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From Matrix to Operator Inequalities

Terry A. Loring

Abstract. We generalize Löwner's method for proving that matrix monotone functions are operator monotone. The relation $x \leq y$ on bounded operators is our model for a definition of C^*-relations being residually finite dimensional.

Our main result is a meta-theorem about theorems involving relations on bounded operators. If we can show there are residually finite dimensional relations involved and verify a technical condition, then such a theorem will follow from its restriction to matrices.

Applications are shown regarding norms of exponentials, the norms of commutators, and "positive" noncommutative $*$-polynomials.

1 Introduction

This paper is about bounded operators that satisfy relations that involve algebraic relations, the operator norm, functional calculus, and positivity. The word positive, when applied to matrices, shall mean positive semidefinite.

The $*$-strong topology can bridge the gap between representations of relations by bounded operators on Hilbert space and representations by matrices. This feature of the $*$-strong topology has been noted before, for example by Löwner in [2], or in the context of residually finite dimensional C^*-algebras in [9].

This article is essentially independent of our previous paper [13] on C^*-relations. We minimize the role of universal C^*-algebras. Perhaps someone will see how to strip out the C^*-algebras and get a result that works for norms other than the operator norm.

Theorem 3.6, our main result, is a meta-theorem. We first define C^*-relations, and define for C^*-relations the concepts of closed and residually finite dimensional (RFD). The meta-theorem is that, given a theorem about matrices that states that an RFD C^*-relation implies a closed C^*-relation, we may conclude the same implication holds for all bounded operators.

2 C^*-Relations

Definition 2.1 Suppose $\mathcal{X}$ is a set. A statement R about functions $f\colon \mathcal{X} \to A$ into various C^*-algebras is a C^**-relation* if the following four axioms hold:

(R1) the unique function $\mathcal{X} \to \{0\}$ satisfies R;
(R2) if $\varphi\colon A \hookrightarrow B$ is an injective $*$-homomorphism and $\varphi \circ f\colon \mathcal{X} \to B$ satisfies R for some f, then $f\colon \mathcal{X} \to A$ also satisfies R;

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(R3) if $\varphi\colon A \to B$ is a $*$-homomorphism and $f\colon \mathcal{X} \to A$ satisfies R, then $\varphi \circ f\colon \mathcal{X} \to B$ also satisfies R;

(R4f) if $f_j\colon \mathcal{X} \to A_j$ satisfy R for $j = 1, \dots n$, then so does $f = \prod f_j$, where $f\colon \mathcal{X} \to \prod_{j=1}^{n} A_j$ sends x to $\langle f_1(x), \dots, f_n(x)\rangle$.

Examples of C^*-relations include the zero-sets of $*$-polynomials in noncommuting variables, henceforth called NC $*$-polynomials.

When $\mathcal{X} = \{1, 2, \dots, m\}$, the case we really care about, we use $a_1, \dots, a_m$ in place of the function notation $j(q) = a_q$. Given $p(x_1, \dots, x_m)$, an NC $*$-polynomial with constant-term zero, its zero-set is the C^*-relation $p(a_1, \dots, a_m) = 0$. Other C^*-relations associated with p include

$$p(a_1, \dots, a_m) \geq 0 \quad \text{and} \quad \left\| p(a_1, \dots, a_m) \right\| \leq C$$

for a constant $C > 0$, as well as $\|p(a_1, \dots, a_m)\| < C$.

Given a set $\mathcal{R}$ of C^*-relations, a function $f\colon \mathcal{X} \to A$ to a C^*-algebra A is called a *representation of* $\mathcal{R}$ *in* A if every statement in $\mathcal{R}$ is true for f. A function $\iota\colon \mathcal{X} \to U$ into a C^*-algebra is *universal* for $\mathcal{R}$ if ι is a representation of $\mathcal{R}$ and for every representation $f\colon \mathcal{X} \to A$ of $\mathcal{R}$ there is a unique $*$-homomorphism $\varphi\colon U \to A$ so that $\varphi \circ \iota = f$.

It is important to note that often there is no universal C^*-algebra and no universal representation. See [13].

We use the notation $C^*\langle \mathcal{X} \mid \mathcal{R}\rangle$ for U and call it the universal C^*-algebra. Notice that universal representation ι is usually what we should be talking about. Notice also that ι need not be injective, but still we often say that the representation $f\colon \mathcal{X} \to A$ of $\mathcal{R}$ extends to a unique $*$-homomorphism $\varphi\colon U \to A$ with the requirement $\varphi(\iota(x)) = f(x)$. A good exercise is to show that $\iota(\mathcal{X})$ must generate $C^*\langle \mathcal{X} \mid \mathcal{R}\rangle$ as a C^*-algebra. Alternately, this is clear from the proof of [13, Theorem 2.6].

