6

Adiabatic limit

If we assume that the mass of an electron is purely electromagnetic, then by equating its rest energy and electrostatic Coulomb energy the charge distribution must be concentrated in a ball of radius

$$r_{\rm cl} = \frac{e^2}{m_{\rm e}c^2} = 3 \times 10^{-13}\ \text{cm} \tag{6.1}$$

which is the so-called classical electron radius. Quantum mechanically one argues that on the basis of light scattering the electron appears to have an effective size of the order of the Compton wavelength $\lambda_{\rm c} = \hbar m_{\rm e}/c = (e^2/\hbar c)^{-1}\, r_{\rm cl} = 137\, r_{\rm cl}$. Thus empirically R_φ is limited to $r_{\rm cl} \leq R_\varphi \leq 137 r_{\rm cl}$. Electromagnetic fields which can be manipulated in the laboratory vary little over that length scale. $r_{\rm cl}$ defines a time scale through the time span for light to travel across the diameter of the charge distribution,

$$t_{\rm cl} = r_{\rm cl}/c = 10^{-23}\ \text{s}\,, \quad \text{equivalently a frequency}\ \ \omega_{\rm cl} = 10^{23}\ \text{Hz}\,. \tag{6.2}$$

Again, manufactured frequencies are much smaller than $\omega_{\rm cl}$. Space-time variations as fast as (6.1) and (6.2) lead us deeply into the quantum regime. Thus it is natural and physically compelling to study the dynamics of a charged particle under external potentials which vary *slowly* on the scale of the charge radius R_φ, which is the only length scale available. This means we have to introduce a scale of potentials and enquire about an approximately autonomous particle dynamics with an error depending on the scale under consideration. We will introduce such a scheme in the following section. The resulting problem has many similarities with the derivation of hydrodynamics from Newtonian particle dynamics – with the most welcome bonus that it is simpler mathematically by many orders of magnitude. Still, the comparison is instructive.

https://doi.org/10.1017/9781009402286.007 Published online by Cambridge University Press

6.1 Scaling limit for external potentials of slow variation

For the Abraham model, see Eq. (2.41), the Lorentz force has in addition to the dynamical fields $\boldsymbol{E}(\boldsymbol{x}, t)$, $\boldsymbol{B}(\boldsymbol{x}, t)$ also prescribed external fields, which are the gradients of the external potentials $\phi_{\mathrm{ex}}(\boldsymbol{x})$, $\boldsymbol{A}_{\mathrm{ex}}(\boldsymbol{x})$.

We want to impose the condition that ϕ_{ex} and $\boldsymbol{A}_{\mathrm{ex}}$ are slowly varying on the scale of R_φ. Formally we introduce a small *dimensionless* parameter ε and consider the potentials

$$\phi_{\mathrm{ex}}(\varepsilon\boldsymbol{x}) \,, \quad \boldsymbol{A}_{\mathrm{ex}}(\varepsilon\boldsymbol{x}) \,, \tag{6.3}$$

which are slowly varying in the limit $\varepsilon \to 0$. Most of our results extend to potentials which vary also slowly in time. For simplicity we restrict ourselves to time-independent potentials here. Clearly, ε appears as a parameter of the potential, just like ω_0 is a parameter of the harmonic oscillator potential $\frac{1}{2} m\omega_0^2 x^2$. But ε should really be thought of as a bookkeeping device which orders the magnitude of the various terms and the space-time scales according to the powers of ε. Such a scheme is familiar from very diverse contexts and appears whenever one has to deal with a problem involving scale separation.

So how small is ε? From the discussion above one might infer that if ϕ_{ex}, $\boldsymbol{A}_{\mathrm{ex}}$ vary over a scale of 1 mm, then $\varepsilon = 10^{-12}$. This is a totally meaningless statement, because $e\phi_{\mathrm{ex}}$, $e\boldsymbol{A}_{\mathrm{ex}}$ have the dimension of energy and thus the variation depends on the adopted energy scale. In (6.3) we merely stretch the spatial axes by a factor ε^{-1} and fix the energy scale. Since from experience this point is likely to be confusing, let us consider the specific example of a charge revolving in the uniform magnetic field $\boldsymbol{B}_{\mathrm{ex}} = (0, 0, B_0)$. The corresponding vector potential is linear in $\boldsymbol{x}$, and to introduce ε as in (6.3) just means that the magnetic field strength equals εB_0. The limit $\varepsilon \to 0$ is a limit of small magnetic field strength *relative* to some reference field B_0. Thus to obtain ε we first have to determine the reference field and compare it with the magnetic field of interest. This shows that in order to fix ε we have to specify the physical situation in detail, in particular the external potentials, the mass of the particle, the charge of the particle, $\gamma(\boldsymbol{v})$, and the time span of interest.

The scaling scheme (6.3) has the great advantage that the analysis can be carried out in generality. In a second step one has to figure out ε for a *concrete* situation, which leads to a quantitative estimate of the error terms. For instance, if in the case above we consider an electron with velocities such that $\gamma \leq 10$, then, by comparing the Hamiltonian term and the friction term, the reference field turns out to be $B_0 = 10^{17}$ gauss. Laboratory magnetic fields are less than 10^5 gauss and thus $\varepsilon < 10^{-12}$. In this and many other concrete examples, ε is very small, less than 10^{-10}, which implies that, firstly, all corrections *beyond* radiation reaction are negligible. Secondly, we do not have to go each time through the scheme indicated above and may as well set $\varepsilon = 1$ thereby returning to conventional units. Still on a theoretical level the use of the scale parameter ε is very convenient. In an appendix to this section we will work out the example of a constant magnetic field more explicitly. If the reader feels uneasy about the scaling limit, (s)he should consult this example first.

