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LOCALLY COMPACT TIGHT RIESZ GROUPS

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1. Introduction

It is known that a strongly archimedean locally compact tight Riesz group without pseudozeros is essentially R^m with the usual topology and tight order. We show that a locally compact tight Riesz group, $(G, \leqq)$, without pseudozeros, is algebraically and topologically isomorphic with $R^m \oplus D$, where D is discrete. $R^m \oplus \{0\}$ is a clopen o-ideal; and we give necessary and sufficient conditions for G to be isomorphic with $R^m \oplus D$ in all respects. Further $(G, \leqq)$ contains an o-ideal isomorphic with $R^m \oplus \sum_{\aleph} Z$ and G is isomorphic with it if and only if $(G, \leqq, \mathscr{U})$ is interval-compact.

2. Preliminaries

Let $(G, \leqq)$ be an abelian pogroup. Denote the set $\{x: x \geqq 0\}$ by P and the set $\{x: x > 0\}$ by P^*. We say x is *pseudopositive* if $x + p > 0$ for all $p > 0$ and $x \ngeqq 0$. We say x is a *pseudozero* if x and $-x$ are pseudopositive. The intervals

$$(a, b) = \{x: a < x < b\} \quad a, b \in G$$

form a subbase for the *open-interval topology* $\mathscr{U}$ on G.

If $(G, \leqq)$ is dense then $(G, \mathscr{U})$ is a topological group. Also $(G, \mathscr{U})$ is T_1 if and only if $(G, \leqq)$ has no pseudozeros. Denote the closure of $S \subseteq G$ in $\mathscr{U}$ by S^-. We say $(G, \leqq, \mathscr{U})$ is *interval-compact* Cameron and Miller (to appear) if $(a, b)^-$ is compact for every $a < b$.

If $(G, \leqq)$ has no pseudozeros we write $x \succ 0$ to mean that $x > 0$ or that x is pseudopositive. Then $(G, \preccurlyeq)$ is a pogroup and $\preccurlyeq$ is called the *associated order*. We write $a \gg b$ if $a \preccurlyeq nb$ for all integers n, and $(G, \leqq)$ is said to be *archimedean* if $a \gg b$ implies that $b = 0$. If $(G, \leqq)$ is an l-group, we say $(G, \leqq)$ is a *complete* l-group if each bounded subset of G has a sup and an inf. Every complete l-group is archimedean. If, given $a, b > 0$ there exists some integer n such that $na > b$, we say $(G, \leqq)$ is *strongly archimedean* (Loy and Miller (1972) call this archimedean). A strongly archimedean l-group is archimedean. We say $(G, \leqq)$ has the

TR(m, n) *interpolation property*, Cameron and Miller (to appear), if, given $a_1, \cdots, a_m, b_1, \cdots, b_n \in G$ such that $a_i < b_j$ for $i = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, n$ there exists $c \in G$ such that $a_i < c < b_j$ for $i = 1, \cdots, m$ and $j = 1, \cdots, n$. A *tight Riesz group* (abbreviated TRG) is a directed abelian pogroup satisfying TR(2, 2). We say a pogroup has *small elements* if given $a > 0$ and n, a positive integer, there exists $b > 0$ such that $a > nb$. Divisible isolated pogroups and pogroups satisfying TR(1, 2) have small elements.

Let $\{G_i, \preccurlyeq\}_{i \in I}$ be a family of l-groups, denote their (small) direct sum by $\sum G_i$ and define $\preccurlyeq$ on $\sum G_i$ by, $(g_i) \succcurlyeq 0$ if $g_i \succcurlyeq 0$ for all $i \in I$. Denote the direct sum of two groups G_1 and G_2 by $G_1 \oplus G_2$. If $(G, \preccurlyeq)$ is an l-group and S a subset of G denote the set

$$\{x : |x| \wedge |s| = 0 \text{ for all } s \in S\}$$

by $S^{\perp}$. Let Z, R denote the abelian groups of integers and reals. If $x = (x_1, \cdots, x_n) \in R^n$ we say $x > 0$ if $x_i > 0$ for $i = 1, \cdots, n$. We call this the *tight order*. Then the open-interval topology is precisely the euclidean topology $\mathscr{E}$, and $x \succcurlyeq 0$ if $x_i \geqq 0$ for $i = 1, \cdots, n$.

3. Locally compact and interval-compact TRGs

Let $(G, \leqq)$ be a non-trivially ordered abelian pogroup. We recall two results of Loy and Miller (1972), in fact Theorem 1 is the main result of that paper.