Given a set $\mathcal{R}$ of C^*-relations on a set $\mathcal{X}$, we let $\operatorname{rep}_{\mathcal{R}}(\mathcal{X}, A)$ denote the set of all representations of $\mathcal{R}$ in A. If $\mathbb{H}$ is a Hilbert space, then we set

$$\operatorname{rep}_{\mathcal{R}}(\mathcal{X}, \mathbb{H}) = \operatorname{rep}_{\mathcal{R}}(\mathcal{X}, \mathbb{B}(\mathbb{H})).$$

The notation $\prod_{\lambda \in \Lambda} A_\lambda$ shall denote the C^*-algebra product consisting of all bounded sequences or families $\langle a_\lambda \rangle_{\lambda \in \Lambda}$ that have a_λ in A_λ. Given a family of functions $f_\lambda\colon \mathcal{X} \to A_\lambda$ we say it is *bounded* if $\sup_\lambda \|f_\lambda(x)\|$ is finite for all x in $\mathcal{X}$. For such a bounded family we define their *product* to be the function

$$\prod_{\lambda \in \Lambda} f_\lambda\colon \mathcal{X} \to \prod_{\lambda \in \Lambda} A_\lambda$$

that sends x to the family $\langle f_\lambda(x)\rangle$.

It could be argued that the above should be called the sum of the representations, see [5, Section II.6.1]. Notice that, given Hilbert spaces $\mathbb{H}_\lambda$, we have the inclusion (block diagonal)

$$\prod_\lambda \mathbb{B}(\mathbb{H}_\lambda) \subseteq \mathbb{B}\Big(\bigoplus_\lambda \mathbb{H}_\lambda\Big).$$

A potential source of confusion is that we are talking about representations of relations in C^*-algebras, but then often want to represent various C^*-algebras on a Hilbert space. Sometimes we cut out the middle man and talk about representations of relations on a Hilbert space. However, what we allow to be called C^*-relations on a set of operators are properties that can be determined by how they sit in the C^*-algebra they generate.

Definition 2.2 Suppose $\mathcal{R}$ is a set of C^*-relations on a set $\mathcal{X}$. We say $\mathcal{R}$ is *closed* if:

(R4b) for every bounded family $f_\lambda\colon \mathcal{X} \to A_\lambda$ of representations of $\mathcal{R}$ the product function $\prod f_\lambda$ is also a representation of $\mathcal{R}$.

We say $\mathcal{R}$ is *compact* if it satisfies the following even stronger axiom.

(R4) Every family $f_\lambda\colon \mathcal{X} \to A_\lambda$ of representations of $\mathcal{R}$ is bounded and the associated product function $\prod f_\lambda$ is a representation of $\mathcal{R}$.

Following Hadwin, Kaonga, Mathes ([11]), Phillips ([17]) and others, we showed in [13] that $\mathcal{R}$ is compact if and only if there is a universal C^*-algebra for $\mathcal{R}$.

For example $\{x^2 - x = 0,\ x^* - x = 0\}$ is compact and has universal C^*-algebra isomorphic to $\mathbb{C}$. Also compact is

$$\left\{ \|x^2 - x\| \leq \tfrac{1}{8},\ x^* - x = 0 \right\},$$

and this also has a universal C^*-algebra that is commutative. On the other hand $\{x^2 - x = 0\}$ is closed but not compact, while

$$\left\{ \|x^2 - x\| < \tfrac{1}{8},\ x^* - x = 0 \right\}$$

is not even closed.

It is sometimes easier to look at unital relations and unital C^*-algebras. Everything here carries over. Notice we are not putting 1 in $\mathcal{X}$, but use it symbolically in relations to stand for the unit in A when considering a function $f\colon \mathcal{X} \to A$.

The relations associated with NC $*$-polynomials are not the only interesting C^*-relations. The relation $0 \leq x \leq 1$ is compact, with universal C^*-algebra $C_0(0,1]$. Another relation on $\{x, y, z\}$ is $0 \leq \left[\begin{smallmatrix} y & x^* \\ x & z \end{smallmatrix}\right]$, which is to be interpreted so that X, Y, and Z in a C^*-algebra A form a representation if and only if the matrix $\left[\begin{smallmatrix} Y & X^* \\ X & Z \end{smallmatrix}\right]$ is a positive element in $\mathbf{M}_2(A)$.