Adopting (6.3), Newton's equations of motion now read

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(m_{\mathrm{b}}\gamma\boldsymbol{v}(t)\big) = e\big(\boldsymbol{E}_\varphi(\boldsymbol{q}(t),t) + \varepsilon\boldsymbol{E}_{\mathrm{ex}}(\varepsilon\boldsymbol{q}(t)) + \boldsymbol{v}(t)\times\big(\boldsymbol{B}_\varphi(\boldsymbol{q}(t),t) + \varepsilon\boldsymbol{B}_{\mathrm{ex}}(\varepsilon\boldsymbol{q}(t))\big)\big)\,, \tag{6.4}$$

where

$$\boldsymbol{E}_{\mathrm{ex}} = -\nabla\phi_{\mathrm{ex}}\,, \quad \boldsymbol{B}_{\mathrm{ex}} = \nabla\times\boldsymbol{A}_{\mathrm{ex}}\,. \tag{6.5}$$

Note that if $\boldsymbol{E}_{\mathrm{ex}}$, $\boldsymbol{B}_{\mathrm{ex}}$ are smeared by φ, as would be proper, the resulting error in (6.4) is of order ε^3, which can be ignored for our purposes.

Equation (6.4) has to be supplemented with Maxwell's equations (2.39), (2.40). Our goal is to understand the structure of the solution for small ε, and as a first qualitative step one should discuss the rough order of magnitudes in powers of ε. But before that we have to specify the initial data. We give ourselves $\boldsymbol{q}^0$, $\boldsymbol{v}^0$ as the initial position and velocity of the charge. The initial fields are assumed to be Coulombic, i.e. of the form of a charge soliton centered at $\boldsymbol{q}^0$ with velocity $\boldsymbol{v}^0$, compare with (4.28), which we formalize as

Condition (I):

$$Y(0) = S_{\boldsymbol{q}^0,\boldsymbol{v}^0}\,. \tag{6.6}$$

Equivalently, according to (4.31), (4.32), we may say that the particle has traveled freely with velocity $\boldsymbol{v}^0$ for the infinite time span $(-\infty, 0]$. At time $t = 0$ the external potentials are turned on. Geometrically, our initial data are exactly on the soliton manifold $\mathcal{S}$ considered as a submanifold of the phase space $\mathcal{M}$. If there are no external forces, the solution stays on $\mathcal{S}$ and moves along a straight line. For slowly varying external potentials as in (6.3) we will show that the solution stays ε-close to $\mathcal{S}$ in the local energy distance.

On general grounds one may wonder whether such specific initial data are really required. In analogy to hydrodynamics, we call this the initial slip problem. In times of order t_φ $(= R_\varphi/c)$, the fields close to the charge acquire their Coulombic form while the external forces are still negligible; compare with figure 6.1. However, during that period the particle might gain or lose in momentum and energy through the interaction with its own field and the data at time t_φ close to the particle

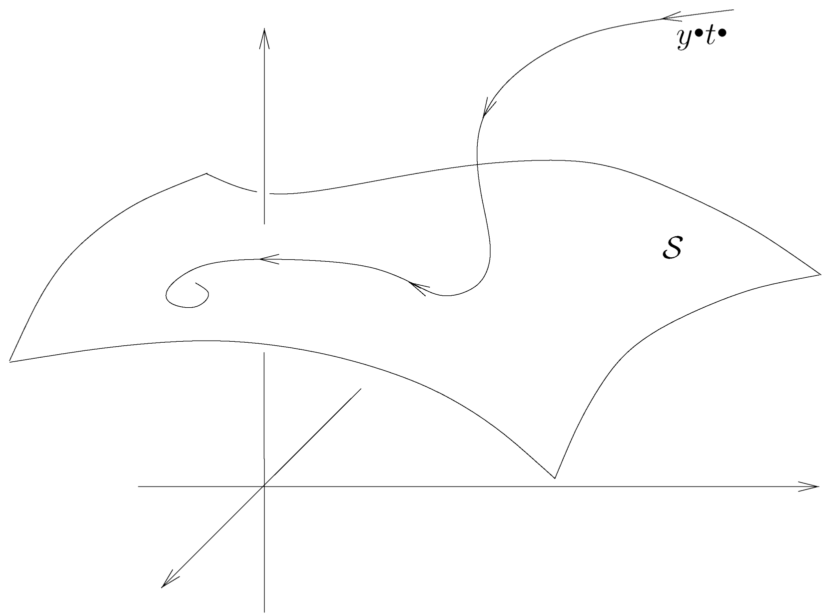

Figure 6.1: Schematic phase space with attractive soliton manifold $\mathcal{S}$. Away from $\mathcal{S}$ the motion is fast, on $\mathcal{S}$ it is slow.

are approximately of the form $S_{\tilde{\boldsymbol{q}},\tilde{\boldsymbol{v}}}$, where $\tilde{\boldsymbol{q}}$ and $\tilde{\boldsymbol{v}}$ are to be computed from the full solution. Of course, at a distance ct away from the charge, the field still remembers its $t = 0$ data. Thus we see that the initial slip problem translates into the long-time asymptotics of a charge at *zero* external potentials but with general initial field data. We refer to section 5.2, where this point has been studied in detail. At the moment we just circumvent the initial slip by fiat.

Let us discuss the three relevant time scales, where we recall that $t_\varphi = R_\varphi/c$.

(i) *Microscopic scale*, $t = \mathcal{O}(t_\varphi)$, $\boldsymbol{q} = \mathcal{O}(R_\varphi)$. On this scale the particle moves along an essentially straight line. The electromagnetic fields adjust themselves to their comoving Coulombic form. As we will see, they do this with a precision $\mathcal{O}(\varepsilon)$ in the energy norm.

(ii) *Macroscopic potential scale*, $t = \mathcal{O}(\varepsilon^{-1}t_\varphi)$, $\boldsymbol{q} = \mathcal{O}(\varepsilon^{-1}R_\varphi)$. This scale is defined by the variation of the potentials, i.e. on this scale the potentials are $\phi_{\mathrm{ex}}(\boldsymbol{x})$, $\boldsymbol{A}_{\mathrm{ex}}(\boldsymbol{x})$. The particle follows the external forces. Since it is in company with almost Coulombic fields, the particle responds to the forces according to the effective energy–momentum relation, which we determined in chapter 4. On the macroscopic scale the motion is Hamiltonian up to errors of order ε. There is no dissipation of energy and momentum.

(iii) *Macroscopic friction scale*. Accelerated charges lose energy through radiation, which means that there must be friction corrections to the effective

Hamiltonian motion. According to Larmor's formula the radiation losses are proportional to $\dot{\boldsymbol{v}}(t)^2$. Since the external forces are of the order ε, these losses are proportional to ε^2 when measured in microscopic units. Integrated over a time span $\varepsilon^{-1}t_\varphi$ the friction results in an effect of order ε. Thus we expect order ε dissipative corrections to the conservative motion on the macroscopic scale. Followed over the even longer time scale $\varepsilon^{-2}t_\varphi$, the radiation reaction results in $\mathcal{O}(1)$ deviations from the Hamiltonian trajectory.

