $\{x_{t,i-1},\ldots,x_{t,1}\}$, for $1 \le t \le k$. Let $w_t(a_{k,i-1},\ldots,a_{k,1}) = \sum_{s=1}^{i-1} a_{k,s}x_{t,s}$, where $(a_{k,i-1},\ldots,a_{k,1})$ describes an element $w_k \in W_k^{i-1}$. Using the same element, we describe a vector space of dimension $k-1$, $V^{k-1}(1,w_k) = \langle x_{k-1,m} + w_{k-1}(a_{k,i-1},\ldots,a_{k,1}),\ldots,x_{1,m} + w_1(a_{k,i-1},\ldots,a_{k,1})\rangle$. For elements of $V^{k-1}(1,w_k)$ we write $v(w_k)$. Now we are ready to describe the invariant polynomial we mentioned above.

$$\prod_{w_k\in W_k^{i-1}}\ \prod_{v(w_k)\in V^{k-1}(1,w_k)} \big(x_{k,i} + w_k + v(w_k)\big)x_{k,i} + w_k + v(w_k) \tag{3}$$
$$= z_{(k-1)m+i} + \sum_{s=1}^{i-1} a_{k,s} z_{(k-1)m+s} \sum_{t=1}^{k-1} b_t\Big(z_{(t-1)m+i} + \sum_{s=1}^{i-1} a_{k,s} z_{(t-1)m+s}\Big)$$
$$= \sum_{t=1}^{k} b_t(z_{(t-1)m+i}) + \sum_{t=1}^{k} b_t\Big(\sum_{s=1}^{i-1} a_{k,s} z_{(t-1)m+s}\Big)$$

and the non-zero b_t's define a partition as in (2).

Let $(g,h) \in U_n \times U_m$. Then $(g,h)\big(x_{k,i}+w_k+v(w_k)\big) = (g,h)x_{k,i}+(g,h)w_k+(g,h)v(w_k)$ and the action is given by:

i) From (2), $(g,h)w_k = h(w_k) \in W_k^{i-1}$.

ii) From (2), $(g,h)v(w_k) = (g(v))\big(h(w_k)\big)$.

iii) From (1), $(g,h)x_{k,i} = x_{k,i} + w'_k + v'(w'_k)$, where $w'_k \in W_k^{i-1}$ and $v'(w'_k) \in V^{k-1}(1,w'_k)$.

From I), ii), and iii) above we see that (3) is $U_n \times U_m$ invariant.

It remains to show that $V^{(n,m)}_{km+i}$ and the polynomial (3) above are equal and this is obvious from (1).

$$V^{(n,m)}_{km+i} = \prod_{w_k\in W_k^{i-1}}\ \prod_{v(w_k)\in V^{k-1}(1,w_k)} \big(x_{k,i} + w_k + v(w_k)\big) \tag{4}$$

■

It is known that generators for U_n are given by its subset $\{u_{ij} \mid 1 \le i,j \le n\}$, where $u_{ij} - I_n$ is the zero matrix except at position (i,j) where there is a one. Hence the following is obvious.

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LEMMA 5. *Let $U_n \times U_m$ be considered as a subgroup of GL_{nm} under the inclusion described above. Then the set $\{u_{ij}^{(n,m)} w_{tm+1,sm+1}^{(n,m)} \mid 1 \le i,j \le n, t < s\}$ provides a generating set for $U_n \times U_m$.*

Here $u_{ij}^{(n,m)}$ is the matrix with 1's along the main diagonal and at positions $(km+i, km+j)$ for all appropriate k; and $w_{tm+1,sm+1}^{(n,m)}$ consists of 1's along the main diagonal and at positions $(tm+r, sm+r)$ for $1 \le r \le m$.

$U_n \times U_m$ is not generated by pseudoreflections. Thus, $S(M_{n,m})^{U_n \times U_m}$ is not a polynomial algebra.