Many results about operators relative to the operator norm, or positivity, can be stated in the form where one set of C^*-relations implies another. For example,

$$x^*x = xx^* \implies \begin{bmatrix} |x| & x^* \\ x & |x| \end{bmatrix} \geq 0$$

or

$$0 \leq k \leq 1, \quad \|h\| \leq 1, \quad \|hk - kh\| \leq \epsilon \implies \left\| hk^{\frac{1}{2}} - k^{\frac{1}{2}}h \right\| \leq \tfrac{5}{4}\epsilon$$

or

$$x^*x = xx^*, \quad yx = xy \implies yx^* = x^*y.$$

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In many cases, such a theorem will follow from its restriction to the matrix case. Whether this leads to a new result, a shorter proof of a known result, or a new but harder proof of a known result, depends on the example. What seems interesting is just how many theorems in the literature involve C^*-relations that are residually finite dimensional.

3 Residually Finite Dimensional C^*-Relations

We certainly want to look at relations that are compact and have a universal C^*-algebra that is residually finite dimensional (RFD). For example, for $0 < \epsilon \leq 2$ the relations

$$u^*u = uu^* = v^*v = vv^* = 1, \quad \|uv - vu\| \leq \epsilon$$

have a universal C^*-algebra that has been dubbed "the soft torus" by Exel, and this C^*-algebra has been shown to be residually finite dimensional by Eilers and Exel in [8].

A C^*-algebra is *residually finite dimensional* if there is a separating family of representations of A on finite dimensional Hilbert spaces.

The restriction to compact relations is artificial in operator theory. A very important example is the relation $\|xy - yx\| \leq \epsilon$ on $\{x, y\}$. (The obvious "and" operation turns a set of C^*-relations into a single relation, so we tend to use "relation" and "set of relations" interchangeably.) We could discuss RFD σ-C^*-algebras, but prefer to take as our starting point an alternate characterization of RFD C^*-algebras described in [9].

On $\operatorname{rep}_{\mathcal{R}}(\mathcal{X}, \mathbb{H})$ we will consider the pointwise $*$-strong topology, where $\mathbb{H}$ is a Hilbert space and $\mathcal{R}$ is a C^*-relation on a set $\mathcal{X}$. Compare this to

$$\operatorname{rep}(A, \mathbb{H}) = \left\{ \pi \colon A \to \mathbb{B}(\mathbb{H}) \mid \pi \text{ is a } *\text{-homomorphism} \right\}$$

for a C^*-algebra A with the pointwise $*$-strong topology. Equivalently, consider this with the pointwise strong topology. A representation π is said to be *finite dimensional* if its essential subspace is finite dimensional. The relevant result from [9] is that A is RFD if and only if for all $\mathbb{H}$ the finite dimensional representations are dense in $\operatorname{rep}(A, \mathbb{H})$. For more characterizations of a C^*-algebra being RFD, see [1].

We need a definition of finite dimensional for $f \in \operatorname{rep}_{\mathcal{R}}(\mathcal{X}, \mathbb{H})$. We define the *essential subspace* of f to be

$$\left\{ \xi \in \mathbb{H} \mid f(x)\xi = \left(f(x) \right)^* \xi = 0, \ \forall x \in \mathcal{X} \right\}^{\perp}.$$

We say f is *finite dimensional* if its essential subspace is finite dimensional. Notice this property has nothing to do with $\mathcal{R}$.

If $\mathcal{R}$ is a compact set of C^*-relations, then the essential space of f is the same as the essential space of the associated representation π of $C^*\langle \mathcal{X} \mid \mathcal{R} \rangle$. Thus f is finite dimensional if and only if π is finite dimensional.

Definition 3.1 A set of C^*-relations $\mathcal{R}$ on $\mathcal{X}$ is *residually finite dimensional (RFD)* if there are finite constants $C(x, r)$, $(x \in X, \ r \in [0, \infty))$ so that, for every $\mathbb{H}$ and

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any choice of nonnegative constants r_x, every $f \in \operatorname{rep}_{\mathcal{R}}(\mathcal{X}, \mathbb{H})$ satisfying $\|f(x)\| \leq r_x$ ($\forall x \in \mathcal{X}$) is in the $*$-strong closure of

$$\{g \in \operatorname{rep}_{\mathcal{R}}(\mathcal{X}, \mathbb{H}) \mid g \text{ is finite dimensional and } g(x) \leq C(x, r_x)\ \forall x \in \mathcal{X}\}.$$

We say f in $\operatorname{rep}_{\mathcal{R}}(\mathcal{X}, \mathbb{H})$ is *cyclic* if there is a vector ξ so that $A\xi$ is dense in $\mathbb{H}$, where $A = C^*(f(\mathcal{X}))$. Even if $A\xi$ is not dense, we call its closure a *cyclic subspace* for f. We say f is *unitarily equivalent* to g in $\operatorname{rep}_{\mathcal{R}}(\mathcal{X}, \mathbb{K})$ if there is a unitary $U\colon \mathbb{K} \to \mathbb{H}$ so that $g(x) = U^{-1} f(x) U$. If $\mathbb{K}$ is a reducing subspace for all operators in $f(\mathcal{X})$, then let U be the inclusion of $\mathbb{K}$ in $\mathbb{H}$ and let $g(x) = U^* f(x) U$. By (R3) this is also a representation and we call g a *subrepresentation of* f.