On the friction time scale the motion either comes to a standstill or stays uniform. In addition, as will be shown, the dissipative effective equation has the same long-time behavior as the true solution. Thus we expect no further qualitatively distinct time scale beyond the friction scale.

From our description, in a certain sense, the most natural scale is the macroscopic scale and we transform Maxwell's and Newton's equations to this new scale by setting

$$t' = \varepsilon t\,, \quad \boldsymbol{x}' = \varepsilon \boldsymbol{x}\,. \tag{6.7}$$

We have the freedom of how to scale the amplitudes of the dynamic part of the electromagnetic fields. We require that their energy is independent of ε. Then

$$\boldsymbol{E}'(\boldsymbol{x}', t') = \varepsilon^{-3/2}\,\boldsymbol{E}(\boldsymbol{x}, t)\,, \quad \boldsymbol{B}'(\boldsymbol{x}', t') = \varepsilon^{-3/2}\,\boldsymbol{B}(\boldsymbol{x}, t)\,. \tag{6.8}$$

Finally the new position and velocity are

$$\boldsymbol{q}'(t') = \varepsilon \boldsymbol{q}(t)\,, \quad \boldsymbol{v}'(t') = \boldsymbol{v}(t)\,, \tag{6.9}$$

so that $\frac{\mathrm{d}}{\mathrm{d}t'}\boldsymbol{q}' = \boldsymbol{v}'$. There is little risk of confusion in omitting the prime. We then denote

$$\boldsymbol{q}^\varepsilon(t) = \varepsilon \boldsymbol{q}(\varepsilon^{-1}t)\,, \quad \boldsymbol{v}^\varepsilon(t) = \boldsymbol{v}(\varepsilon^{-1}t)\,, \quad \varphi_\varepsilon(\boldsymbol{x}) = \varepsilon^{-3}\,\varphi(\varepsilon^{-1}\boldsymbol{x})\,, \tag{6.10}$$

which means that $\int \mathrm{d}^3x\, \varphi_\varepsilon(\boldsymbol{x}) = 1$ independent of ε and that φ_ε is supported in a ball of radius εR_φ. In the macroscopic coordinates the coupled Maxwell's and Newton's equations read

$$\begin{aligned}
&\partial_t \boldsymbol{B}(\boldsymbol{x}, t) = -\nabla \times \boldsymbol{E}(\boldsymbol{x}, t)\,,\\
&\partial_t \boldsymbol{E}(\boldsymbol{x}, t) = \nabla \times \boldsymbol{B}(\boldsymbol{x}, t) - \sqrt{\varepsilon} e \varphi_\varepsilon(\boldsymbol{x} - \boldsymbol{q}^\varepsilon(t))\boldsymbol{v}^\varepsilon(t)\,,\\
&\frac{\mathrm{d}}{\mathrm{d}t}\big(m_\mathrm{b}\gamma \boldsymbol{v}^\varepsilon(t)\big) = e\big(\boldsymbol{E}_\mathrm{ex}(\boldsymbol{q}^\varepsilon(t)) + \boldsymbol{v}^\varepsilon(t) \times \boldsymbol{B}_\mathrm{ex}(\boldsymbol{q}^\varepsilon(t))\big)\\
&\qquad\qquad + \sqrt{\varepsilon}\, e\big(\boldsymbol{E}_{\varphi_\varepsilon}(\boldsymbol{q}^\varepsilon(t), t) + \boldsymbol{v}^\varepsilon(t) \times \boldsymbol{B}_{\varphi_\varepsilon}(\boldsymbol{q}^\varepsilon(t), t)\big)
\end{aligned} \tag{6.11}$$

together with the constraints

$$\nabla \cdot \boldsymbol{E} = \sqrt{\varepsilon}\, e \varphi_\varepsilon(\cdot - \boldsymbol{q}^\varepsilon(t))\,, \quad \nabla \cdot \boldsymbol{B} = 0\,. \tag{6.12}$$

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On the macroscopic scale the conserved energy is

$$\mathcal{E}_{\mathrm{mac}} = m_{\mathrm{b}}\gamma(\boldsymbol{v}) + e\phi_{\mathrm{ex}}(\boldsymbol{q}) + \frac{1}{2}\int \mathrm{d}^3x \left(\boldsymbol{E}(\boldsymbol{x})^2 + \boldsymbol{B}(\boldsymbol{x})^2\right). \tag{6.13}$$

Also the initial data have to be transformed and become

Condition (I_ε):

$$Y^\varepsilon(0) = S^\varepsilon_{\boldsymbol{q}^0,\boldsymbol{v}^0} = (\boldsymbol{E}^\varepsilon_{\boldsymbol{v}^0}(\boldsymbol{x}-\boldsymbol{q}^0), \boldsymbol{B}^\varepsilon_{\boldsymbol{v}^0}(\boldsymbol{x}-\boldsymbol{q}^0), \boldsymbol{q}^0, \boldsymbol{v}^0) \tag{6.14}$$

with

$$\boldsymbol{E}^\varepsilon_{\boldsymbol{v}} = -\nabla\phi^\varepsilon_{\boldsymbol{v}} + \boldsymbol{v}(\boldsymbol{v}\cdot\nabla\phi^\varepsilon_{\boldsymbol{v}}), \quad \boldsymbol{B}^\varepsilon_{\boldsymbol{v}} = -\boldsymbol{v}\times\nabla\phi^\varepsilon_{\boldsymbol{v}}, \tag{6.15}$$

where now

$$\widehat{\phi}^\varepsilon_{\boldsymbol{v}}(\boldsymbol{k}) = \frac{\sqrt{\varepsilon}\, e\widehat{\varphi}(\varepsilon\boldsymbol{k})}{\boldsymbol{k}^2 - (\boldsymbol{v}\cdot\boldsymbol{k})^2}. \tag{6.16}$$

On the macroscopic scale, the scaling parameter ε can be absorbed into the "effective" charge distribution $\sqrt{\varepsilon}e\widehat{\varphi}_\varepsilon$. Its electrostatic energy,

$$m_{\mathrm{e}} = \frac{1}{2}e^2\int \mathrm{d}^3k\, \varepsilon|\widehat{\varphi}_\varepsilon(\boldsymbol{k})|^2\frac{1}{\boldsymbol{k}^2} = \frac{1}{2}\int \mathrm{d}^3k|\widehat{\varphi}(\boldsymbol{k})|^2\frac{1}{\boldsymbol{k}^2}, \tag{6.17}$$

is independent of ε, while its charge

$$e\int \mathrm{d}^3x\, \sqrt{\varepsilon}\,\varphi_\varepsilon(\boldsymbol{x}) = \sqrt{\varepsilon}\, e \tag{6.18}$$

vanishes as $\sqrt{\varepsilon}$. Recall that ε is a "bookkeeping device".