THEOREM 6. *The algebra, $S[V_i^{(n,m)} \mid 1 \le i \le nm]$, generated by $A = \{V_1^{(n,m)}, \ldots, V_{nm}^{(n,m)}\}$ is a polynomial subalgebra of $F_p[z_1, \ldots, z_{nm}]^{U_n \times U_m}$.*

PROOF. Let $H_{nm} = \{(a_{i,j}) \mid a_{i,j} \in F_p, a_{i,j} = 0 \text{ if } i < j \text{ or } (i \bmod m) > (j \bmod m)\}$ such that $H_{nm} \subseteq U_{nm}$.

It is easy to see that H_{nm} is a subgroup of U_{nm}. Moreover, H_{nm} is generated by pseudoreflections and hence $F_p[z_1, \ldots, z_{nm}]^{H_{nm}}$ is a polynomial algebra.

The order of H equals $|H_{nm}| = |U_m|^{\frac{(n+1)n}{2}} p^{\frac{m \cdot n \cdot (n-i)}{2}}$. Calculating the degree of the elements of A, we observe that $|H_{nm}| = \prod_{i=1}^{nm} |V_i^{(n,m)}|$. It remains to show that elements of A are algebraically independent. We proceed by inducting on the length nm. For $n = 1$, the assertion follows from the generators of $F_p[z_1, \ldots, z_m]^{U_m}$. Next, we prove the assertion for $m+1$ (H_{m+1} as above).$V_{m+1}^{(n,m)} = \prod_{a \in F_p}(z_{m+1} + az_1)$. Assume, there exists a polynomial f such that $f(V_1, \ldots, V_{m+1}) = 0$. Letting $z_{m+1} = 0$, we get a polynomial g such that $g(V_1, \ldots, V_m) = 0$. The general step is the same as the one above. ∎

To demonstrate the ideas above, let us work out a particular example (see [4]).

EXAMPLE 7. Let $g = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ and $h = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$. Then $(g,h) = \begin{pmatrix} 1 & b & a & ab \\ 0 & 1 & 0 & a \\ 0 & 0 & 1 & b \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

The following polynomials are easily seen to be $U_2 \times U_2$-invariants.

i) $V_1 = z_1 = x_{1,1} = V_1(y_1)$.

ii) $V_2 = z_2(z_2 + z_1) = x_{1,2}(x_{1,2} + x_{1,1}) = V_1(y_2)V_1(y_2 + y_1)$.

iii) $V_3^{(2,2)} = z_3(z_3 + z_1) = x_{2,1}(x_{2,1} + x_{1,1}) = V_2(y_1)$.

iv) $V_4^{(2,2)} = z_4(z_4 + z_3)(z_4 + z_2)(z_4 + z_3 + z_2 + z_1) = x_{2,2}(x_{2,2} + x_{2,1})(x_{2,2} + x_{1,2})(x_{2,2} + x_{2,1} + x_{1,2} + x_{1,1}) = V_2(y_2)V_2(y_2 + y_1)$.

v) $\det = z_4z_1 + z_2z_3 = V_2(y_2) + V_2(y_2 + y_1) + V_2(y_1)$.

We give here the relation between the invariants above and the usual polynomial invariant generators of U_4.

$V_3 = V_3^{(2,2)}(V_3^{(2,2)} + V_2)$.

$V_4 = V_4^{(2,2)}(V_4^{(2,2)} + V_1^2V_2 + \det(\det + V_1^2) + V_1^2V_3^{(2,2)} + V_2V_3^{(2,2)})$.

It is obvious that the polynomial algebra generated by $\{V_1, V_2, V_3, V_4\}$ is not a subalgebra of the polynomial algebra generated by $\{V_1, V_2, V_3^{(2,2)}, V_4^{(2,2)}\}$.

A) $Y_1 = D_{2,1}(y_1) = x_{11}^2 + x_{11}x_{21} + x_{21}^2 = V_1^2 + V_3^{2,2}$.

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B) $Y_2 = D_{2,1}(y_2) = x_{12}^2 + x_{12}x_{22} + x_{22}^2$.