Lemma 3.2 *Suppose $\mathcal{R}$ is a set of C^*-relations on $\mathcal{X}$. Every Hilbert space representation of $\mathcal{R}$ is unitarily equivalent to a product of cyclic representations.*

Proof The proof is almost identical to that of the same result for representations of C^*-algebras. ■

Lemma 3.3 *A set $\mathcal{R}$ of C^*-relations is RFD if and only if there are finite constants $C(x, r)$ for $x \in X$ and $r \in [0, \infty))$ so that, for every $\mathbb{H}$, every cyclic $f \in \operatorname{rep}_{\mathcal{R}}(\mathcal{X}, \mathbb{H})$ is in the $*$-strong closure of*

$$\left\{g \in \operatorname{rep}_{\mathcal{R}}(\mathcal{X}, \mathbb{H}) \mid g \text{ is finite dimensional and } g(x) \leq C(x, \|f(x)\|)\right\}.$$

Proof The forward implication is obvious, so assume the condition on the cyclic representations holds for some choice of $C(x, r)$. Without loss of generality, $C(x, r) \neq 0$.

We may as well assume f equals $\prod_{\gamma \in \Gamma} f_\gamma$, where f_λ is a cyclic representation on $\mathbb{H}_\gamma$ and $\mathbb{H} = \bigoplus \mathbb{H}_\gamma$. Suppose $\epsilon > 0$ and $\xi = \langle \xi_\gamma \rangle$ is a unit vector. There is a finite set Γ_0 so that when we define $\eta = \langle \eta_\gamma \rangle$ by

$$\eta_\gamma = \begin{cases} \xi_\gamma & \text{if } \gamma \in \Gamma_0, \\ 0 & \text{if } \gamma \notin \Gamma_0, \end{cases}$$

we have $\|\xi - \eta\| \leq \delta$ for

$$\delta = \frac{\epsilon}{2} \big(\|f(x)\| + C(x, \|f(x)\|) \big)^{-1}.$$

Suppose Γ_0 has q elements. For each γ in Γ_0 there is a finite dimensional representation $g_\gamma\colon \mathcal{X} \to \mathbb{H}_\gamma$ so that $\|g_\gamma(x)\| \leq C(x, \|f(x)\|)$ and

$$\|g_\gamma(x)\xi_\gamma - f_\gamma(x)\xi_\gamma\|, \quad \big\| \big(g_\gamma(x)\big)^* \xi_\gamma - \big(f_\gamma(x)\big)^* \xi_\gamma \big\| \leq \frac{\epsilon}{2q}.$$

For $\gamma \notin \Gamma_0$ set $g_\gamma(x) = 0$. Let $g = \prod_\gamma g_\gamma$, which is a representation, first in $\prod_{\gamma \in \Gamma_0} \mathbb{B}(\mathbb{H}_\gamma)$ by (R4f), and then on $\mathbb{H}$ by (R3). It satisfies the norm condition, since

$$\|g(x)\| = \sup_\gamma \|g_\gamma(x)\|.$$

The essential space of g is just the sum of the orthogonal essential spaces of the g_γ for $\gamma \in \Gamma_0$, and so g is also finite dimensional. For each x we have

$$\begin{aligned}\big\|g(x)\xi - f(x)\xi\big\| &\leq \big\|g(x) - f(x)\big\|\,\|\xi - \eta\| + \big\|g(x)\eta - f(x)\eta\big\| \\ &\leq \big(\|f(x)\| + C(x, \|f(x)\|)\big)\,\delta + \sum_{\gamma\in\Gamma_0} \big\|g_\gamma(x)\xi_\gamma - f_\gamma(x)\xi_\gamma\big\| \\ &\leq \big(\|f(x)\| + C(x, \|f(x)\|)\big)\,\delta + q\rho = \epsilon\end{aligned}$$

and similarly $\|(g(x))^*\xi - (f(x))^*\xi\| \leq \epsilon$. ■

Proposition 3.4 *If $\mathcal{R}$ is a compact set of C^*-relations on a set $\mathcal{R}$, then $\mathcal{R}$ is RFD if and only if $C^*\langle \mathcal{X} \mid \mathcal{R}\rangle$ is RFD.*

Proof By the discussion above, this follows directly from [9, Theorem 2.4]. ■

Every C^*-algebra is isomorphic to the universal C^*-algebra of some C^*-relations; see [13, Section 2]. There is an abundant supply of RFD C^*-algebras and so an abundant supply of RFD C^*-relations. Examples include the subhomogeneous C^*-algebras.