We argued that on the macroscopic scale the response to external potentials in the motion of the charges is of order one. We thus expect that $\boldsymbol{q}^\varepsilon(t)$ tends to a nondegenerate limit as $\varepsilon\to 0$, i.e.

$$\lim_{\varepsilon\to 0}\boldsymbol{q}^\varepsilon(t) = \boldsymbol{r}(t), \quad \lim_{\varepsilon\to 0}\boldsymbol{v}^\varepsilon(t) = \boldsymbol{u}(t). \tag{6.19}$$

The position $\boldsymbol{r}(t)$ and velocity $\boldsymbol{u}(t)$ should be governed by an effective Lagrangian. In section 4.1 we determined the effective inertial term. If the potentials add in as usual, one has

$$L_{\mathrm{eff}}(\boldsymbol{q},\dot{\boldsymbol{q}}) = T(\dot{\boldsymbol{q}}) - e\big(\phi_{\mathrm{ex}}(\boldsymbol{q}) - \dot{\boldsymbol{q}}\cdot\boldsymbol{A}_{\mathrm{ex}}(\boldsymbol{q})\big), \tag{6.20}$$

which results in the equations of motion

$$\dot{\boldsymbol{r}} = \boldsymbol{u}, \quad m(\boldsymbol{u})\dot{\boldsymbol{u}} = e(\boldsymbol{E}_{\mathrm{ex}}(\boldsymbol{r}) + \boldsymbol{u}\times\boldsymbol{B}_{\mathrm{ex}}(\boldsymbol{r})). \tag{6.21}$$

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The velocity-dependent mass $m(\boldsymbol{u})$ has a bare and a field contribution. From (4.12) we conclude that

$$m(\boldsymbol{u}) = \frac{\mathrm{d}P_{\mathrm{s}}(\boldsymbol{u})}{\mathrm{d}\boldsymbol{u}} \tag{6.22}$$

as a 3×3 matrix. If instead of the velocity we introduce the canonical momentum, $\boldsymbol{p}$, then the effective Hamiltonian reads

$$H_{\mathrm{eff}}(\boldsymbol{r}, \boldsymbol{p}) = E_{\mathrm{eff}}(\boldsymbol{p} - e\boldsymbol{A}_{\mathrm{ex}}(\boldsymbol{r})) + e\phi_{\mathrm{ex}}(\boldsymbol{r}) \tag{6.23}$$

with Hamilton's equations of motion

$$\dot{\boldsymbol{r}} = \nabla_{\boldsymbol{p}} H_{\mathrm{eff}}\,, \quad \dot{\boldsymbol{p}} = -\nabla_{\boldsymbol{r}} H_{\mathrm{eff}}\,. \tag{6.24}$$

Our plan is to establish the limit (6.19) and to investigate the corrections due to radiation losses.

6.1.1 Appendix 1: How small is ε?

We consider an electron moving in an external magnetic field oriented along the z-axis, $\boldsymbol{B}_{\mathrm{ex}} = (0, 0, B_0)$. The corresponding vector potential is $\boldsymbol{A}_{\mathrm{ex}}(\boldsymbol{x}) = \frac{1}{2} B_0(-x_2, x_1, 0)$. According to our convention the slowly varying vector potential is given by $\boldsymbol{A}_{\mathrm{ex}}(\varepsilon\boldsymbol{x}) = \frac{1}{2}\varepsilon B_0(-x_2, x_1, 0)$. Thus B_0 is a reference field strength, which is to be determined, and $B = \varepsilon B_0$ is the physical field strength in the laboratory. The motion of the electron is assumed to be in the 1–2 plane and we set $\boldsymbol{v} = (\boldsymbol{u}, 0)$. According to section 9.2, example (iii), within a good approximation the motion of the electron is governed by

$$\gamma\dot{\boldsymbol{u}} = \omega_{\mathrm{c}}(\boldsymbol{u}^{\perp} - \beta\omega_{\mathrm{c}}\boldsymbol{u})\,. \tag{6.25}$$

Here $\boldsymbol{u}^{\perp} = (-u_2, u_1)$, $\omega_{\mathrm{c}} = eB/m_0c$ is the cyclotron frequency, and $\beta = e^2/6\pi c^3 m_0$. The first term is the Lorentz force and the second term accounts for the radiation reaction.

We now choose the reference field B_0 such that the two terms balance, i.e.

$$B_0 = (\beta e/m_0 c)^{-1}\,. \tag{6.26}$$

For electrons

$$B_0 = 1.1 \times 10^{17}\ \text{gauss} \tag{6.27}$$

and even larger by a factor $(1836)^2$ for protons. For a laboratory field of 10^5 gauss this yields

$$\varepsilon = 10^{-12}\,. \tag{6.28}$$

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Written in units of B_0, (6.25) becomes

$$\gamma \dot{\boldsymbol{u}} = \varepsilon \omega_c^0 (\boldsymbol{u}^{\perp} - \varepsilon \boldsymbol{u}) \tag{6.29}$$

with $\beta\omega_c^0 = 1$, i.e. $\omega_c^0 = eB_0/m_0c = 1.6 \times 10^{28}\ \mathrm{s}^{-1}$. Thus friction is of relative order ε and higher-order corrections would be of relative order ε^2. As will be demonstrated, the dimensionless scaling parameter ε serves as a bookkeeping device to track the relative order of the various terms contributing to the dynamics.

