M. Hellus, L. T. Hoa and J. Stückrad
Nagoya Math. J.
Vol. 196 (2009), 67–85

GRÖBNER BASES OF SIMPLICIAL TORIC IDEALS

MICHAEL HELLUS, LÊ TUÂN HOA AND JÜRGEN STÜCKRAD

Abstract. Bounds for the maximum degree of a minimal Gröbner basis of simplicial toric ideals with respect to the reverse lexicographic order are given. These bounds are close to the bound stated in Eisenbud-Goto's Conjecture on the Castelnuovo-Mumford regularity.

Introduction

Let I be a homogeneous ideal of a polynomial ring R. The coarsest measure of the complexity of a Gröbner basis (w.r.t. to a term order $\leq$) of an ideal I is its maximum degree, which is the highest degree of a generator of the initial ideal $\text{in}_{\leq}(I)$. However, this quantity is not easy to be handled with. One way to study it is to use a better-behaved invariant, the Castelnuovo-Mumford regularity $\text{reg}(I)$ of I. This invariant can be defined as the maximum over all i of the degree minus i of any minimal i-th syzygy of I, treating generators as 0-th syzygies. In the generic coordinates and with respect to the reverse lexicographic order, the maximum degree in a minimal Gröbner basis of I is bounded by $\text{reg}(I)$ (see [BS, Corollary 2.5]). Unfortunately, this is not true for arbitrary coordinates (see, e.g., the example after [HHy, Lemma 14]). On the other hand, a famous conjecture by Eisenbud and Goto states that $\text{reg}(I) \leq \deg(R/I) - \text{codim}(R/I) + 1$, provided I is a prime ideal containing no linear form (see [EG]). Here $\deg(R/I)$ and $\text{codim}(R/I)$ denote the multiplicity and the codimension of R/I, respectively. Thus, in the generic coordinates the *Eisenbud-Goto bound* $\deg(R/I) - \text{codim}(R/I) + 1$ is an expected bound for the maximum degree in a minimal Gröbner basis of I w.r.t. the reverse lexicographic order of a prime ideal containing no linear form. We may hope that this expectation still holds for some other coordinates.

Received May 18, 2008.
Revised March 13, 2009, April 3, 2009.
Accepted April 14, 2009.
2000 Mathematics Subject Classification: 13P10, 14M25.
The second author was supported by NAFOSTED (Vietnam) and Max-Planck Institute for Mathematics in the Sciences (Germany).

https://doi.org/10.1017/S002776300000979X Published online by Cambridge University Press

In this paper we are interested in estimating the degree-complexity of Gröbner bases of simplicial toric ideals. Toric ideals are nice, particularly because they are prime ideals and in the natural coordinates they are generated by binomials. In order to find a minimal Gröbner basis of such an ideal, it is therefore natural to try to keep the original coordinates, so that elements of such a Gröbner basis can be taken as binomials - which are cheap to compute and to restore. On the other hand in [HS] the last two authors have shown that for a large class of simplicial toric ideals I, the Castelnuovo-Mumford regularity $\operatorname{reg}(I)$ is bounded by the *Eisenbud-Goto bound* $\deg(R/I) - \operatorname{codim}(R/I) + 1$. From these phenomena we believe that following conjecture holds:

CONJECTURE. *Assume that I is the toric ideal associated with a homogeneous simplicial affine semigroup S over an arbitrary field K. The maximum degree in a minimal Gröbner basis of I in the natural coordinates and w.r.t. the reverse lexicographic order is bounded above by* $\deg K[S] - \operatorname{codim} K[S] + 1$.

Note that this is not true for an arbitrary term order (see Example 1.2). For the rest of the paper, if not otherwise stated, we consider only the natural coordinates and the reverse lexicographic order. Although we are still not able to solve the above problem, we can establish the upper bound $2(\deg K[S] - \operatorname{codim} K[S])$. In order to do that we first establish an upper bound in terms of the reduction number $r(S)$ of $K[S]$. Then, combining with a bound of [HS] on $r(S)$, we get the main result, see Theorem 1.1. We also provide another bound in terms of the codimension $c = \operatorname{codim} K[S]$ and the total degree α of monomials defining S (Theorem 1.4). In a lot of examples bounds in Theorems 1.1 and 1.4 are even much smaller then the Eisenbud-Goto bound.

In Section 2 we solve the above conjecture for certain classes of simplicial toric ideals. Ideals of first type come from a simple observation that the maximum degrees in their minimal Gröbner bases are bounded by the Castelnuovo-Mumford regularity if the corresponding rings $K[S]$ are generalized Cohen-Macaulay rings. Ideals of second type are raised by certain properties of the parameter set $\mathcal{A}$ (see Propositions 2.4 and 2.6). In this situation, by using Theorem 1.4 we can restrict ourselves to few exceptional cases when the codimension is very big. Then the main technique is to refine bounds on the reduction number or to calculate its exact value, so that one

https://doi.org/10.1017/S002776300000979X Published online by Cambridge University Press

can apply Theorem 1.1. In particular, we show that the conjecture holds for all simplicial toric ideals in [HS], for which the Eisenbud-Goto conjecture is known to be true.

Notation. In this paper we use bold letters to denote a vector, while their coordinates are written in the normal style. Thus a_i, e_{1i} are the i-th coordinates of vectors $\mathbf{a}$, $\mathbf{e}_1$, respectively; $\mathbf{x}^{\mathbf{m}} = x_1^{m_1} \cdots x_c^{m_c}$, $\mathbf{y}^{\mathbf{n}} = y_1^{n_1} \cdots y_d^{n_d}$ and $\mathbf{t}^{\mathbf{n}} = t_1^{n_1} \cdots t_d^{n_d}$. The ordering of variables is always assumed to be $x_1 > \cdots > x_c > y_1 > \cdots > y_d$. We always write a binomial in such a way that its first term is bigger than the second one.

§1. Bounds

Let $S \subseteq \mathbb{N}^d$ be a homogeneous, simplicial affine semigroup generated by a set of elements of the following type:

$$\begin{aligned} \mathcal{A} &= \{\mathbf{e}_1, \ldots, \mathbf{e}_d, \mathbf{a}_1, \ldots, \mathbf{a}_c\} \\ &\subseteq M_{\alpha,d} = \{(x_1, \ldots, x_d) \in \mathbb{N}^d \mid x_1 + \cdots + x_d = \alpha\}, \end{aligned}$$

where $c \geq 2$, $\alpha \geq 2$ are natural numbers and $\mathbf{e}_1 = (\alpha, 0, \ldots, 0), \ldots, \mathbf{e}_d = (0, \ldots, 0, \alpha)$. Moreover, if $\mathbf{a}_i = (a_{i1}, \ldots, a_{id})$, we can assume that the integers a_{ij}, where $i = 1, \ldots, c$, $j = 1, \ldots, d$, are relatively prime. Note that $\dim K[S] = d$ and $\operatorname{codim} K[S] = c$. Let $I_{\mathcal{A}}$ be the kernel of the homomorphism

$$\begin{aligned} K[\mathbf{x}, \mathbf{y}] := K[x_1, \ldots, x_c, y_1, \ldots, y_d] &\longrightarrow K[S] \\ &\equiv K[t_1^{\alpha}, \ldots, t_d^{\alpha}, \mathbf{t}^{\mathbf{a}_1}, \ldots, \mathbf{t}^{\mathbf{a}_c}] \subseteq K[\mathbf{t}]; \\ x_i \longmapsto \mathbf{t}^{\mathbf{a}_i};\ y_j \longmapsto t_j^{\alpha},&\ i = 1, \ldots, c;\ j = 1, \ldots, d. \end{aligned}$$

We call $I_{\mathcal{A}}$ a *simplicial toric ideal defined by* $\mathcal{A}$ (or S). We will consider the standard grading on $K[\mathbf{x}, \mathbf{y}]$ and $K[S]$, i.e. $\deg(x_i) = \deg(y_j) = 1$ and if $\mathbf{b} \in S$, then $\deg(\mathbf{b}) = (b_1 + \cdots + b_d)/\alpha$.

Note that $I_{\mathcal{A}}$ always has a minimal Gröbner basis consisting of binomials (see, e.g., [St1, Chapter 1]). We are interested in bounding its maximum degree.