Given a specific C^*-algebra it can be difficult to find a nice universal set of generator and relations. Conversely, given a set of C^*-relations, it can be difficult to get a description of its universal C^*-algebra that is more useful than the given universal property. For present purposes it is best to work directly with representations of C^*-relations.

Lemma 3.5 *Suppose $\mathcal{R}$ is a set of C^*-relations $\mathcal{R}$ on $\mathcal{X}$. If $\mathcal{R}$ is closed and $\langle f_\lambda\rangle_{\lambda\in\Lambda}$ is a bounded net in* $\operatorname{rep}(A, \mathbb{H})$ *that converges to the function $f\colon \mathcal{X} \to \mathbb{B}(\mathbb{H})$, then $f \in$* $\operatorname{rep}(A, \mathbb{H})$.

Proof The key is noticing inside $\prod_{\lambda\in\Lambda} \mathbb{B}(\mathbb{H})$ the C^*-algebra A of all bounded nets $\langle a_\lambda\rangle$ indexed by Λ that have $*$-strong limits

$$L\big(\langle a_\lambda\rangle\big) = \lim_\lambda a_\lambda \quad (*\text{-strong}).$$

Recall, say from [5, I.3.2.1], that we need boundedness to gain joint continuity of multiplication in the $*$-strong topology. Here we let λ range over the directed set Λ that indexes the net f_λ.

The f_λ form a bounded family of representations, so $\prod f_\lambda$ determines a representation of $\mathcal{R}$ in A. Now we use the naturality property for C^*-relations and conclude $f = L \circ \prod f_\lambda$ is a representation. ■

Now we present the main theorem.

Theorem 3.6 *Suppose $\mathcal{R}$ and $\mathcal{S}$ are C^*-relations on $\mathcal{X}$. If $\mathcal{R}$ is residually finite dimensional and $\mathcal{S}$ is closed, and if every finite-dimensional representation of $\mathcal{R}$ is a representation of $\mathcal{S}$, then every representation of $\mathcal{R}$ is a representation of $\mathcal{S}$.*

Proof Given a representation $f\colon \mathcal{X} \to \mathbb{B}(\mathbb{H})$ of $\mathcal{R}$, the fact that $\mathcal{R}$ is RFD tells us that there are functions $f_\lambda\colon \mathcal{X} \to \mathbb{B}(\mathbb{H})$ with finite dimensional essential spaces that are representations of $\mathcal{R}$ and so that $f_\lambda(x)$ converges $*$-strongly to $f(x)$. By assumption, the f_λ are also representations of $\mathcal{S}$ and, since $\mathcal{S}$ is closed, we conclude that f is a representation of $\mathcal{S}$. ■

4 Examples

It is easy to find many closed C^*-relations. Here are enough to keep us busy. Blackadar noted in [6] the importance of "softening" a relation $p(x_1,\dots,x_n) = 0$ to $\|p(x_1,\dots,x_n)\| \leq \epsilon$.

Proposition 4.1 *Suppose $\epsilon > 0$ is real number. If p is a NC $*$-polynomial in $x_1,\dots,x_n$, with constant-term zero, then each of*

$$p(x_1,\dots,x_n) = 0, \quad p(x_1,\dots,x_n) \geq 0, \quad \textit{and} \quad \|p(x_1,\dots,x_n)\| \leq \epsilon$$

is a closed C^*-relation.

Proof Consider the last relation, for illustration. The other parts of the proof are similar. Certainly

$$\big\| p(0,0,0,\dots,0)\big\| = \|0\| = 0 \leq \epsilon.$$

The evaluation of an NC $*$-polynomial does not depend on the ambient algebra. Therefore axiom (R2) holds. NC $*$-polynomials are natural, so (R3) holds. Given bounded families $\langle x_j^{(\lambda)}\rangle$ we get elements in the product C^*-algebra $\prod A_\lambda$. All NC $*$-polynomials respect products, so

$$\begin{aligned}
\Big\| p\Big(\big\langle x_1^{(\lambda)}\big\rangle, \big\langle x_2^{(\lambda)}\big\rangle, \big\langle x_3^{(\lambda)}\big\rangle, \dots, \big\langle x_n^{(\lambda)}\big\rangle\Big)\Big\| &= \Big\|\Big\langle p\big(x_1^{(\lambda)}, x_2^{(\lambda)}, x_3^{(\lambda)}, \dots, x_n^{(\lambda)}\big)\Big\rangle\Big\| \\
&= \sup_\lambda \Big\| p\big(x_1^{(\lambda)}, x_2^{(\lambda)}, x_3^{(\lambda)}, \dots, x_n^{(\lambda)}\big)\Big\| \\
&\leq \epsilon.
\end{aligned}$$