6.1.2 Appendix 2: Adiabatic protection

The adiabatic limit, as discussed above, relies on the fact that photons have zero mass. If they had finite mass, radiation damping would be hindered. This point can be most easily argued in the context of a scalar wave field. Moreover, rather than having a particle interacting with the field, it suffices to have a source fixed at the origin.

The scalar wave field is denoted by ϕ with canonically conjugate momentum field π. They are governed by

$$\partial_t \phi(\boldsymbol{x}, t) = \pi(\boldsymbol{x}, t)\,, \quad \partial_t \pi(\boldsymbol{x}, t) = \Delta\phi(\boldsymbol{x}, t) - \kappa^2 \phi(\boldsymbol{x}, t) + \alpha(t)\delta(\boldsymbol{x})\,. \tag{6.30}$$

$\alpha(t)$ is a smooth function vanishing outside the interval $[0, T]$. Assuming that $\phi = 0$, $\pi = 0$ initially we want to determine how much energy is radiated in the long-time limit.

The local field energy is given by

$$e(\boldsymbol{x}, t) = \frac{1}{2}\big(\pi(\boldsymbol{x}, t)^2 + (\nabla\phi(\boldsymbol{x}, t))^2 + \kappa^2\phi(\boldsymbol{x}, t)^2\big) \tag{6.31}$$

from which, using (6.30), the energy current

$$\boldsymbol{j}_{\mathrm{e}} = -\pi\nabla\phi \tag{6.32}$$

follows. The energy flow through a sphere of radius R is given by

$$\begin{aligned} -\int_0^\infty \mathrm{d}t\, R^2 \int \mathrm{d}^2\omega\, \pi(\boldsymbol{\omega} R, t)\boldsymbol{\omega} \cdot \nabla\phi(\boldsymbol{\omega} R, t) & \\ = -4\pi R^2 \int_0^\infty \mathrm{d}t\, \pi(R, t)\phi'(R, t) & \\ = -4\pi \int_0^\infty \mathrm{d}t\, R\pi(R, R+t) R\phi'(R, R+t)\,. & \end{aligned} \tag{6.33}$$

The first step uses radial symmetry of the solution to (6.30), while retaining the notation for the radial fields and setting $\phi'(R, t) = \partial_R\phi(R, t)$, and the second step uses the condition that the solution is supported inside the light cone. To separate

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between near and far field one still has to take the limit $R \to \infty$ in (6.33). Thus

$$E_{\mathrm{diss}} = \lim_{R\to\infty} -4\pi \int_0^\infty \mathrm{d}t\, R\pi(R, R+t) R\phi'(R, R+t)\,. \tag{6.34}$$

The fundamental solution of (6.30) is

$$\begin{pmatrix}\phi(t)\\ \pi(t)\end{pmatrix} = \begin{pmatrix}\partial_t G & G\\ \partial_t^2 G & \partial_t G\end{pmatrix}\begin{pmatrix}\phi\\ \pi\end{pmatrix}. \tag{6.35}$$

G is the propagator for $t \geq 0$,

$$G(\boldsymbol{x}, t) = \frac{1}{4\pi|\boldsymbol{x}|}\delta(t - |\boldsymbol{x}|) - \frac{1}{4\pi}\kappa^2 F\big(\kappa\sqrt{t^2 - \boldsymbol{x}^2}\big)\chi(\boldsymbol{x}^2 \leq t^2) \tag{6.36}$$

with

$$F(z) = \frac{1}{z} J_1(z) \tag{6.37}$$

and J_1 the integer Bessel function of order 1. For the initial conditions $\phi = 0, \pi = 0$ the solution to (6.30) is then

$$\phi(\boldsymbol{x}, t) = \int_0^t \mathrm{d}s\, G(\boldsymbol{x}, t-s)\alpha(s)\,, \quad \pi(\boldsymbol{x}, t) = \int_0^t \mathrm{d}s\, \partial_t G(\boldsymbol{x}, t-s)\alpha(s)\,. \tag{6.38}$$

Before inserting them in (6.34) both terms have to be somewhat simplified through partial integrations using the condition that $\alpha(0) = 0$. For the momentum field one obtains

$$4\pi R\pi(R, R+t) = \dot\alpha(t) - \kappa^2 R \int_0^t \mathrm{d}s\, F\left(\kappa\sqrt{(t-s)(2R+t-s)}\right)\dot\alpha(s)\,. \tag{6.39}$$

For the scalar field there are two subleading contributions, which vanish as $R \to \infty$, and the leading term

$$4\pi R\phi'(R, R+t) = -\dot\alpha(t) + \kappa^2 R \int_0^t \mathrm{d}s\, F\left(\kappa\sqrt{(t-s)(2R+t-s)}\right) \times \frac{R}{R+t-s}\dot\alpha(s) + \mathcal{O}\Big(\frac{1}{R}\Big)\,. \tag{6.40}$$

We insert (6.39) and (6.40) into (6.33), which results in four terms. The first one is clearly $(4\pi)^{-1}\int_0^\infty \mathrm{d}t\dot\alpha(t)^2$. For the cross-term the integral involving F converges to $\dot\alpha(t)$ as $R \to \infty$. Thus the cross-terms add up to $-(2\pi)^{-1}\int_0^\infty \mathrm{d}t\dot\alpha(t)^2$. The fourth term requires more work. The t-integration of (6.33) is split into $[0, T]$ and $[T, \infty]$. The first integral yields $(4\pi)^{-1}\int_0^\infty \mathrm{d}t\dot\alpha(t)^2$, thereby cancelling terms

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1 to 3. The remainder is

$$E_{\text{diss}} = \lim_{R\to\infty} \frac{1}{4\pi} \int_0^T \mathrm{d}s \dot{\alpha}(s) \int_0^T \mathrm{d}s' \dot{\alpha}(s') \int_T^\infty \mathrm{d}t \kappa^4 RF\Big(\kappa\sqrt{(t-s)(2R+t-s)}\Big)$$
$$\times RF\Big(\kappa\sqrt{(t-s')(2R+t-s')}\Big) \frac{R}{R+t-s'} . \tag{6.41}$$

At this point one can use the asymptotics of J_1 for large arguments leading through oscillating integrands to

$$E_{\text{diss}} = \frac{1}{2\pi} \int_\kappa^\infty \mathrm{d}\omega \frac{1}{\omega} \sqrt{\omega^2 - \kappa^2} |\omega \widehat{\alpha}(\omega)|^2 . \tag{6.42}$$