Let $A = A_0 \oplus A_1 \oplus \cdots$, where $A_0 = K$, be a standard graded K-algebra of dimension d. A minimal reduction of A is a graded ideal I generated by d linear forms such that $[IA]_n = A_n$ for $n \gg 0$. The least integer n such that $[IA]_{n+1} = A_{n+1}$ is called the reduction number of A

w.r.t. I and will be denoted by $r_I(A)$. Note that $(t_1^\alpha, \dots, t_d^\alpha)$ is a minimal reduction of $K[S]$. We denote by $r(S)$ the reduction number of $K[S]$ w.r.t. this minimal reduction. Then $r(S)$ is the least positive integer r such that $(r+1)\mathcal{A} = \{e_1, \dots, e_d\} + r\mathcal{A}$, where for two subsets B and C of $\mathbb{Z}^d$ we denote by $B \pm C$ the set of all elements of the form $b \pm c$, $b \in B$, $c \in C$, and $nB = B + \cdots + B$ (n times). This reduction number was used in [HS] to bound the Castelnuovo-Mumford regularity of $K[S]$.

THEOREM 1.1. *The maximum degree in a minimal Gröbner basis of* $I_{\mathcal{A}}$ *is bounded by*

$$\max\{r(S)+1,\, 2r(S)-1\} \le \max\{2,\, 2(\deg K[S] - \operatorname{codim} K[S]) - 1\}.$$

Proof. Let $s = \max\{r(S)+1, 2r(S)-1\}$ and set

$$G = \{\mathbf{x}^{\mathbf{m}}\mathbf{y}^{\mathbf{n}} - \mathbf{x}^{\mathbf{p}}\mathbf{y}^{\mathbf{q}} \in I_{\mathcal{A}} \mid \deg(\mathbf{x}^{\mathbf{m}}\mathbf{y}^{\mathbf{n}}) = \deg(\mathbf{x}^{\mathbf{p}}\mathbf{y}^{\mathbf{q}}) \le s\}.$$

By [HS, Theorem 1.1], $r(S) \le \deg K[S] - \operatorname{codim} K[S]$. Hence, it suffices to show that G is a Gröbner basis of $I_{\mathcal{A}}$. In particular this also implies that $G \neq \emptyset$. Assume that this is not the case. Then one can find a binomial $b = \mathbf{x}^{\mathbf{m}}\mathbf{y}^{\mathbf{n}} - \mathbf{x}^{\mathbf{p}}\mathbf{y}^{\mathbf{q}} \in I_{\mathcal{A}}$ of the smallest degree $\deg b > s$ such that $\operatorname{in}(g) \nmid \mathbf{x}^{\mathbf{m}}\mathbf{y}^{\mathbf{n}}$ for all $g \in G$.

If $\deg(\mathbf{x}^{\mathbf{m}}) \ge r(S)+1$, then we can write $\mathbf{x}^{\mathbf{m}} = \mathbf{x}^{\mathbf{m}'}\mathbf{x}^{\mathbf{m}''}$, where $\deg(\mathbf{x}^{\mathbf{m}'}) = r(S)+1$. By the definition of $r(S)$ we can find $\mathbf{m}^*$, $\mathbf{n}^*$ such that $\deg(\mathbf{x}^{\mathbf{m}^*}) = r(S)$ and $g := \mathbf{x}^{\mathbf{m}'} - \mathbf{x}^{\mathbf{m}^*}\mathbf{y}^{\mathbf{n}^*} \in I_{\mathcal{A}}$ (note that $\mathbf{x}^{\mathbf{m}'} > \mathbf{x}^{\mathbf{m}^*}\mathbf{y}^{\mathbf{n}^*}$). Then $g \in G$ and $\operatorname{in}(g) = \mathbf{x}^{\mathbf{m}'} \mid \mathbf{x}^{\mathbf{m}}\mathbf{y}^{\mathbf{n}}$, a contradiction. Thus $\deg(\mathbf{x}^{\mathbf{m}}) \le r(S)$.

If $\deg(\mathbf{x}^{\mathbf{p}}) \ge r(S)+1$, then as above, we can find $\mathbf{p}'$, $\mathbf{p}''$ such that $\mathbf{x}^{\mathbf{p}}\mathbf{y}^{\mathbf{q}} - \mathbf{x}^{\mathbf{p}'}\mathbf{y}^{\mathbf{p}''+\mathbf{q}} \in I_{\mathcal{A}}$ and $\deg(\mathbf{x}^{\mathbf{p}'}) = r(S) < \deg(\mathbf{x}^{\mathbf{p}})$. Then

$$\mathbf{x}^{\mathbf{m}}\mathbf{y}^{\mathbf{n}} - \mathbf{x}^{\mathbf{p}'}\mathbf{y}^{\mathbf{p}''+\mathbf{q}} = (\mathbf{x}^{\mathbf{m}}\mathbf{y}^{\mathbf{n}} - \mathbf{x}^{\mathbf{p}}\mathbf{y}^{\mathbf{q}}) + (\mathbf{x}^{\mathbf{p}}\mathbf{y}^{\mathbf{q}} - \mathbf{x}^{\mathbf{p}'}\mathbf{y}^{\mathbf{p}''+\mathbf{q}}) \in I_{\mathcal{A}},$$

and $\mathbf{x}^{\mathbf{m}}\mathbf{y}^{\mathbf{n}} > \mathbf{x}^{\mathbf{p}}\mathbf{y}^{\mathbf{q}} > \mathbf{x}^{\mathbf{p}'}\mathbf{y}^{\mathbf{p}''+\mathbf{q}}$. Hence, replacing $\mathbf{x}^{\mathbf{p}}\mathbf{y}^{\mathbf{q}}$ by $\mathbf{x}^{\mathbf{p}'}\mathbf{y}^{\mathbf{p}''+\mathbf{q}}$, we may assume from the beginning that $\deg(\mathbf{x}^{\mathbf{p}}) \le r(S)$.

Now, since $\mathbf{x}^{\mathbf{m}}\mathbf{y}^{\mathbf{n}} - \mathbf{x}^{\mathbf{p}}\mathbf{y}^{\mathbf{q}} \in I_{\mathcal{A}}$, we have

$$\sum_{i=1}^{c} m_i \mathbf{a}_i + \sum_{j=1}^{d} n_j \mathbf{e}_j = \sum_{i=1}^{c} p_i \mathbf{a}_i + \sum_{j=1}^{d} q_j \mathbf{e}_j.$$

From the minimality of $\deg(\mathbf{x}^{\mathbf{m}}\mathbf{y}^{\mathbf{n}})$ we may assume that $\mathbf{x}^{\mathbf{m}}\mathbf{y}^{\mathbf{n}}$ and $\mathbf{x}^{\mathbf{p}}\mathbf{y}^{\mathbf{q}}$ have no common variable. That means if we set $C = \{i \mid m_i \neq 0\}$ and $D = \{j \mid n_j \neq 0\}$, then the above equality can be rewritten as

$$\sum_{i \in C} m_i \mathbf{a}_i + \sum_{j \in D} n_j \mathbf{e}_j = \sum_{i \notin C} p_i \mathbf{a}_i + \sum_{j \notin D} q_j \mathbf{e}_j.$$

Hence

$$\sum_{j \in D} \sum_{i \in C} m_i a_{ij} + \sum_{j \in D} n_j \alpha = \sum_{j \in D} \sum_{i \notin C} p_i a_{ij} = \sum_{i \notin C} p_i \sum_{j \in D} a_{ij} \leq \sum_{i \notin C} p_i \alpha.$$

This implies

$$\sum_{j=1}^{d} n_j = \sum_{j \in D} n_j \leq \sum_{i \notin C} p_i = \deg(\mathbf{x}^{\mathbf{p}}). \tag{1}$$

The equality holds if and only if $m_i a_{ij} = 0$ for all $(i,j) \in C \times D$ and $p_i a_{ij} = 0$ for all (i,j) such that $i \notin C$ and $j \notin D$. This yields $\sum_{i \in C} m_i \mathbf{a}_i = \sum_{j \notin D} q_j \mathbf{e}_j$, which means $\mathbf{x}^{\mathbf{m}} - \mathbf{y}^{\mathbf{q}} \in I_{\mathcal{A}}$. Since $\mathbf{x}^{\mathbf{m}} > \mathbf{y}^{\mathbf{q}}$ and $\deg(\mathbf{x}^{\mathbf{m}}) \leq r(S)$, $g := \mathbf{x}^{\mathbf{m}} - \mathbf{y}^{\mathbf{q}} \in G$. But this is impossible because $\operatorname{in}(g) \mid \mathbf{x}^{\mathbf{m}}\mathbf{y}^{\mathbf{n}}$. Hence, by (1), we must have $\sum_{j=1}^{d} n_j < \deg(\mathbf{x}^{\mathbf{p}}) \leq r(S)$, and so

$$\deg(b) = \deg(\mathbf{x}^{\mathbf{m}}) + \sum_{j=1}^{d} n_j \leq 2r(S) - 1 \leq s,$$

a contradiction. The theorem is proved. ∎

It should be noted that if S is not necessarily a simplicial semigroup, then Sturmfels [St2] showed that w.r.t. any term order, the maximum degree in a minimal Gröbner basis of $I_{\mathcal{A}}$ is bounded by $c \cdot \deg K[S]$.