C) $D_{2,1}(y_1 + y_2) = x_{11}^2 + x_{12}^2 + (x_{11} + x_{12})(x_{21} + x_{22}) + x_{21}^2 + x_{22}^2$.

D) $D_{2,1}(D_{2,1}) = Y_1^2 + Y_1Y_2 + Y_2^2 + (Y_1 + Y_2)\sum_{u\in W} D_{2,1}(u) = x_{11}^4 + x_{11}^2x_{21}^2 + x_{21}^4 + +x_{11}^3x_{22} + x_{11}^2x_{12}x_{21} + x_{22}^4 + x_{11}x_{22}^3 + x_{12}^2x_{22}^2 + x_{11}x_{12}x_{22}^2 + x_{21}^2x_{22}^2 + x_{11}x_{21}x_{12}x_{22} + x_{11}x_{21}x_{22}^2 + x_{21}^2x_{12}x_{22} + x_{12}^2x_{21}x_{22} + x_{12}x_{21}x_{22}^2x_{12}^4 + x_{11}^2x_{12}^2 + x_{12}^2x_{21}^2 + x_{11}^2x_{22}^2x_{11}x_{22}x_{12}^2 + x_{12}^3x_{21} + x_{11}x_{21}x_{12}^2 + x_{11}^2x_{12}x_{22}x_{11}x_{22}x_{21}^2 + x_{11}x_{12}x_{21}^2 + x_{12}x_{21}^3 + x_{11}^2x_{21}x_{22} = V_1^4 + V_2^2 + (V_3^{(2,2)})^2 + V_4^{(2,2)} + \det^2 + \det(V_1^2 + V_2 + V_3^{(2,2)})$.

F) $\sum_{u\in W} D_{2,1}(u) = \sum_{v\in V} v(D_{2,1}) = \det = x_{11}x_{22} + x_{12}x_{21}$.

Polynomials D) and F) in the last example are $\mathrm{GL}_2 \times \mathrm{GL}_2$-invariants.

It is obvious that the polynomial algebra generated by $\{D_{2,1}(D_{2,1}), D_{2,1}(D_{2,0}), D_{2,0}(D_{2,1}), D_{2,0}(D_{2,0})\}$ (see [4]) is not a subalgebra of the polynomial algebra generated by $\{V_1, V_2, V_3^{(2,2)}, V_4^{(2,2)}\}$.

Hence, both $F_p[z_1, \ldots, z_{nm}]^{U_n\times U_m}$ and $S[c_{n,i}(c_{m,j}) \mid 0 \leq i \leq n-1, 0 \leq j \leq m-1]$ are not subalgebras of $S[V_i^{(n,m)} \mid 1 \leq i \leq nm]$.

PROPOSITION 8. *The relation between V_{km+1} and $V_{km+1}^{(n,m)}$ is described by the following inductive relations: Let $Y_{km+1,1} := V_{km+1}^{(n,m)} = \prod_{a\in\langle z_{im+1}, 0\leq i\leq k-1\rangle}(z_{km+1} + a)$ and $Y_{km+1,s} := Y_{km+1,s-1}Y_{km+1,s-1}(z_s)$, then $V_{km+1} = Y_{km+1,m}$.*

PROOF. Let us recall the definition and expression of $V_{km+1} = \prod_{a\in\langle z_t, 0\leq t\leq km\rangle}(z_{km+1} + a) = \sum_{i=0}^{km}(-1)^{km-i}z_{km+1}^{p^i}D_{km,i}$. We construct the last product by adding more basis elements in the corresponding index vector space.