LEMMA 1. (Loy and Miller). *Let $(G, \leqq)$ be a locally compact TRG such that $(G, \preccurlyeq)$ is an l-group, then $(G, \mathscr{U})$ is algebraically and topologically isomorphic with $R^m \oplus D$, where m is a positive integer and D is discrete.*

PROOF. This follows immediately from §4, 6° of Loy and Miller (1972) and a theorem of G. W. Mackey's (see Loy and Miller (1972)).

THEOREM 1. (Loy and Miller) *Let $(G, \leqq, \mathscr{U})$ be a strongly archimedean locally compact TRG without pseudozeros, then it is isomorphic in all respects with $(R^m, \leqq, \mathscr{E})$ for some positive integer m.*

The result of Lemma 1 can be generalized, without changing the method of proof, to the following Lemma.

LEMMA 2. *Let $(G, \leqq, \mathscr{U})$ be an isolated locally compact pogroup with small elements and without pseudozeros, then $(G, \mathscr{U})$ is algebraically and topologically isomorphic with $R^m \oplus D$, where m is a non-negative integer and D is discrete.*

We also give a modification of Theorem 1 in terms of small elements.

THEOREM 2. *Let $(G, \leqq, \mathscr{U})$ be a strongly archimedean locally compact po-group with small elements such that $(G, \preccurlyeq)$ is an l-group, then it is isomorphic in all respects with $(R^m, \leqq, \mathscr{E})$ for some positive integer m.*

PROOF. Firstly we show that $(G, \leqq)$ is isolated. If $nx > 0$ for some positive integer n, then for some $p > 0$, $nx > np$. So certainly $x \succcurlyeq p$, since $(G, \preccurlyeq)$ is isolated, and so $x > 0$. Now we show that $(G, \leqq)$ is TR(1,2). Let $a, b > 0$, then for some positive n, $na > b$ and for some $c > 0$, $b > nc$. So $a, b > c > 0$, and hence $(G, \leqq)$ is TR(1,2). It can easily be shown that if $(G, \leqq)$ is TR(1,2) and $(G, \preccurlyeq)$ is an l-group then in fact $(G, \leqq)$ is TR(2,2). Clearly $(G, \leqq)$ is directed, so it is a TRG. Now apply Theorem 1.

If we remove "strongly archimedean" from the hypothesis of Theorem 1, we can still show that at least an o-ideal (convex directed subgroup) of $(G, \leqq)$ is isomorphic with R^m.

THEOREM 3. *Let $(G, \leqq, \mathscr{U})$ be an isolated locally compact TRG without pseudozeros, then there exists a clopen o-ideal H and a positive integer m such that:*

(i) *$(H, \leqq, \mathscr{U})$ is isomorphic in all respects with $(R^m, \leqq, \mathscr{E})$,*

(ii) *if $x \in P^*$ and $y \in H \cap P^*$ then $nx > y$ for some positive integer n,*

(iii) *if $(G, \preccurlyeq)$ is an l-group then $(H, \preccurlyeq)$ is an l-ideal.*

PROOF. Let θ denote the isomorphism indicated in Lemma 2. If $\alpha \in R^m$ and $\alpha > 0$ then for some $a \in P^*$, we have

$$\theta\{(-a, a)\} \subseteq (-\alpha, \alpha) \oplus \{0\}.$$

In a TRG $(0, na) = (0, a) + \cdots + (0, a)$. (§2, 4° of Loy and Miller (1972)), so $\theta\{(0, na)\} \subseteq R^m \oplus \{0\}$ and $\theta\{(-na, 0)\} \subseteq R^m \oplus \{0\}$. Let

$H = \{x\colon -na < x < na$ for some positive integer $n\}$, then it is easily shown that $\theta(H) \subseteq R^m \oplus \{0\}$. Also for some positive integer k,

$$\theta\{(-a, a)\} \supseteq \left(-\frac{\alpha}{k}, \frac{\alpha}{k}\right) \oplus \{0\}.$$

So $\theta(H) = R^m \oplus \{0\}$. It is easily shown that H is an o-ideal in $(G, \leqq)$. H is an open subgroup of a topological group and hence closed. Also the induced topology on H is homeomorphic with the open-interval topology on $(H, \leqq)$, hence $(H, \leqq)$ has no pseudozeros and is locally compact in its open-interval topology.

If $b \in H \cap P^*$ then for some $\beta \in R^m$, $\beta > 0$, we have

$$\theta\{(-b, b)\} \supseteq (-\beta, \beta) \oplus \{0\},$$

so $H = \{x\colon -nb < x < nb$ for some positive integer $n\}$. Hence $(H, \leqq)$ is strongly archimedean. So by Theorem 1, (i) follows.

If $x \in P^*$ and $y \in H \cap P^*$ then there exists $z \in H \cap P^*$ such that $x, y > z > 0$. By (i) $nz > y$ for some positive n, so $nx > y$. The proof of (iii) is trivial.