The following example shows that estimations in Theorem 1.1 do not hold for an arbitrary term order.

Example 1.2. Let $\mathcal{A} = \{(4,0), (3,1), (1,3), (0,4)\}$. Then

$$I_{\mathcal{A}} = (x_1x_2 - y_1y_2, x_1^3 - x_2y_1^2, x_2^3 - x_1y_2^2, x_2^2y_1 - x_1^2y_2).$$

This is also a minimal Gröbner basis of $I_{\mathcal{A}}$ w.r.t. the reverse lexicographic order. W.r.t. the lexicographic order we get the following minimal Gröbner basis:

$$\{x_1x_2 - y_1y_2, x_1^3 - x_2y_1^2, x_1y_2^2 - x_2^3, x_1^2y_2 - x_2^2y_1, x_2^4 - y_1y_2^3\}.$$

https://doi.org/10.1017/S002776300000979X Published online by Cambridge University Press

In this example $r(S) = \deg K[S] - \operatorname{codim} K[S] = 2$ and both bounds in Theorem 1.1 are equal to 3.

The above example also provides a case when upper bounds in Theorems 1.1 are tight. However if $\deg K[S] - \operatorname{codim} K[S] \geq 3$ we believe that the second bound is never attained (see the conjecture mentioned in the introduction). Similarly, we don't think that the first bound is attained if $r(S)$ is big. However, the following example shows that in general it is at most twice of the best possible bound.

EXAMPLE 1.3. Given $d \geq 2$ and $\alpha \geq \max\{4, d+1\}$. Let

$$\mathcal{A} = M_{\alpha,d} \setminus \{(\beta, \alpha - \beta, 0, \dots, 0) \mid 2 \leq \beta \leq \alpha - 2\}. \tag{2}$$

We may assume $\mathbf{a}_1 = (\alpha - 1, 1, 0, \dots, 0)$ and $\mathbf{a}_2 = (1, \alpha - 1, 0, \dots, 0)$. If $S \ni (\alpha - 2)\mathbf{a}_1 = \sum m_i \mathbf{a}_i + \sum n_j \mathbf{e}_j$ with $\sum n_j > 0$, comparing $d-1$ last coordinates, one should have $\alpha - 2 = m_1 + m_2(\alpha - 1) + n_2\alpha$. This implies $n_2 = m_2 = 0$ and $m_1 = \alpha - 2$, which is impossible, since $m_1 = (\alpha - 2) - \sum n_j < \alpha - 2$. Hence $(\alpha - 2)\mathbf{a}_1 \notin \{\mathbf{e}_1, \dots, \mathbf{e}_d\} + (\alpha - 3)\mathcal{A}$ and $r(S) \geq \alpha - 2$.

Let $\mathbf{b} = (b_1, \dots, b_d) \in \mathbb{N}^d$ such that $\alpha \mid b_1 + \dots + b_d$ and $b_3 + \dots + b_d > 0$. By induction on $\deg(\mathbf{b}) := (b_1 + \dots + b_d)/\alpha$, we show that $\mathbf{b} \in S$. The case $\deg(\mathbf{b}) = 1$ follows from (2). Let $\deg(\mathbf{b}) \geq 2$. If $b_1 \geq \alpha$, then $\mathbf{b} = \mathbf{e}_1 + \mathbf{b}'$ with $b'_3 + \dots + b'_d > 0$. By the induction hypothesis, $\mathbf{b}' \in S$ and hence $\mathbf{b} \in S$. The same holds if $b_2 \geq \alpha$. Hence we may assume that $b_1, b_2 < \alpha$. In this case $b_2 + b_3 + \dots + b_d \geq \alpha + 1$, and we can find $b'_2 = b_2, b'_3 \leq b_3, \dots, b'_d \leq b_d$ such that $b'_2 + \dots + b'_d = \alpha$. Let $b'_1 = 0$. Then both elements $\mathbf{b}'$ and $\mathbf{b} - \mathbf{b}'$ satisfy the induction hypothesis, which implies $\mathbf{b} = \mathbf{b}' + (\mathbf{b} - \mathbf{b}') \in S$.

Further, let $\mathbf{b} = (b_1, b_2, 0, \dots, 0)$ with $b_1 + b_2 = \alpha(\alpha - 2)$. We show that also $\mathbf{b} \in S$. Indeed, we can write $b_2 = p\alpha + q$, where $p \leq \alpha - 2$, $q \leq \alpha - 1$. Note that $p = \alpha - 2$ implies $q = 0$ and $\mathbf{b} = (\alpha - 2)\mathbf{e}_2 \in S$.

Let $p \leq \alpha - 3$. If $p + q \geq \alpha - 1$, then

$$\mathbf{b} = 0\mathbf{a}_1 + (\alpha - q)\mathbf{a}_2 + (\alpha - 3 - p)\mathbf{e}_1 + (p + q - \alpha + 1)\mathbf{e}_2 \in S.$$

Otherwise $(p + q \leq \alpha - 2)$,

$$\mathbf{b} = q\mathbf{a}_1 + 0\mathbf{a}_2 + (\alpha - 2 - p - q)\mathbf{e}_1 + p\mathbf{e}_2 \in S.$$

Summarizing the above arguments we get that $\mathbf{b} \in S$ if $\deg(\mathbf{b}) = \alpha - 2$.

Now let $\mathbf{a} \in (\alpha-1)\mathcal{A}$. Since $\alpha \geq d+1$, $a_1+\cdots+a_d = \alpha(\alpha-1) \geq d\alpha$ and there is an index i such that $a_i \geq \alpha$. Note that $\deg(\mathbf{a}-\mathbf{e}_i) = \alpha-2$. By the above result $\mathbf{a}-\mathbf{e}_i \in S$. Hence $\mathbf{a} = \mathbf{e}_i + (\mathbf{a}-\mathbf{e}_i) \in \{\mathbf{e}_1,\dots,\mathbf{e}_d\}+S$, which implies $r(S) \leq \alpha-2$.

Summing up we get $r(S) = \alpha - 2$.

On the other hand, $x_1^{\alpha-1} - x_2 y_1^{\alpha-2} \in I_\mathcal{A}$, $x_1^{\alpha-1} > x_2 y_1^{\alpha-2}$ and there is no other binomial of $I_\mathcal{A}$ whose first term divides $x_1^{\alpha-1}$. Therefore the binomial $x_1^{\alpha-1} - x_2 y_1^{\alpha-2}$ must be contained in the reduced Gröbner basis of $I_\mathcal{A}$. The degree of this binomial is $\alpha-1$, while the first bound of Theorem 1.1 is $2\alpha-5$. Note that the Eisenbud-Goto bound in this example is $\alpha^{d-1}+\alpha+d-\binom{\alpha+d-1}{d-1}-2$.

It was also shown that the Castelnuovo-Mumford regularity of $\operatorname{reg}(I_\mathcal{A})$ is bounded by $c(\alpha-1)+1$ (see [HS, Theorem 3.2(i)]). In the following theorem we obtain a similar result for Gröbner bases.