First step: $1 \leq s \leq m$. $Y_{km+1,2}(z_2) = \prod_{a\in\langle z_{im+1}, 0\leq i\leq k-1\rangle}(z_{km+1} + z_2 + a)$. Using the expression above, the last product can be written as $V_{km+1}^{(n,m)} + \prod_{a\in\langle z_{im+1}, 0\leq i\leq k-1\rangle}(z_2 + a)$. We rewrite the last product $V_2 \prod_{i=1}^{k-1}\prod_{a\in\langle z_{jm+1}, 0\leq j<i\rangle}(z_{im+1} + z_2 + a)$. By induction hypothesis $\prod_{a\in\langle z_{jm+1}, 0\leq j<i\rangle}(z_{im+1} + z_2 + a)$ can be expressed as an algebraic combination of generators of $S[V_i^{(n,m)} \mid 1 \leq i \leq nm]$. For the general step: $Y_{km+1,s-1}(z_s) = Y_{km+1,s-1} + V_s \prod_{i=[\frac{s}{m}]+1}^{k-1}\prod_{a\in\langle z_{jm+1}, 0\leq j<i\rangle}(z_{im+1} + z_s + a)$ and $V_{km+1} = Y_{km+1,m}$. ∎

4. **Chern classes of U_{n+m}.** Let $V_i \in F_p[x_1, \ldots, x_n]^{U_n}$ and $y_k \in W^*$, then $V_i \otimes (y_k + w) \in S(M_{n,m})$. Let $ß_{i,k} = \{(x_i + u) \otimes (y_k + w) \mid u \in V^{i-1}, w \in (W^{k-1})^*\}$ for $1 \leq k \leq n$ and $1 \leq i \leq m$.

The coefficients of

$$P_{i,k}(x) := \prod_{w\in(W^{k-1})^*}\big(x - V_i \otimes (y_k + w)\big) = \prod_{y\in ß_{i,k}}(x - y)$$

provide an invariant set for the ring $S(M_{n,m})^{U_n\times U_m}$, for $1 \leq k \leq n$ and $1 \leq i \leq m$.

The following is obvious.

PROPOSITION 9. *i) The polynomial $\prod_{u\in V^{i-1}}\prod_{w\in(W^{k-1})^*}\big(x - (x_i + u) \otimes (y_k + w)\big)$ is a $U_n \times U_m$ invariant.*

ii) For $x = 0$ the above is $V_{(k-1)m+i}^{(n,m)}$.

iii) The polynomial $\sum_{w_1,\ldots,w_s\in(W^{k-1})^*}\prod_{t=1}^{s} V_i(y_k + w_t)$ is a $U_n \times U_m$ invariant, for $1 \leq s \leq p^{k-1}$.

A good source for Chern classes and modular invariants of finite groups is [3].

Let $N = p^{nm}$ and $P: U_n \times U_m \longrightarrow \Sigma_N \longrightarrow U(N)$ be the representation induced by the action of $U_n \times U_m$ on $M_{n,m} \cong F_p^N$. Let U_{n+m} be given as the semidirect product: $M_{n,m} \odot (U_n \times U_m)\ \big(\big(h, (g_1, g_2)\big) \circ \big(h', (g_1', g_2')\big) = \big(g_1 h' g_2^{-1} + h, (g_1 g_1', g_2 g_2')\big)\big)$.

The following composition provides a representation

$$\rho_{\text{ß}}: M_{n,m} \hookrightarrow U_{n+m} \longrightarrow F_p^N \odot \Sigma_N \longrightarrow U(1)^N \odot \Sigma_N \subset U(N).$$

Here $\beta \subset (M_{n,m})^*$ is a $U_n \times U_m$ invariant set, $F_p \longrightarrow U(1)$ is given by mapping the generator on $e^{\frac{2\pi i}{p}}$ and $\prod_{\mu\in\beta}(-u): M_{n,m} \longrightarrow F_P^N$. Here $U(N)$ is the unitary group. The total Chern class of ρ_β satisfies $\rho_\beta^*(C) = \prod_{u\in cC}(1-u)$ (see [3]).

THEOREM 10. *Let $\beta_{i,k} \subset (M_{n,m})^*$, then the coefficients of $P_{i,k}(x)$ are Chern classes of $\rho_{\beta_{i,k}}$ belonging to the image of the restriction*

$$i^*: H^*(U_{n+m}) \longrightarrow H^*(M_{n,m})^{U_n \times U_m}$$

for $1 \le k \le n$ and $1 \le i \le m$.

The proof is a routine checking of the proof given in [4].

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