Therefore (R4b) holds. ■

Proposition 4.2 *If f is a continuous, real-valued function on $[0,\infty)$, then*

$$\big\{x^* = x,\ y^* = y,\ f(x) \leq f(y)\big\}$$

is a closed set of C^*-relations.

Proof This follows from trivial facts such as $f(0) \leq f(0)$ and well-known facts about the functional calculus. ■

Proposition 4.3 *The relation $x = x^*$ is a closed C^*-relation.*

Proof This a special case of Proposition 4.1 ■

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Proposition 4.4 *The relation* $0 \leq x$ *is a closed* C^**-relation.*

Proof A key fact about C^*-algebras is that positivity ($x = y^* y$ for some y) does not depend on the ambient C^*-algebra. See, for example, [7, Section 1.6.5]. ■

Proposition 4.5 *If* f *is a holomorphic function on the complex plane, and if* ϵ *is real number such that* $\epsilon \geq |f(0)|$, *then* $\|f(x)\| \leq \epsilon$ *is a closed* C^**-relation.*

Proof The functional calculus is to be applied in the unitization of the ambient C^*-algebra. We know $f(0) = f(0)1$ for the zero operator and so $\{0\}$ is a representation. The holomorphic functional calculus is natural, does not depend on the surrounding C^*-algebra, and respects finite products since polynomials do. ■

Proposition 4.6 *The union of two closed sets of* C^**-relations on the same set is closed. If a set of* C^**-relations on a set* $\mathcal{X}$ *is closed, the same properties applied to a larger set* $\mathcal{Y} \supseteq \mathcal{X}$ *form a closed set of* C^**-relations.*

Proof These statements should be obvious. ■

Proposition 4.7 *If* p *is an NC* $*$*-polynomial in* $x_1, \dots, x_n$, *with constant-term zero, and if we are given* $t_1, \dots, t_m \geq 0$ *and* $j_1, \dots, j_m \in \mathbb{N}$ *(possibly repeated), then*

$$\left\{ x_j^* = x_j, \, p(x_{j_1}^{t_1}, \dots, x_{j_m}^{t_m}) \geq 0 \right\}$$

is a closed set of C^**-relations. If only integer powers of* x_r *are used, then the relation* $x_r^* = x_r$ *may be dropped and the result is still a closed set of* C^**-relations.*

Proof Pile the functional calculus higher and deeper. Just to illustrate, we are talking about a relation such as $x^{\frac{1}{3}} y x^{\frac{2}{3}} \geq 0$ applied to pairs (x, y) of operators, where x is positive. ■

The task of finding RFD relations is harder. We start with the classic that kicked off this investigation.

Theorem 4.8 *The set of* C^**-relations* $\{x^* = x, \ y^* = y, \ x \leq y\}$ *is RFD.*

Proof We dress Löwner's argument from [2] in categorical clothing.

By Lemma 3.3, we need only consider representations on a separable Hilbert space $\mathbb{H}$. Suppose x and y are bounded operators on $\mathbb{H}$ that are self-adjoint and that $x \leq y$. Let p_n be the projection onto the first n elements in some fixed orthonormal basis. Define $x_n = p_n x p_n$ and $y_n = p_n y p_n$ so that x_n and y_n have norms bounded by $\|x\|$ and $\|y\|$ and $0 \leq x_n \leq y_n$. The operators x and y form finite dimensional relations, and they converge $*$-strongly to x and y. ■

Theorem 4.9 *Suppose* $\beta > 0$ *is a real number. The* C^**-relation* $\mathrm{Re}\, x \leq \beta$ *is RFD.*

Proof This proof is almost identical to the last. ■

Theorem 4.10 *Suppose* $\beta \geq 1$ *is a real number. The* C^**-relation* $\|e^{\mathrm{Re}\, x}\| \leq \beta$ *is RFD.*

Proof Since the real part of x is Hermitian, the exponential of the real part is Hermitian with spectrum contained in the positive real line. Therefore, $\|e^{\operatorname{Re} x}\| \leq e^{\beta}$ holds if and only if $e^{\operatorname{Re} x} \leq e^{\beta}$, which holds if and only if $\operatorname{Re} x \leq \ln(\beta)$. This relation has the same representations as the relation in Lemma 4.9, so must itself be RFD. ■