THEOREM 1.4. *The maximum degree in a minimal Gröbner basis of* $I_\mathcal{A}$ *is bounded by* $\max\{c,\ \alpha,\ c(\alpha-1)-1\} \leq c(\alpha-1)$.

Proof. The proof is similar to that of Theorem 1.1. Let $s = \max\{c,\ \alpha,\ c(\alpha-1)-1\}$ and set

$$G = \big\{\mathbf{x^m y^n} - \mathbf{x^p y^q} \in I_\mathcal{A} \mid \deg(\mathbf{x^m y^n}) = \deg(\mathbf{x^p y^q}) \leq s\big\}.$$

Assume that G is not a Gröbner basis. Then one can find a binomial $b = \mathbf{x^m y^n} - \mathbf{x^p y^q} \in I_\mathcal{A}$ of the smallest degree $\deg b > s$ such that $\operatorname{in}(g) \nmid \mathbf{x^m y^n}$ for all $g \in G$. Since $\alpha\mathbf{a}_i = a_{i1}\mathbf{e}_1+\cdots+a_{id}\mathbf{e}_d$, $x_i^\alpha - \mathbf{y}^{\mathbf{a}_i} \in G$ for all $i = 1,\dots,c$. Note that $x_i^\alpha > \mathbf{y}^{\mathbf{a}_i}$. Since $\operatorname{in}(x_i^\alpha - \mathbf{y}^{\mathbf{a}_i}) \nmid \mathbf{x^m y^n}$, we must have $m_i \leq \alpha-1$ for all $i \leq c$.

If $p_i \geq \alpha$, then

$$\mathbf{x^m y^n} - \frac{\mathbf{x^p}}{x_i^\alpha}\mathbf{y}^{\mathbf{q}+\mathbf{a}_i} = (\mathbf{x^m y^n} - \mathbf{x^p y^q}) + (x_i^\alpha - \mathbf{y}^{\mathbf{a}_i})\frac{\mathbf{x^p}}{x_i^\alpha}\mathbf{y^q} \in I_\mathcal{A}.$$

Note that $\mathbf{x^m y^n} > \mathbf{x^p y^q} > \frac{\mathbf{x^p}}{x_i^\alpha}\mathbf{y}^{\mathbf{q}+\mathbf{a}_i}$. Replacing b by $\mathbf{x^m y^n} - \frac{\mathbf{x^p}}{x_i^\alpha}\mathbf{y}^{\mathbf{q}+\mathbf{a}_i}$ and repeating this procedure, we may also assume that $p_i \leq \alpha-1$ for all $i \leq c$.

As in the proof of Theorem 1.1, let $C = \{i \mid m_i \neq 0\}$ and $D = \{j \mid n_j \neq 0\}$. Then we can also conclude that

$$\sum_{j\in D} n_j \leq \sum_{i \notin C} p_i \leq (c-\sharp C)(\alpha-1), \tag{3}$$

https://doi.org/10.1017/S002776300000979X Published online by Cambridge University Press

and that $\sum_{j\in D} n_j = (c - \sharp C)(\alpha - 1)$ implies $\mathbf{x}^{\mathbf{m}} - \mathbf{y}^{\mathbf{q}} \in I_{\mathcal{A}}$. Hence

$$\deg(\mathbf{x}^{\mathbf{m}}\mathbf{y}^{\mathbf{n}}) = \sum_{i\in C} m_i + \sum_{j\in D} n_j \leq \sharp C(\alpha - 1) + (c - \sharp C)(\alpha - 1) = c(\alpha - 1).$$

Since $\deg(\mathbf{x}^{\mathbf{m}}\mathbf{y}^{\mathbf{n}}) = \deg(b) \geq c(\alpha - 1)$, we must have $\deg(\mathbf{x}^{\mathbf{m}}\mathbf{y}^{\mathbf{n}}) = c(\alpha - 1)$. Therefore $\sum_{j\in D} n_j = (c - \sharp C)(\alpha - 1)$ and $m_i = \alpha - 1$ for all $i \in C$. By (3) we have $\mathbf{x}^{\mathbf{m}} - \mathbf{y}^{\mathbf{q}} \in I_{\mathcal{A}}$. If $C \neq \{1, \ldots, c\}$, then $\deg(\mathbf{x}^{\mathbf{m}}) \leq s$ and $\mathbf{x}^{\mathbf{m}} - \mathbf{y}^{\mathbf{q}} \in G$, which is impossible because $\mathbf{x}^{\mathbf{m}} \mid \mathrm{in}(b)$. Thus $C = \{1, \ldots, c\}$. This yields $D = \emptyset$ and

$$b = (x_1 \cdots x_c)^{\alpha - 1} - \mathbf{y}^{\mathbf{q}}.$$

Let $\mathbf{a} = \mathbf{a}_1 + \cdots + \mathbf{a}_c$ and $\mathbf{a} := (a_1, \ldots, a_d)$. The above equality assures that $\alpha \mid (\alpha - 1)a_i$ for all $i = 1, \ldots, c$. This implies $a_i = q_i'\alpha$ for some $q_i' \in \mathbb{N}$. But then $g := x_1 \cdots x_c - y_1^{q_1'} \cdots y_d^{q_d'} \in I_{\mathcal{A}}$. Since $\deg(x_1 \cdots x_c) = c \leq s$, $g \in G$ and we get a contradiction that $\mathrm{in}(g) = x_1 \cdots x_c \mid \mathrm{in}(b) = (x_1 \cdots x_c)^{\alpha - 1}$. The proof of the theorem is completed.

Remark 1.5. Theorem 1.4 shows that our conjecture holds if the codimension is not too big. This includes the case $c = 2$ and $\deg K[S] > \alpha$, because we always have $\alpha \mid \deg K[S]$ (see [HS, Lemma 3.4]). Note that the case $c = 2$ (even if $\deg K[S] = \alpha$) was completely solved by Peeva and Sturmfels (see [PS, Theorem 7.3 and Proposition 8.3]). Another proof was recently given in [BGM] (see Theorems 2.1, 2.8 and 3.5 there).

Note that an ideal is usually given by its generating set and this set serves as the input data for computing a Gröbner basis. In the case of a toric ideal $I_{\mathcal{A}}$ the input data is $\mathcal{A}$, and before computing a Gröbner basis of $I_{\mathcal{A}}$ we have to compute a generating set of this ideal. However, in many algorithms we get a Gröbner basis of $I_{\mathcal{A}}$ as a by-product of computing a generating set. The last result of this section shows that, by using a suitable term order, the computation of simplicial toric ideals runs rather quickly. In order to compute $I_{\mathcal{A}}$, a standard procedure is the following (see, e.g., [St1], Algorithm 4.5):

1. Form the ideal $J_{\mathcal{A}} = (x_1 - \mathbf{t}^{\mathbf{a}_1}, \ldots, x_c - \mathbf{t}^{\mathbf{a}_c}, y_1 - t_1^{\alpha}, \ldots, y_d - t_d^{\alpha}) \subset K[\mathbf{t}, \mathbf{x}, \mathbf{y}]$.
2. Compute a Gröbner basis G' of $J_{\mathcal{A}}$ by Buchberger's algorithm, using an elimination order $\preceq$ with respect to the variables $t_1, \ldots, t_d$. Here we assume $t_1 \succeq \cdots \succeq t_d \succeq x_1 \succeq \cdots \succeq y_d$.

3. From G' get a Gröbner basis $G = G' \cap K[\mathbf{x},\mathbf{y}]$ of $I_{\mathcal{A}} = J_{\mathcal{A}} \cap K[\mathbf{x},\mathbf{y}]$.

Though this algorithm is not the best one, the following result says that it requires not too many steps. Moreover, by Theorems 1.1 and 1.4, in order to compute G it is sufficient to compute those elements of G' which have degrees up to $\min\{2r(S),\, c(\alpha-1)\}$, that means we can do truncation in the above algorithm.