We show in Theorem 4 that the result of Theorem 3 can be improved; in fact the condition that $(G, \leqq)$ is isolated can be deleted.

If $(G, \leqq)$ is a locally compact TRG such that $(G, \preccurlyeq)$ is an l-group, is the isomorphism referred to in Lemma 1 also an order isomorphism? That is, is $(D, \preccurlyeq)$ an l-group such that, if $a \in G$ and $\theta(a) = \langle \alpha, d \rangle$ we have $a > 0$ if and only if $\alpha > 0$ and $d \succcurlyeq 0$, and $a \succcurlyeq 0$ if and only if $\alpha \succcurlyeq 0$ and $d \succcurlyeq 0$?

LEMMA 3. *Let $(G, \preccurlyeq)$ be an l-group with an l-ideal $(H, \preccurlyeq)$ isomorphic with $(R^m, \preccurlyeq)$ for some positive integer m. Then $G \equiv H \oplus H^{\perp}$ if and only if there does not exist $x \in G$ and $h \in H$, $h \succ 0$ such that $x \gg h$.*

PROOF. Certainly the condition is necessary. Now suppose that $x \gg h$ is false for all $x \in G$ and all $h \in H$, $h \succ 0$. Let $a \in G$, $a \succ 0$, and consider the set $\{a \wedge h : h \in H\}$. This is a subset of H and so it is bounded above in H since otherwise, $a \gg e_i$, for some i, where $e_i = (0, 0, \cdots, 0, 1, 0, \cdots)$, $e_i \in R^m \equiv H$. Hence $\bigvee \{a \wedge h : h \in H\}$ exists for all $a \succ 0$, and belongs to H. It follows that $G \equiv H \oplus H^{\perp}$ (see proof of Theorem 16, Fuchs (1963; page 91)).

COROLLARY. *If $(G, \leqq)$ is a locally compact TRG such that $(G, \preccurlyeq)$ is an l-group, then G is isomorphic in all respects with $R^m \oplus D$, if and only if there does not exist an integer i and $x \in G$ with $x \gg e_i$.*

Denote the set $\{x \geqq 0; (0, x)^- \text{ is compact}\}$ by C. It is a consequence of a result in Cameron and Miller (to appear) that if $(G, \leqq)$ is an interval-compact TRG without pseudozeros, then $(G, \preccurlyeq)$ is a complete l-group. Below we characterize all such groups completely.

THEOREM 4. *Let $(G, \leqq, \mathscr{U})$ be a locally compact TRG without pseudozeros, then:*

(i) *C is a convex subsemigroup of $(P, \leqq)$,*

(ii) *the subgroup $(C - C, \preccurlyeq)$, generated by C, is an l-group isomorphic in all respects with $R^m \oplus \sum_{\aleph} Z$, for some positive integer m and cardinal $\aleph$,*

(iii) *the condition in Theorem 3 that $(G, \leqq)$ is isolated is redundant.*

PROOF. (i) It is obvious that C is convex. In §2, 14° Loy and Miller (1972) it is shown that C is a subsemigroup of P.

(ii) It follows easily that $(C - C, \leqq)$ is interval-compact in its open-interval topology. Also $(C - C, \leqq)$ is a TRG without pseudozeros, so $(C - C, \preccurlyeq)$ is a complete l-group. We now apply Lemma 3 Corollary. So $C - C$ is isomorphic in all respects with $R^m \oplus D_1$, for some discrete l-group $(D_1, \preccurlyeq)$. Further the strictly positive cone of $(D_1, \preccurlyeq)$ can have no infinite descending chains. For suppose that $d_1 \succ d_2 \succ \cdots \succ 0$, $d_i \in D_1$, and let $\alpha \in R^m$, $\alpha > 0$, and $a = \theta^{-1}\langle \alpha, d_1 \rangle$. By Lemma 3 Corollary $a > 0$ so $(0,a)^-$ is compact. Let $b = \theta^{-1}\langle \alpha, 0 \rangle$, and cover $(0,a)^-$ by sets $(x-b, x+b)$ where $x \in (0,a)^{-1}$. Then clearly $\theta^{-1}\langle 0, d_i \rangle$ all belong to different covers, contradicting the compactness of $(0,a)^-$. It follows from Birkhoff (1967; page 299) that $(D_1, \preccurlyeq)$ is isomorphic with $(\Sigma_{\aleph} Z, \preccurlyeq)$ for some cardinal $\aleph$.

(iii) Write $(H, \leqq)$ for $(R^m, \leqq)$ in (ii).

COROLLARY. *Let $(G, \leqq, \mathscr{U})$ be an interval-compact TRG, then G is isomorphic in all respects with $R^m \oplus \Sigma_{\aleph} Z$, for some positive integer m and some cardinal $\aleph$, and conversely.*

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