Theorem 4.11 *The empty set of relations on any set $\mathcal{X}$ is a closed C^*-relation.*

Proof This is more or less contained in [10]. Given a Hilbert space $\mathbb{H}$ and operators a_j on $\mathbb{H}$ with $j \in \mathcal{X}$, take any net of finite-rank projections p_λ converging strongly to the identity. Then $p_\lambda a_j p_\lambda$ is bounded in norm by $\|a_j\|$ and converges $*$-strongly to a_j. ■

Many amalgamated products $A *_C B$ turn out to be RFD when A and B are RFD. The simplest theorem of this sort, proved in [9], is that A and B being RFD implies $A * B$ is RFD. This generalizes easily here to something very useful.

Theorem 4.12 *Suppose $\mathcal{X}$ and $\mathcal{Y}$ are disjoint sets and that $\mathcal{R}$ is an RFD set of C^*-relations on $\mathcal{X}$ and $\mathcal{S}$ is an RFD closed set of C^*-relations on $\mathcal{Y}$. If we regard both sets as relations on $\mathcal{X} \cup \mathcal{Y}$, then the set $\mathcal{R} \cup \mathcal{S}$ is an RFD set of C^*-relations.*

Proof All we need to know is that if A is a set of operators on $\mathbb{H}$ that are zero on the orthogonal complement of the finite dimensional subspace $\mathbb{H}_1$, and if B is a set of operators on $\mathbb{H}$ that are zero on the orthogonal complement of the finite dimensional subspace $\mathbb{H}_2$, then the union is a set of operators that is zero on the orthogonal complement of the subspace $\mathbb{H}_1 + \mathbb{H}_2$, which is also finite dimensional. ■

Proposition 4.13 *If $p_1, \dots, p_m$ are NC $*$-polynomials in $x_1, \dots, x_n$ that are homogeneous of degrees that can vary, and $\epsilon_s > 0$ are real constants, and $0 \leq n_1 \leq n_2 \leq n$, then*

$$\begin{aligned}
&0 \leq x_j, && (j = 1, \dots, n_1)\\
&x_j^* = x_j, && (j = n_1 + 1, \dots, n_2)\\
&\big\| p_s(x_1, \dots, x_n) \big\| \leq \epsilon_s && (s = 1, \dots, m)
\end{aligned}$$

form a closed set of C^*-relations.

Proof Assume $x_1, \dots, x_n$ are in $\mathbb{B}(\mathbb{H})$, where $\mathbb{H}$ is separable, and that these operators satisfy the above relations. Let u_k be a countable approximate identity for the compact operators, with $0 \leq u_k \leq 1$, that is quasi-central for $x_1, \dots, x_n$. Such an approximate identity exists by [15, Corollary 3.12.16]. Applying a decreasing perturbation to each u_k we may further assume each u_k is finite-rank.

Let $x_{j,k} = u_k x_j u_k$. Clearly $\|x_{j,k}\| \leq \|x_j\|$, and $0 \leq x_{j,k}$ for $j \leq n_1$ and $x_{j,k}^* = x_{j,k}$ for $n_1 < j \leq n_2$. Also $x_{j,k} \to x_j$ in the $*$-strong topology. For fixed k the $x_{j,k}$ all act as zero on the complement of range of u_k, which is finite dimensional. However, we need to modify the $x_{j,k}$ to make the last line of relations hold.

Suppose p_s is homogeneous of degree d_s. This means u_k appears $2d_s$ times in each monomial in p_s. Since u_k is quasi-central for the x_j, we have

$$\lim_{k \to \infty} \big\| p_s(x_{1,k}, \dots, x_{n,k}) - u_k^{2d_s} p_s(x_1, \dots, x_n) \big\| = 0.$$

Therefore,

$$\limsup_{k\to\infty}\left\| p_s(x_{1,k},\dots,x_{n,k})\right\| = \limsup_{k\to\infty}\left\| u_k^{2d_s} p_s(x_1,\dots,x_n)\right\| \le \left\| p_s(x_1,\dots,x_n)\right\|.$$

It is easy to show that if $a_\lambda \to a$ in the strong topology, then

$$\liminf_{\lambda\to\infty}\|a_\lambda\| \ge \|a\|.$$

The $x_{j,k}$ are bounded sequences converging to x_j in the $*$-strong topology, so

$$\lim_k p_s(x_{1,k},\dots,x_{n,k}) = p_s(x_1,\dots,x_n) \quad (*\text{-strong}),$$

which means

$$\lim_{k\to\infty}\left\| p_s(x_{1,k},\dots,x_{n,k})\right\| = \left\| p_s(x_1,\dots,x_n)\right\|.$$