PROPOSITION 1.6. *Assume that the restriction of the elimination order $\preceq$ on $K[\mathbf{x},\mathbf{y}]$ is the reverse lexicographic order. Then the maximum degree in a minimal Gröbner basis of $J_{\mathcal{A}}$ is bounded by $d(\alpha-1)+\min\{2r(S),\, c(\alpha-1)\}$.*

Proof. The proof is similar to that of Theorem 1.1. We give here a sketch. Let $s = d(\alpha-1) + \min\{2r(S),\, c(\alpha-1)\}$ and set

$$G = \left\{\mathbf{t}^{\mathbf{p}}\mathbf{x}^{\mathbf{m}}\mathbf{y}^{\mathbf{n}} - \mathbf{t}^{\mathbf{p}'}\mathbf{x}^{\mathbf{m}'}\mathbf{y}^{\mathbf{n}'} \in J_{\mathcal{A}} \mid \deg(\mathbf{t}^{\mathbf{p}}\mathbf{x}^{\mathbf{m}}\mathbf{y}^{\mathbf{n}} - \mathbf{t}^{\mathbf{p}'}\mathbf{x}^{\mathbf{m}'}\mathbf{y}^{\mathbf{n}'}) \le s\right\}.$$

Assume that G is not a Gröbner basis. Then one can find a binomial $b = \mathbf{t}^{\mathbf{p}}\mathbf{x}^{\mathbf{m}}\mathbf{y}^{\mathbf{n}} - \mathbf{t}^{\mathbf{p}'}\mathbf{x}^{\mathbf{m}'}\mathbf{y}^{\mathbf{n}'} \in J_{\mathcal{A}}$ of the smallest degree $\deg b > s$ such that $\mathrm{in}(g) \nmid \mathbf{t}^{\mathbf{p}}\mathbf{x}^{\mathbf{m}}\mathbf{y}^{\mathbf{n}}$ for all $g \in G$. Since $t_i^{\alpha} - y_i \in G$ and $t_i^{\alpha} \succ y_i$, as in the proof of Theorem 1.4, we can assume that $p_i, p_i' \le \alpha - 1$ for all $i \le d$.

Using arguments in the proof of Theorem 1.1 we may assume that $\deg(\mathbf{x}^{\mathbf{m}}),\, \deg(\mathbf{x}^{\mathbf{m}'}) \le r(S)$. Note that $J_{\mathcal{A}}$ is the kernel of the epimorphism $K[\mathbf{t},\mathbf{x},\mathbf{y}] \to K[\mathbf{t}]$ mapping $t_j,\, x_i,\, y_j$ to $t_j,\, \mathbf{t}^{\mathbf{a}_i},\, t_j^{\alpha}$, respectively. Therefore $b \in J_{\mathcal{A}}$ if and only if

$$p_j + \sum_{i \in C} m_i a_{ij} + n_j \alpha = p_j' + \sum_{i \notin C} m_i' a_{ij} + n_j' \alpha \tag{4}$$

for all $j \le d$, where C and D are the same as in the proof of Theorem 1.1. Then, instead of (1) we get

$$\sum_{j \in D} p_j + \sum_{j=1}^{d} n_j \le \sum_{j \in D} p_j' + \deg(\mathbf{x}^{\mathbf{m}'}) \le (\sharp D)(\alpha-1) + r(S),$$

which implies

$$\sum_{j=1}^{d} p_j + \sum_{j=1}^{d} n_j \le d(\alpha-1) + r(S).$$

Hence $\deg(\mathbf{t}^{\mathbf{p}}\mathbf{x}^{\mathbf{m}}\mathbf{y}^{\mathbf{n}}) \le d(\alpha-1) + 2r(S)$. Similarly, $\deg(\mathbf{t}^{\mathbf{p}'}\mathbf{x}^{\mathbf{m}'}\mathbf{y}^{\mathbf{n}'}) \le d(\alpha - 1) + 2r(S)$, and so $\deg(b) \le d(\alpha-1) + 2r(S)$.

Now, applying arguments in the proof of Theorem 1.4 to (4) we can also conclude that $\deg(b) \le d(\alpha-1)+c(\alpha-1)$.

Summing up, we get $\deg(b) \le s$, a contradiction. ∎

§2. Eisenbud-Goto bound

In this section we will provide some partial positive answers to our conjecture.

Recall that a quotient ring R/I modulo a homogeneous ideal I is said to be a *generalized Cohen-Macaulay* ring if all local cohomology modules $H^i_{\mathfrak{m}}(R/I)$, $i < \dim R/I$, with the support in the maximal homogeneous ideal $\mathfrak{m}$ of R/I are of finite length (see the Appendix in [SV1]). The Castelnuovo-Mumford regularity of a finitely generated graded R-module M is the number

$$\operatorname{reg}(M) = \max\{n \mid [H^i_{\mathfrak{m}}(M)]_{n-i} \neq 0 \text{ for } i \geq 0\}.$$

Note that $\operatorname{reg}(I) = \operatorname{reg}(R/I)+1$. The following result is a simple observation, but has some interesting consequences.

LEMMA 2.1. *Assume that $K[S]$ is a generalized Cohen-Macaulay ring. Then the maximum degree in a minimal Gröbner basis of $I_{\mathcal{A}}$ is bounded by* $\operatorname{reg} I_{\mathcal{A}}$.

Proof. Note that $y_1,\dots,y_d$ is a system of parameters of $K[S]$. Since $K[S] \cong K[\mathbf{x},\mathbf{y}]/I_{\mathcal{A}}$ is a generalized Cohen-Macaulay ring, the ideal $I_{\mathcal{A}}$ and all ideals $(I_{\mathcal{A}}, y_d,\dots,y_i)$, $i = d, d-1,\dots,1$, are unmixed up to $\mathfrak{m}$-primary components (see [SV1, Proposition 3 in the Appendix]). In particular, y_{i-1} is a non-zero divisor on the ring $K[\mathbf{x},\mathbf{y}]/(I_{\mathcal{A}}, y_d,\dots,y_i)^{\text{sat}}$, where $J^{\text{sat}} = \bigcup_{n\geq 1} J : \mathfrak{m}^n$ denotes the saturation of J. This means $y_d,\dots,y_1$ is a generic sequence of $K[S]$ in the sense of [BS, Definition 1.5]. By [BS, Corollary 2.5], the maximum degree in a minimal Gröbner basis of $I_{\mathcal{A}}$ is bounded by $\operatorname{reg}(I_{\mathcal{A}})$. ∎

Remark 2.2. Note that $y_d,\dots,y_1$ is always a system of parameters of $\operatorname{in}(I_{\mathcal{A}})$. This follows from the fact that $x_i^{\alpha} \in \operatorname{in}(I_{\mathcal{A}})$ for all $i \le c$ (since $x_i^{\alpha} - \mathbf{y}^{\mathbf{a}_i} \in I_{\mathcal{A}}$). However, if $K[S]$ is not a generalized Cohen-Macaulay ring, it maybe no more a generic sequence of $K[S]$. For example, let $d=\alpha=3$ and

$$\mathcal{A} = \{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3,\ \mathbf{a}_1=(2,0,1),\ \mathbf{a}_2=(1,2,0),\ \mathbf{a}_3=(1,1,1),$$
$$\mathbf{a}_4=(1,0,2),\ \mathbf{a}_5=(0,2,1),\ \mathbf{a}_6=(0,1,2)\}.$$

https://doi.org/10.1017/S002776300000979X Published online by Cambridge University Press

Then

$$\text{in}(I_{\mathcal{A}}) = (x_1x_2, x_2x_3, x_2x_5, x_1^2, x_1x_3, x_3^2, x_2x_4, x_2x_6, x_3x_5, x_5^2, x_1x_4, x_3x_4, x_4x_5, x_4^2, x_3x_6, x_5x_6, x_4x_6, x_6^2, x_2^3, x_1x_6y_2).$$

Clearly $(\text{in}(I_{\mathcal{A}}), y_3)^{sat} = (\text{in}(I_{\mathcal{A}}), y_3)$ and y_2 is a zero divisor of $K[\mathbf{x}]/(\text{in}(I_{\mathcal{A}}), y_3)$. Hence, by [BS, Theorem 2.4(a)], y_3, y_2, y_1 is not a generic sequence of $K[\mathbf{x}]/I_{\mathcal{A}}$.