In the limit $\kappa \to 0$ one obtains the familiar analog of the Larmor formula as

$$E_{\text{diss}} = \frac{1}{4\pi} \int \mathrm{d}t \dot{\alpha}(t)^2 . \tag{6.43}$$

If α has slow time variation, incorporated as $\alpha(\varepsilon t)$, $\varepsilon \ll 1$, then

$$E_{\text{diss}} = \varepsilon \frac{1}{4\pi} \int \mathrm{d}t \dot{\alpha}(t)^2 , \tag{6.44}$$

which in our working example would determine the time scale for radiation damping. On the other hand, for $\kappa > 0$

$$E_{\text{diss}} = \varepsilon \frac{1}{2\pi} \int_{\kappa/\varepsilon}^\infty \mathrm{d}\omega \frac{1}{\omega} \sqrt{\omega^2 - (\kappa/\varepsilon)^2} |\omega \widehat{\alpha}(\omega)|^2 . \tag{6.45}$$

If $\widehat{\alpha}$ has exponential decay, $\widehat{\alpha}(\omega) \cong \mathrm{e}^{-\gamma|\omega|}$ for large $|\omega|$, then $E_{\text{diss}} = \varepsilon \mathrm{e}^{-\gamma\kappa/\varepsilon}$. The low frequencies of the source do not couple to the medium.

If photons were massive, the adiabatic motion of charges would be protected in the sense that radiation damping is of order $\mathrm{e}^{-1/\varepsilon}$ rather than of order ε^2 as is the case for photons with dispersion $\omega(\boldsymbol{k}) = c|\boldsymbol{k}|$.

6.2 Comparison with the hydrodynamic limit

In hydrodynamics one assumes that a small droplet of fluid with center $\boldsymbol{r}$ has its intrinsic velocity, $\boldsymbol{u}(\boldsymbol{r})$, and that relative to the moving frame the particles are distributed according to thermal equilibrium with density $\rho(\boldsymbol{r})$ and temperature $T(\boldsymbol{r})$. For such notions to be reasonably well defined, the hydrodynamic fields $\rho, \boldsymbol{u}, T$ must be slowly varying on the scale of the typical interparticle distance. This is how the analogy with the Maxwell–Newton equations arises. As for them we have three characteristic space-time scales.

(i) *Microscopic scale*. The microscopic scale is measured in units of a collision time, respectively interatomic distance. On that scale the hydrodynamic fields are frozen. Possible deviations from local equilibrium relax through collisions. To prove such behavior one has to establish a sufficiently fast relaxation to equilibrium. For Newtonian particles no general method is available. For the Maxwell field the situation is much simpler. Local deviations from the Coulomb field are transported off to infinity and are no longer seen.

(ii) *Macroscopic Euler scale*. The macroscopic space-time scale is defined by the variation of the hydrodynamic fields. If, as before, we introduce the dimensionless scaling parameter ε, then space-time is $\mathcal{O}(\varepsilon^{-1})$ in microscopic units. On the macroscopic scale the time between collisions is $\mathcal{O}(\varepsilon)$, the interparticle distance $\mathcal{O}(\varepsilon)$, and the pair potential for the particle at position $\boldsymbol{q}_i$ and the one at $\boldsymbol{q}_j$ is $V(\varepsilon^{-1}(\boldsymbol{q}_i - \boldsymbol{q}_j))$. On the macroscopic scale the hydrodynamic fields evolve according to the Euler equations. These are first-order equations, which must be so, since space and time are scaled in the same way. The Euler equations are of Hamiltonian form. There is no dissipation, and no entropy is produced. In fact, there is a slight complication here. Even for smooth initial data the Euler equations develop shock discontinuities. There the assumption of slow variation fails and shocks are a source of entropy.

(iii) *Macroscopic friction scale*. In a real fluid there are frictional forces which are responsible for the relaxation to global equilibrium. One adds to the Euler equations diffusive-like terms, which are second order in spatial derivatives, and obtains the compressible Navier–Stokes equations incorporating the shear and volume viscosity resulting from friction in momentum transport and thermal conductivity resulting from friction in energy transport. On the macroscopic scale these corrections are of order ε. In the same spirit, based on the full Maxwell–Newton equations, there will be dissipative terms of order ε which have to be added to (6.21). Of course, in this context one has to deal only with ordinary differential equations as effective dynamics.

6.3 Point-charge limit, negative bare mass

The conventional point-charge limit is to let the diameter of the charge distribution tend to zero under the condition that the total charge remains fixed. Accordingly, let us consider now R_φ as a reference scale and let $R/R_\varphi \to 0$. Then for the point charge one sets

$$\varphi_R(\boldsymbol{x}) = R^{-3}\varphi(\boldsymbol{x}/R) \tag{6.46}$$

and takes the limit $R \to 0$. This means that the charge diameter is small in units of the variation of the external potential, since this is the only other length scale available. At first sight, one just seems to say that the potentials vary slowly on

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the scale set by the charge diameter and that hence the point-charge limit and the adiabatic limit coincide. To see the difference let us consider the electrostatic energy

$$\frac{1}{2} e^2 \int d^3k |\widehat{\varphi}_R(\boldsymbol{k})|^2 \frac{1}{\boldsymbol{k}^2} = \frac{1}{R} m_f . \tag{6.47}$$

In particular, the ratio of field mass to bare mass grows as R^{-1} in the point-charge limit and remains constant in the adiabatic limit.

