Introduction

This book has been devised for students who, having perhaps taken a first course in functional analysis but not being ready for a second, more advanced one, would like to take another look at the subject. To accommodate their needs, we do go through basic notions, but the stress on a systematic study is considerably lighter than in other monographs. Instead, our main goal is to show how the notion of *completeness* permeates functional analysis and modern mathematical analysis as a whole.

Hence, this is not a basic course in functional analysis. Neither is it a second course; we do not even dare to touch more advanced topics; we want to spend more time thinking about the basics. It is thus an in-between course, an intermediate course: ‘Functional Analysis Revisited.’ Perhaps a preparation for a second course, perhaps not.

At the same time, the book is a testimony to the author’s fascination with the simplicity and beauty of the notion of completeness. Nearly every chapter contains a main result, an important one for a branch of mathematics, or science in general, that crucially hinges on the completeness of a metric space involved. We learn thus that, were it not for completeness, Achilles would not catch the tortoise, a functional series would not converge, a contraction mapping would not have a fixed point, and differential equation would not have a solution; neither would a renewal equation, and as a result a model of population growth would be rendered useless. To continue the list: a continuous function would not be Riemann integrable, and thus the fundamental theorem of calculus would cease to be true; polynomials would not approximate continuous functions well; we would not be able to project on subspaces of Hilbert spaces; and we would have an issue with solving the all-important heat equation. Without the notion of completeness, mathematics, as we know it today, would not exist, or at least it would be in a very pitiful state.

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It is thus no surprise that Banach spaces, that is, normed linear spaces that are complete, play a central role in this little book. Banach spaces are appealing blends of algebraic and topological notions, with a structure simple enough to be found everywhere around us, and at the same time rich enough to lead to satisfactory, deep theory including the uniform boundedness principle of Banach and Steinhaus, and the open mapping and closed graph theorems. For a mathematician, they are a pleasure to look at and a joy to discover in the heart of applied problems.

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Technically – though corrected, expanded, rearranged and rethought – the book is a translation of my *Analiza funkcjonalna jeden i pół. Szkic o zupełności*, published (in Polish) by Lublin University of Technology Press in 2015. The original has served as a textbook for several courses in functional analysis at this university, and was used by my esteemed colleagues at Silesian University and Łódź University of Technology. I am very grateful for all the encouragement and critical remarks that have made the present edition better than the original.

As compared with the Polish edition, besides new figures and pictures, new material added or corrected in certain sections (like the Hausdorff Moment Problem) and a few new sections (like the ones devoted to convergence of Fourier series, the Fejér theorem (Section 14.4) and uniform convergence (Section 15.4) in particular), there are two completely new chapters. These are Chapters 15 and 16.

In Chapter 15 we stop skimming the surface and finally go a bit deeper into the structure of complete spaces. Namely, we discuss the Baire Category theorem and see the Banach–Steinhaus uniform boundedness principle as a consequence of this fundamental result. This leads us naturally to the open mapping and closed graph theorems, and the perspective that these provide on linear operators is so appealing that we are not able to resist the temptation of subsequently touring the land of semigroups of operators in Chapter 16. I hope readers will enjoy this new material, and will find it useful in their own pursuit of mathematics.

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Every chapter ends with a short, non-technical summary of its most important results, preceded by plenty of reasonably-pitched exercises; a few that are a little more demanding are marked with a sign. Some of the exercises are taken directly from the original Polish edition; some were created for the sake

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of midterm and final exams that took place after the Polish book's publication, and thus in part are due to Adam Gregosiewicz.

I am grateful to many colleagues, including R. Bogucki and B. Przeradzki, who notified me about a number of typos in the Polish version of the book; these errors were of course corrected in this edition. I owe special thanks to W. Chojnacki for his constant efforts to make my English, and the English of this book in particular, correct and more readable (to make it simple is *mission impossible*). E. Ratajczyk diligently read the entire apparently ready-to-be-published text, and still found a great number of misprints. Due to her efforts, there is hope that the set of errors is nowhere dense in this book – at most a set of the first category.

Finally, I would like to thank the CUP team, including R. Astely, A. Jacobsen and C. Dennison for their professional support.