Corollary 2.3. *The maximum degree in a minimal Gröbner basis of $I_{\mathcal{A}}$ is bounded by $\deg K[S] - \operatorname{codim} K[S] + 1$ in the following cases:*

(*i*) *$d = 2$,*

(*ii*) *$K[S]$ is a so-called Buchsbaum ring,*

(*iii*) *$K[S]$ is a simplicial semigroup ring with isolated singularity, or equivalently, $\mathcal{A}$ contains all points of $M_{\alpha,d}$ of type $(0, \dots, \alpha-1, \dots, 1, \dots, 0)$, where $\alpha - 1$, 1 stay in the i-th and j-th positions, respectively, and the other coordinates are zero.*

Proof. In all these cases, $K[S]$ is a generalized Cohen-Macaulay ring and it is known that $\text{reg}(I_{\mathcal{A}}) \le \deg K[S] - \operatorname{codim} K[S] + 1$ (the case (i) is proved in [GLP], (ii) in [SV2, Theorem 1] and (iii) is [HH, Corollary 2.2]). Hence the statement follows from Lemma 2.1. □

Recall that a semigroup S is said to be *normal* if $S = \mathbb{Z}(S) \cap \mathbb{N}^d$. Under this condition, it is well-known that $\text{reg}(I_{\mathcal{A}}) \le d$ (this holds even without the assumption S being simplicial, see [St1, Proposition 13.14]). Hence, by Lemma 2.1, the maximum degree in a minimal Gröbner basis of $I_{\mathcal{A}}$ is bounded by d. This gives a partial answer to the following question posed by Sturmfels in [St1, p. 136]: If the semigroup S is normal, does the toric ideal $I_{\mathcal{A}}$ posses a Gröbner basis of degree at most d?

Under the assumption of the following result it was shown in [HS, Proposition 3.7] that $\text{reg}(I_{\mathcal{A}}) \le \deg K[S] - \operatorname{codim} K[S] + 1$. Unfortunately we cannot use it to derive the corresponding result for a Gröbner basis, because $K[S]$ is in general not a generalized Cohen-Macaulay ring.

Proposition 2.4. *Assume that $\deg K[S] = \alpha^{d-1}$ and $\alpha \le d - 1$. Then the maximum degree in a minimal Gröbner basis of $I_{\mathcal{A}}$ is bounded by $\deg K[S] - \operatorname{codim} K[S] + 1$.*

https://doi.org/10.1017/S002776300000979X Published online by Cambridge University Press

In order to prove this proposition we need to recall a result from [HS]. Let $\mathcal{P}$ denote the convex polytope spanned by $\mathcal{A} \subset \mathbb{R}^d$. Note that $\mathcal{P}$ is a $(d-1)$-dimensional polytope whose faces are spanned by

$$\mathcal{A}_I = \{\mathbf{a} \in \mathcal{A} \mid a_i = 0 \text{ for all } i \in I\},$$

where $I \subseteq \{1, \dots, d\}$. Let $\mathcal{P}_I$ denote the corresponding face of $\mathcal{P}$. (For short, we will also write $\mathcal{A}_i$, $\mathcal{P}_i$ instead of $\mathcal{A}_{\{i\}}$, $\mathcal{P}_{\{i\}}$.) We say that a face $\mathcal{P}_I$ is *full* if $\mathcal{A}_I$ contains all points of $M_{\alpha,d}$ lying on this face, i.e. if $\mathcal{A}_I = \mathcal{P}_I \cap M_{\alpha,d}$.

LEMMA 2.5. ([HS, Lemma 1.2]) *If $\mathcal{P}$ has a full face of dimension i, then $r(S) \leq \alpha^{d-1-i} + i - 1$.*

Proof of Proposition 2.4. If $\mathcal{A} = M_{\alpha,d}$, then by Corollary 2.3(iii) we are done. Hence we may assume that

$$c \leq \sharp M_{\alpha,d} - 1 - d = \binom{\alpha+d-1}{d-1} - d - 1.$$

If $\alpha \geq 3$, $d \geq 6$ or $\alpha = 4$, $d = 5$, then by [HS, Claim 1, p. 141], $c \leq \alpha^{d-2}$. Hence

$$\deg K[S] - c + 1 > \alpha^{d-1} - \alpha^{d-2} = \alpha^{d-2}(\alpha - 1) \geq c(\alpha - 1),$$

and by Theorem 1.4 we are done. Thus the left cases are: $\alpha = 2$, $d \geq 3$; $\alpha = 3$, $d = 4$ and $\alpha = 3$, $d = 5$. We consider these cases separately.

Case 1: $\alpha = 2$, $d \geq 3$. Then $c \leq d(d+1)/2 - (d+1) = (d-2)(d+1)/2$. It is easy to verify that $(d-2)(d+1)/2 - 1 \leq 2^{d-2}$. Hence, if $c \leq (d-2)(d+1)/2 - 1$ we have $c \leq 2^{d-1} - c = \deg K[S] - c$. By Theorem 1.4 we are done. The left case is $c = (d-2)(d+1)/2$, i.e. $\mathcal{A}$ is obtained from $M_{2,d}$ by deleting exactly one point. We may assume $\mathcal{A} = M_{2,d} \setminus \{\mathbf{b} := (1, 1, 0, \dots, 0)\}$. Note that $2\mathbf{a}_i \in \{\mathbf{e}_1, \dots, \mathbf{e}_d\} + \{\mathbf{e}_1, \dots, \mathbf{e}_d\}$ for all $i \leq c$. Moreover, if $\mathbf{a}_i$, $\mathbf{a}_j$ are two different points and $\mathbf{a}_i$, $\mathbf{a}_j$, $\mathbf{b}$ do not lie in the same 2-dimensional face of $\mathcal{P}$, then $\mathbf{a}_i + \mathbf{a}_j \in \mathcal{A} + \{\mathbf{e}_1, \dots, \mathbf{e}_d\}$ (see Fig. 1). From this it follows that $r(S) = 2$. By Theorem 1.1, $I_{\mathcal{A}}$ has a Gröbner basis of degree at most $3 \leq 2^{d-1} - (d-2)(d+1)/2 + 1 = \deg K[S] - c + 1$.

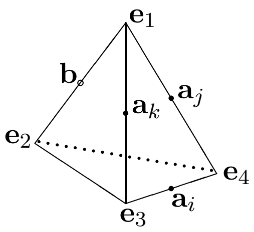

Fig. 1

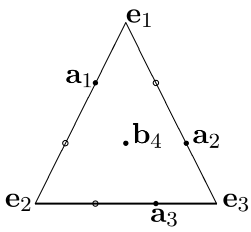

Fig. 2

Case 2: $\alpha = 3$, $d = 4$. Then $\deg K[S] = 27$ and $c \leq 15$. By Theorem 1.4, the statement of the proposition holds true for $c \leq 9$. Let $c \geq 10$, i.e. $\mathcal{A}$ is obtained from $M_{3,4}$ by deleting at most 6 points. We distinguish two subcases.

Subcase 2*a*: Each edge of $\mathcal{P}$ contains exactly one deleting point. In this case $c = 10$. By Theorem 1.1 it suffices to show that $r(S) \leq 8$.

Consider, for example, the facet $\mathcal{P}_4 = \{\mathbf{a} \in \mathcal{P} \mid a_4 = 0\}$. Then $\mathcal{A}_4 = \mathcal{A} \cap \mathcal{P}_4$ has exactly 7 points, say $\mathcal{A}_4 = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{b}_4 := (1,1,1,0)\}$ as shown in Fig. 2, where $\mathbf{a}_1$ can be taken as $(2,1,0,0)$, while there are two choices for each of $\mathbf{a}_2$ and $\mathbf{a}_3$. One can check by computer that in this case the reduction number $r(\langle \mathcal{A}_4 \rangle) \leq 3$. In particular, $\sum_{i=1}^{3} m_i \mathbf{a}_i + n_4 \mathbf{b}_4 \notin S + \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ implies that $m_i, n_4 \leq 2$ and

$$(5) \qquad \sum_{i=1}^{3} m_i + n_4 \leq 3.$$

Moreover, since $2\mathbf{b}_4 + \mathbf{a}_1 = (4,3,2,0) = \mathbf{e}_2 + 2(2,0,1,0) = \mathbf{e}_1 + \mathbf{e}_2 + (1,0,2,0)$ and one of two points $(2,0,1,0)$ and $(1,0,2,0)$ on the edge $\overline{\mathbf{e}_1\mathbf{e}_3}$ must belong to $\mathcal{A}_4$, we get that $2\mathbf{b}_4 + \mathbf{a}_1 \in S + \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. The same is true for $2\mathbf{b}_4 + \mathbf{a}_2$ and $2\mathbf{b}_4 + \mathbf{a}_3$. This means, in addition to (5) we also have $m_1 = m_2 = m_3 = 0$ if $n_4 = 2$.