To display the order of magnitude of the various dynamical contributions we resort again to our standard example of an electron in a uniform magnetic field $\boldsymbol{B}_{ex} = B\widehat{\boldsymbol{n}}$, $\widehat{\boldsymbol{n}} = (0, 0, 1)$ with B of the order of 1 tesla $= 10^4$ gauss, say. It suffices to consider small velocities. In the adiabatic limit we set $B = \varepsilon B_0$ where the reference field is $B_0 = 1.1 \times 10^{17}$ gauss; compare with appendix 1 to section 6.1. Up to higher-order corrections, the motion of the electron is then governed by

$$\Big(m_b + \frac{4}{3} m_f\Big)\dot{\boldsymbol{v}} = \frac{e}{c} \varepsilon B_0(\boldsymbol{v} \times \widehat{\boldsymbol{n}}) + \frac{e^2}{6\pi c^3} \ddot{\boldsymbol{v}} + \mathcal{O}(\varepsilon^3) \tag{6.48}$$

on the microscopic scale. Going over to the macroscopic time scale, $t' = \varepsilon^{-1} t$, (6.48) becomes

$$\Big(m_b + \frac{4}{3} m_f\Big)\dot{\boldsymbol{v}} = \frac{e}{c} B_0(\boldsymbol{v} \times \widehat{\boldsymbol{n}}) + \frac{e^2}{6\pi c^3} \varepsilon \ddot{\boldsymbol{v}} + \mathcal{O}(\varepsilon^2) . \tag{6.49}$$

Setting $m_0 = m_b + \frac{4}{3} m_f$, $\omega_c^0 = e B_0/m_0 c$, $\beta = e^2/6\pi c^3 m_0$, and restricting to the motion on the critical manifold, as will be explained in chapter 9, Eq. (6.49) becomes

$$\dot{\boldsymbol{v}} = \omega_c^0\big(\boldsymbol{v} \times \widehat{\boldsymbol{n}} + \varepsilon\beta\omega_c^0 (\boldsymbol{v} \times \widehat{\boldsymbol{n}}) \times \widehat{\boldsymbol{n}}\big) + \mathcal{O}(\varepsilon^2) , \tag{6.50}$$

equivalently, on the microscopic time scale

$$\dot{\boldsymbol{v}} = \omega_c\big(\boldsymbol{v} \times \widehat{\boldsymbol{n}} + \beta\omega_c(\boldsymbol{v} \times \widehat{\boldsymbol{n}}) \times \widehat{\boldsymbol{n}}\big) + \mathcal{O}(\varepsilon^3) \tag{6.51}$$

with the cyclotron frequency $\omega_c = e\,\varepsilon B_0/m_0 c = eB/m_0 c$.

For the point-charge limit we rely on the Taylor expansion of section 7.2. Then, for small velocities,

$$\Big(m_b + R^{-1} \frac{4}{3} m_f\Big)\dot{\boldsymbol{v}} = \frac{e}{c} B(\boldsymbol{v} \times \widehat{\boldsymbol{n}}) + \frac{e^2}{6\pi c^3} \ddot{\boldsymbol{v}} + \mathcal{O}(R) . \tag{6.52}$$

Since based on the same expansion, as long as no limit is taken, of course, we can switch back and forth between (6.52) and (6.48), respectively (6.49), provided the appropriate units are used. This can be seen more easily if we accept momentarily

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the differential–difference equation

$$m_{\mathrm{b}}\dot{\boldsymbol{v}}(t) = e\big(\boldsymbol{E}_{\mathrm{ex}}(\boldsymbol{q}(t)) + c^{-1}\boldsymbol{v}(t) \times \boldsymbol{B}_{\mathrm{ex}}(\boldsymbol{q}(t))\big) + \frac{e^2}{12\pi c R^2}\big(\boldsymbol{v}(t - 2c^{-1}R) - \boldsymbol{v}(t)\big)\,, \tag{6.53}$$

which is exact for a uniformly charged sphere at small velocities, see section 7.1. If we expand in the charge diameter R, then

$$\Big(m_{\mathrm{b}} + \frac{e^2}{6\pi Rc^2}\Big)\dot{\boldsymbol{v}} = e(\boldsymbol{E}_{\mathrm{ex}} + c^{-1}\boldsymbol{v} \times \boldsymbol{B}_{\mathrm{ex}}) + \frac{e^2}{6\pi c^3}\ddot{\boldsymbol{v}} + \mathcal{O}(R)\,, \tag{6.54}$$

which is the analog of (6.52). On the other hand, if we assume that the external fields are slowly varying, as explained in section 6.1, then on the macroscopic scale

$$\varepsilon m_{\mathrm{b}}\dot{\boldsymbol{v}}(t) = \varepsilon\, e\big(\boldsymbol{E}_{\mathrm{ex}}(\boldsymbol{q}(t)) + c^{-1}\boldsymbol{v}(t) \times \boldsymbol{B}_{\mathrm{ex}}(\boldsymbol{q}(t))\big) + \frac{e^2}{12\pi c R_\varphi^2}\big(\boldsymbol{v}(t - 2\varepsilon c^{-1}R_\varphi) - \boldsymbol{v}(t)\big)\,, \tag{6.55}$$

where R_φ is now regarded as fixed. Taylor expansion in ε yields

$$\Big(m_{\mathrm{b}} + \frac{e^2}{6\pi R_\varphi c^2}\Big)\dot{\boldsymbol{v}} = e(\boldsymbol{E}_{\mathrm{ex}} + c^{-1}\boldsymbol{v} \times \boldsymbol{B}_{\mathrm{ex}}) + \varepsilon\,\frac{e^2}{6\pi c^3}\ddot{\boldsymbol{v}} + \mathcal{O}(\varepsilon^2) \tag{6.56}$$

which is the analog of (6.49).

As can be seen from (6.52), in the point-charge limit the total mass becomes so large that the particle hardly responds to the magnetic field. The only way out seems to formally compensate the diverging $R^{-1}\,(4/3)m_{\mathrm{f}}$ by setting

$$m_{\mathrm{b}} = -R^{-1}\,(4/3)\,m_{\mathrm{f}} + m_{\mathrm{exp}} \tag{6.57}$$

with m_{exp} the experimental mass of the charged particle. But this is asking for trouble, since the energy (2.44) is no longer bounded from below and potential energy can be transferred to kinetic mechanical energy without limit. To see this mechanism in detail we consider the Abraham model with $\boldsymbol{B}_{\mathrm{ex}} = 0$ and ϕ_{ex} varying only along the 1-axis. The bare mass of the particle is now $-m_{\mathrm{b}}$, with $m_{\mathrm{b}} > 0$ as before. We set $\boldsymbol{q}(t) = (q_t, 0, 0)$, $\boldsymbol{v}(t) = (v_t, 0, 0)$, $\boldsymbol{E}_{\mathrm{ex}} = (-\phi'(q), 0, 0)$. ϕ is assumed to be strictly convex with a minimum at $q = 0$. Initially the particle is at rest at the minimum of the potential. Thus $\boldsymbol{E}(\boldsymbol{x}, 0) = \boldsymbol{E}_0(\boldsymbol{x})$ from (4.5) and $\boldsymbol{B}(\boldsymbol{x}, 0) = 0$. We now give the particle a slight kick to the right, which means $q_0 = 0$, $v_0 > 0$. By

https://doi.org/10.1017/9781009402286.007 Published online by Cambridge University Press

conservation of energy

$$-m_b c^2 \gamma(v_t) + e\,\phi(q_t) + \frac{1}{2}\int d^3x \left(\boldsymbol{E}(\boldsymbol{x},t)^2 + \boldsymbol{B}(\boldsymbol{x},t)^2\right)$$
$$= -m_b c^2 \gamma(v_0) + e\phi(q_0) + \frac{1}{2}\int d^3x\, \boldsymbol{E}(\boldsymbol{x},0)^2\,. \qquad (6.58)$$