If $p_s(x_1,\dots,x_n) = 0$, let $\alpha_{j,k} = 1$, and otherwise let

$$\alpha_{j,k} = \max\left(1, \left(\frac{\|p_s(x_1,\dots,x_n)\|}{\|p_s(x_{1,k},\dots,x_{n,k})\|}\right)^{\frac{1}{d_s}}\right).$$

Let $y_{j,k} = \alpha_{j,k}x_{j,k}$. The $\alpha_{j,k}$ are at most 1, so $\|y_{j,k}\| \le \|x_j\|$. We are scaling by positive factors, so $0 \le y_{j,k}$ for $j \le n_1$ and $y_{j,k}^* = y_{j,k}$ for $n_1 < j \le n_2$. Since $\alpha_{j,k} \to 1$, we see that $y_{j,k} \to x_j$ in the $*$-strong topology. For fixed k the $y_{j,k}$ still all act as zero on the complement of range of u_k. What we have gained are the final relations,

$$\begin{aligned}\left\| p_s(y_{1,k},\dots,y_{n,k})\right\| &= \left\| \alpha_{j,k}^{d_s} p_s(x_{1,k},\dots,x_{n,k})\right\| \\ &= \alpha_{j,k}^{d_s}\left\| p_s(x_{1,k},\dots,x_{n,k})\right\| \le \epsilon_s.\end{aligned}$$ ■

5 Applications

We have several corollaries to Theorem 3.6. All these results refer to the operator norm or order relations. Of course, we recover the result of Löwner that matrix monotone for all orders implies operator monotone.

Corollary 5.1 *Let a be a bounded operator. Then $\|e^a\| \le \|e^{\mathrm{Re}(a)}\|$.*

Proof Theorem IX.3.1 of [3] tells us this result is true for any matrix, and indeed for any unitarily invariant norm.

Since $\|e^x\| \ge 1$ for any operator x, we can rephrase this to say that for each $\alpha \ge 1$, we have

$$\|e^{\mathrm{Re}(a)}\| \le \alpha \implies \|e^a\| \le \alpha.$$

As the first relation is RFD and the second is closed, we are done by Theorem 3.6. ■

The NC $*$-polynomial version (in the original variables, not their fractional powers) of the following can be proven by Helton's sum-of-squares theorem ([14, Theorem 1.1]), which is essentially in [12].

Corollary 5.2 *Suppose that p is an NC $*$-polynomial in $x_1, \ldots, x_n$ with constant-term zero, and that $t_1, \ldots, t_m$ are positive exponents and $j_1, \ldots, j_m$ are between 1 and n. If*

$$p(x_{j_1}^{t_1}, \ldots, x_{j_m}^{t_m}) \geq 0$$

for all self-adjoint matrices $x_1, \ldots, x_n$, then the same hold true for all self-adjoint operators on a Hilbert space. If only integer powers of x_r are used, the relation $x_r^* = x_r$ may be dropped.

Proof The null set of relations is RFD, so Theorem 3.6 still applies. ■

Corollary 5.3 *Let a, b, and x be a bounded operators, with $a \geq 0$ and $b \geq 0$. Then for $0 \leq \nu \leq 1$,*

$$\|a^\nu x b^{1-\nu} + a^{1-\nu} x b^\nu\| \leq \|ax + xb\|.$$

Proof The proof is in [3, Corollary IX.4.10], restricted to matrices but for any unitarily invariant norm. Using the operator norm version of that result, Proposition 4.13, and Proposition 4.7 we find again that the operator result follows from the matrix result. ■

Corollary 5.4 *If C is a constant so that*

$$\|a\| \leq 1 \text{ and } b \geq 0 \implies \|ab^{\frac{1}{2}} - b^{\frac{1}{2}}a\| \leq C\|ab - ba\|^{\frac{1}{2}} \tag{5.1}$$

for all matrices a and b, then (5.1) is true for all bounded operators on Hilbert space (or C^*-algebra elements).

Proof We hope that the constant $C = 1$ works here; see [4, 16]. Perhaps this reduction to the matrix case will make that easier to prove.

We can rephrase this as

$$\|a\| \leq 1,\ b \geq 0, \quad \|ab - ba\| \leq \delta \implies \|ab^{\frac{1}{2}} - b^{\frac{1}{2}}a\| \leq C\delta^{\frac{1}{2}}.$$

The set of relations on the left is RFD by Proposition 4.13, and those on the right are closed by Proposition 4.7. ■

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Department of Mathematics and Statistics, University of New Mexico, Albuquerque, NM 87131, U.S.A.
e-mail: loring@math.unm.edu