Finally, we can write $\mathcal{A} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{a}_1, \ldots, \mathbf{a}_6, \mathbf{b}_1, \ldots, \mathbf{b}_4\}$, where $\mathbf{b}_i$ is the inner point of the facet $\mathcal{P}_i$. Assume that

$$\sum_{i=1}^{6} m_i \mathbf{a}_i + \sum_{j=1}^{4} n_j \mathbf{b}_j \notin S + \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}.$$

Then inequalities of Type (5) should hold for all facets of $\mathcal{P}$. If $\sharp\{j \mid 1 \leq$

$j \leq 4$, $n_j \leq 1\} \geq 3$, adding all of them we get

$$2\sum_{i=1}^{6} m_i + \sum_{j=1}^{4} n_j + \sum_{j=1}^{4} n_j \leq 12 + 1 + 1 + 1 + 2 = 17.$$

Hence $\sum_{i=1}^{6} m_i + \sum_{j=1}^{4} n_j \leq 8$.

Otherwise, we may assume either (a) $n_1, n_2 \leq 1$ and $n_3 = n_4 = 2$ or (b) $n_2 = n_3 = n_4 = 2$ holds. In the case (a), we may further assume that $m_1 = \cdots = m_5 = 0$ and $m_6 \leq 2$. Then

$$\sum_{i=1}^{6} m_i + \sum_{j=1}^{4} n_j \leq m_6 + 1 + 1 + 2 + 2 \leq 8.$$

In the case (b), we have $m_1 = \cdots = m_6 = 0$. Hence

$$\sum_{i=1}^{6} m_i + \sum_{j=1}^{4} n_j \leq 2 + 2 + 2 + 2 = 8.$$

Thus, in all cases we get that $\sum_{i=1}^{6} m_i + \sum_{j=1}^{4} n_j \leq 8$, which implies $r(S) \leq 8$.

Subcase 2b: At least one edge of $\mathcal{P}$ is full. By Lemma 2.5, $r(S) \leq 9$. Hence, by Theorem 1.1, the statement holds true if $c \leq 11$. Moreover, if $\mathcal{P}$ has a full facet, then again by Lemma 2.5, $r(S) \leq 5$, and by Theorem 1.1 we are done. Hence, we may assume that $c = 12, 13, 14$, and $\mathcal{P}$ has no full facet. This corresponds to the situation when $\mathcal{P}$ has 2, 3 or 4 deleting points.

Assume that $\mathcal{P}$ has a facet, say $\mathcal{P}_4$, which contains exactly one deleting point $\mathbf{b}$, i.e. one can write $\mathcal{A}_4 = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{a}_1, \ldots, \mathbf{a}_6\}$ and $\mathbf{b} \notin \mathcal{A}_4$. By Theorem 1.1, it suffices to show that $r(S) \leq 5$. If this is not the case, then one can find $m_1, \ldots, m_c \in \mathbb{N}$ such that $\sum_{i=1}^{c} m_i = 6$ and

$$\sum_{i=1}^{c} m_i \mathbf{a}_i \notin S + \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}. \tag{6}$$

We follow the idea in the proof of [HS, Lemma 1.2]. Considering 6 partial sums $\mathbf{a}_1, \ldots, m_1\mathbf{a}_1$, $m_1\mathbf{a}_1 + \mathbf{a}_2, \ldots, \sum_{i=1}^{c} m_i\mathbf{a}_i$ we can find either two partial sums whose last coordinates are divisible by 3, or three partial sums whose last coordinates are congruent modulo 3. Taking also the differences of these

https://doi.org/10.1017/S002776300000979X Published online by Cambridge University Press

partial sums, we can find in both cases two partial sums $\mathbf{b}_1 = \sum p_i\mathbf{a}_i$ and $\mathbf{b}_2 = \sum q_i\mathbf{a}_i$ such that $m_i \geq p_i \geq q_i \geq 0$, $(i \leq 3)$, $\deg(\mathbf{b}_1) > \deg(\mathbf{b}_2) \geq 2$ and the last coordinates of $\mathbf{b}_1$, $\mathbf{b}_2$ are divisible by 3. Note that $b_3 := b_1 - b_2 \neq 0$ also is a partial sum of $\sum_{i=1}^{c} m_i\mathbf{a}_i$. Fix $i \in \{2,3\}$. We can write $\mathbf{b}_i = \mathbf{b}'_i + n_i\mathbf{e}_4$, where $\mathbf{b}'_i \in \langle \mathcal{A}_4, \mathbf{b}\rangle$. By (6) we must have $\mathbf{b}_i \notin S + \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$, which yields $0 \neq \mathbf{b}'_i \notin \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} + \langle \mathcal{A}_4\rangle$. Together with the fact $2\mathbf{b} \in \langle \mathcal{A}_4\rangle$, this implies $\mathbf{b}'_i \in \{\mathbf{b}, \mathbf{a}_1, \ldots, \mathbf{a}_6\} + \langle \mathcal{A}_4\rangle$. Since also all elements $2\mathbf{b}, \mathbf{b}+\mathbf{a}_1, \ldots, \mathbf{b}+\mathbf{a}_6 \in \langle \mathcal{A}_4\rangle$, the previous relation assures that $\mathbf{b}'_2 + \mathbf{b}'_3 \in \langle \mathcal{A}_4\rangle \subset S$. By (6) we must have $n_2 = n_3 = 0$, and so $\mathbf{b}_1 = \mathbf{b}_2 + \mathbf{b}_3 = \mathbf{b}'_2 + \mathbf{b}'_3 \in \langle \mathcal{A}_4\rangle$. However it is easy (or using computer) to see that $r(\langle \mathcal{A}_4\rangle) = 2$. Since $\deg(\mathbf{b}_1) \geq 3$, $\mathbf{b}_1 \in \langle \mathcal{A}_4\rangle + \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \subseteq S + \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$, which contradicts (6).

Thus, each facet of $\mathcal{P}$ must have at least two deleting points. In particular, $c = 12$ and $\mathcal{P}$ has exactly 4 deleting points. There are only two situations shown in Fig. 3 and Fig. 4. In the situation of Fig. 3 there is eventually one configuration, and by computer we see that $r(S) = 2$. In the situation of Fig. 4 one can show as in Subcase 2a (or using computer for eight different configurations), that $r(S) \leq 8$. But then by Theorem 1.1, $I_{\mathcal{A}}$ has a Gröbner basis of degree at most $15 < \deg K[S] - c + 1 = 16$. The Subcase 2b is completely solved.

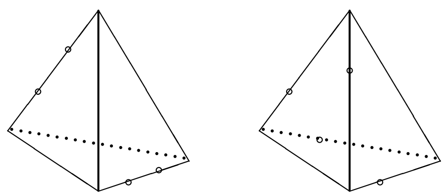

Fig. 3 Fig. 4

Case 3: $\alpha = 3$, $d = 5$. We have $c \leq 29$ and $\deg K[S] = 81$. If $c \leq 27$, then $\deg K[S] - c + 1 \geq 54 \geq 2c$, and by Theorem 1.4 we are done. If $c = 28, 29$, then $\mathcal{A}$ is obtained from $M_{3,5}$ by deleting 1 or 2 points. But then $\mathcal{P}$ has a full 2-dimensional face. By Lemma 2.5, $r(S) \leq 10$. Hence, by Theorem 1.1, we are also done in this subcase. ∎

Finally we show that if on an edge of $\mathcal{P}$ there are enough points belonging to $\mathcal{A}$, then the Eisenbud-Goto bound also holds for the maximum degree in a minimal Gröbner basis of $I_{\mathcal{A}}$. Note that in this setting, the Eisenbud-Goto conjecture on $\mathrm{reg}(I_{\mathcal{A}})$ is still not verified (cf. [HS, Corollary 3.8]).