We split $\boldsymbol{E}$ into longitudinal and transverse components, $\boldsymbol{E} = \boldsymbol{E}_\parallel + \boldsymbol{E}_\perp$, $\widehat{\boldsymbol{E}}_\parallel = \widehat{\boldsymbol{k}}(\widehat{\boldsymbol{k}}\cdot\widehat{\boldsymbol{E}})$. Clearly $\int d^3x\, \boldsymbol{E}_\parallel \cdot \boldsymbol{E}_\perp = 0$ and therefore

$$\int d^3x \boldsymbol{E}(\boldsymbol{x},t)^2 \geq \int d^3x \boldsymbol{E}_\parallel(\boldsymbol{x},t)^2 = \int d^3k (\widehat{\boldsymbol{k}}\cdot\widehat{\boldsymbol{E}}(\boldsymbol{k},t))^2$$
$$= e^2 \int d^3k\, |\boldsymbol{k}|^{-2}\, |\widehat{\varphi}(\boldsymbol{k})|^2 = \int d^3x\, \boldsymbol{E}(\boldsymbol{x},0)^2\,, \qquad (6.59)$$

since the initial field has zero transverse component. Inserting in (6.58) yields

$$\dot{q}_t^2 \geq 1 - \left[\gamma(v_0) + (e/m_b c^2)(\phi(q_t) - \phi(q_0))\right]^{-2}. \qquad (6.60)$$

Since $\gamma(v_0) > 1$, $\dot{q}_t > 0$ for short times. As the particle moves to the right, $(\phi(q_t) - \phi(q_0))$ is increasing and therefore $\dot{q}_t \to 1$ and $q_t \to \infty$ as $t \to \infty$. Note that v_0 and m_b can be arbitrarily small. Not surprisingly, the Abraham model with a negative bare mass behaves rather unphysically. A tiny initial kick suffices to generate a runaway solution.

The point-charge limit is honored by a long tradition, which however seems to have constantly overlooked that physically it is more appropriate to have the external potentials slowly varying on the scale of a fixed-size charge distribution. Then there is no need to introduce a negative bare mass and there are no runaway solutions.

Notes and references

Section 6

The importance of slowly varying external potentials has been emphasized repeatedly. In the early literature slow variation appears as the quasi-stationary hypothesis and quasi-stationary motion (Miller 1997). Such principles remain vague and, interestingly enough, in more mathematical considerations the size R_φ of the charge distribution is taken as an expansion parameter rather than the appropriate parameter in the potential. To me it is rather surprising that, apparently, there is no systematic study of the equations of motion with external potentials of slow variation. We use the notion "adiabatic limit" to correspond to the adiabatic theorem in classical and quantum mechanics which refers to a Hamiltonian with slow time-dependence. More appropriately we should speak of "space-adiabatic limit", since the slow variation is in space, the slow variation in time being a consequence.

Section 6.1

In the context of charges coupled to the Maxwell field the adiabatic limit was first introduced in Komech, Kunze and Spohn (1999) and in Kunze and Spohn (2000a). The fundamental solution (6.36) of the Klein–Gordon equation is discussed in Morse and Feshbach (1953). De Bièvre (private communication) points out that the dissipated energy (6.42) can be guessed also from elementary considerations. In Fourier space the wave equation becomes

$$\partial_t^2 \widehat{\phi}_t(\boldsymbol{k}) = -\omega(\boldsymbol{k})^2 \widehat{\phi}_t(\boldsymbol{k}) + (2\pi)^{-3/2}\alpha(t) \tag{6.61}$$

with $\omega(\boldsymbol{k})^2 = \boldsymbol{k}^2 + \kappa^2$. For a forced harmonic oscillator the equation of motion reads $\ddot{x} = -\omega^2 x + f(t)$ and the energy transferred by the forcing is $\pi|\widehat{f}(\omega)|^2$. Inserting in (6.61) and integrating over all $\boldsymbol{k}$ yields (6.42). Schwinger (1949) uses a similar argument for the radiated energy.

Section 6.2

A more detailed discussion of the hydrodynamic limit can be found in Spohn (1991).

Section 6.3

In the early days of classical electron theory, one simply expanded in R_φ. R_φ was considered to be small, but finite, roughly of the order of the classical electron radius. Schott (1912) pushes the expansion to include the radiation reaction. Apparently, the notion of a point charge is first stated explicitly by Frenkel (1925). The difficulties resulting from the point charge were clearly understood by P. Ehrenfest as stressed by Pauli in his 1933 obituary. The point-charge limit is at the core of the famous Dirac (1938) paper, cf. section 3.3. Since then the limit $m_\mathrm{b} \to -\infty$ has become a standard piece of the theory, reproduced in textbooks and survey articles. The negative bare mass was soon recognized as a source of instability. We refer to the review by Erber (1961). On a linearized level stability is studied by Wildermuth (1955) and by Moniz and Sharp (1977) and Levine, Moniz and Sharp (1977). Bambusi (1996), Bambusi and Noja (1996), and Noja and Posilicano (1998, 1999) discuss the point-charge limit in the dipole approximation and show that then the true solution is well approximated by the linear Lorentz–Dirac equation with the full, both physical and unphysical, solution manifold explored. An extension to the nonlinear theory is attempted by Marino (2002). The bound (6.60) is taken from Bauer and Dürr (2001), which seems to be the only quantitative handling of the instability for the full nonlinear problem.