Proposition 2.6. *Assume that* $\deg K[S] = \alpha^{d-1}$ *and there exists an edge of* $\mathcal{P}$ *such that it is either full or at least* $(\frac{3}{4}+\frac{1}{4d})\alpha+2$ *integer points on it belong to* $\mathcal{A}$*. Then the maximum degree in a minimal Gröbner basis of* $I_{\mathcal{A}}$ *is bounded by* $\deg K[S] - \operatorname{codim} K[S] + 1$.

Proof. By Corollary 2.3 and Proposition 2.4 we may assume that $\alpha \geq d \geq 3$ and

$$c \leq \binom{\alpha+d-1}{d-1} - d - 1.$$

First we consider the case when at least $(\frac{3}{4}+\frac{1}{4d})\alpha+2$ integer points on an edge belong to $\mathcal{A}$. By [HS, Lemma 1.3], $r(S) \leq \frac{d-1}{4d}\alpha^{d-1}$. Hence, by Theorem 1.1, it suffices to show that

$$\alpha^{d-1} - \binom{\alpha+d-1}{d-1} + d + 1 \geq \frac{d-1}{2d}\alpha^{d-1} - 1,$$

or equivalently

$$\frac{d+1}{2d}\alpha^{d-1} + d + 2 \geq \binom{\alpha+d-1}{d-1}. \tag{7}$$

We show this by induction on $d \geq 3$. For $d = 3$ this is equivalent to $\alpha^2 - 9\alpha + 24 \geq 0$. So, assume that the inequality holds for $d \geq 3$. In the dimension $d+1$, by induction we have

$$\binom{\alpha+d}{d} = \frac{\alpha+d}{d}\binom{\alpha+d-1}{d-1} \leq \frac{\alpha+d}{d}\left(\frac{d+1}{2d}\alpha^{d-1} + d + 2\right)$$
$$= \frac{(\alpha+d)(d+1)}{2d^2}\alpha^{d-1} + \frac{d+2}{d}\alpha + d + 2.$$

Hence

$$\begin{aligned}
&\frac{d+2}{2(d+1)}\alpha^d + d + 3 - \binom{\alpha+d}{d}\\
&\geq \alpha^{d-1}\left[\frac{d+2}{2(d+1)}\alpha - \frac{(\alpha+d)(d+1)}{2d^2}\right] - \frac{d+2}{d}\alpha + 1\\
&= \frac{\alpha(d^3+d^2-2d-1) - d(d+1)^2}{2d^2(d+1)}\alpha^{d-1} - \frac{d+2}{d}\alpha + 1\\
&\geq \frac{d(d^3+d^2-2d-1) - d(d+1)^2}{2d^2(d+1)}\alpha^{d-1} - \frac{d+2}{d}\alpha + 1 \text{ (since } \alpha \geq d \geq 3)\\
&= \frac{d^3-4d-2}{2d(d+1)}\alpha^{d-1} - \frac{d+2}{d}\alpha + 1 =: B
\end{aligned}$$

https://doi.org/10.1017/S002776300000979X Published online by Cambridge University Press

If $\alpha = 3$, then $d = 3$ and $B = 7/8$. For $\alpha \geq 4$, since $\alpha^{d-1} \geq 4\alpha$, we further get

$$B \geq \frac{2(d^3 - 4d - 2)}{d(d+1)}\alpha - \frac{d+2}{d}\alpha + 1 = \frac{d[d(2d-1) - 11] - 6}{d(d+1)}\alpha + 1 > 1.$$

Thus we always have $B > 0$, which proves (7).

Now we consider the case when an edge of $\mathcal{P}$ is full, i.e. there are exactly $\alpha + 1$ points on it belonging to $\mathcal{A}$. If $\alpha \geq 6$, then $\alpha \geq \frac{4d}{d-1}$ and the second condition is satisfied, so we are done. Since $\alpha \geq d$, the left cases are $d = 4$, $\alpha \leq 5$ and $d = 3$, $\alpha = 4, 5$. In these cases, by Lemma 2.5, $r(S) \leq \alpha^{d-2}$.

If $d = 4$, $\alpha \leq 5$, then $\deg K[S] - c + 1 \geq \alpha^3 - \binom{\alpha+3}{3} + 6 > 2\alpha^2 \geq 2r(S)$, and by Theorem 1.1 we are done.

If $d = 3$, $\alpha \leq 5$, let $\tilde{c} = \sharp(M_{\alpha,3} \setminus \mathcal{A})$. Then $r(S) \leq \alpha$, and the inequality

$$\deg K[S] - c + 1 = \alpha^2 - \binom{\alpha+2}{2} + \tilde{c} + 4 \geq 2\alpha - 1$$

does not hold only in the following situations: $\alpha = 3, 4$, $\tilde{c} = 1, 2$ and $\alpha = 5$, $\tilde{c} = 1$. By Theorem 1.1, we can restrict ourselves to these situations. By Corollary 2.3, we may assume that one deleting point is $(\alpha - 1, 1, 0)$. Thus, in each case there are only few configurations to consider. Using computer, we can check that $r(S) = 2$ if $\alpha = 3, 5$, and $r(S) \leq 3$ if $\alpha = 4$. But then $\deg K[S] - c + 1 = \alpha^2 - \binom{\alpha+2}{2} + \tilde{c} + 4 \geq 2r(S) - 1$. Again by Theorem 1.1 we are done. $\square$

Acknowledgment. We would like to thank the referee for valuable comments. This paper was initiated during the visit of the second author at Max-Planck Institute for Mathematics in the Sciences (Germany). He would like to thank the MIS for the financial support and the hospitality. All computations in this paper were done by using the package CoCoa [CoCoA].

References

[BS] D. Bayer and M. Stillman, *A criterion for detecting m-regularity*, Invent. Math., **87** (1987), no. 1, 1–11; MR 87k:13019.

[BGM] I. Bermejo, P. Gimenez and M. Morales, *Castelnuovo-Mumford regularity of projective monomial varieties of codimension two*, J. Symb. Comp., **41** (2006), 1105–1124; MR 2007i:14046.

[CoCoA] A. Capani, G. Niesi and L. Robbiano, CoCoA, a system for doing Computations in Commutative Algebra, available via anonymous ftp from: cocoa.dima.unige.it.

[EG] D. Eisenbud and S. Goto, *Linear free resolutions and minimal multiplicity*, J. Algebra, **88** (1984), 89–133; MR 85f:13023.

[GLP] L. Gruson, R. Lazarsfeld and C. Peskine, *On a theorem of Castelnuovo, and the equations defining space curves*, Invent. Math., **72** (1983), 491–506; MR 85g:14033.

[HH] J. Herzog and T. Hibi, *Castelnuovo-Mumford regularity of simplicial semigroup rings with isolated singularity*, Proc. Amer. Math. Soc., **131** (2003), 2641–2647; MR 2004j:13025.

[HHy] L. T. Hoa and E. Hyry, *Castelnuovo-Mumford regularity of initial ideals*, J. Symb. Comp., **38** (2004), 1327–1341; MR 2007b:13028.

[HS] L. T. Hoa and J. Stückrad, *Castelnuovo-Mumford regularity of simplicial toric rings*, J. Algebra, **259** (2003), 127–146; MR 2003j:13024.

[PS] I. Peeva and B. Sturmfels, *Syzygies of codimension* 2 *lattice ideals*, Math. Z., **229** (1998), 163–194; MR 99g:13020.

[St1] B. Sturmfels, Gröbner bases and convex polytopes, University Lecture Series 8, American Mathematical Society, Providence, RI, 1996; MR 97b:13034.

[St2] B. Sturmfels, *Equations defining toric varieties*, Algebraic geometry - Santa Cruz 1995, pp. 437–449, Proc. Sympos. Pure Math., **62**, Part 2, Amer. Math. Soc., Providence, RI, 1997; MR 99b:14058.

[SV1] J. Stückrad and W. Vogel, Buchsbaum rings and applications, An interaction between algebra, geometry and topology, Springer-Verlag, Berlin, 1986; MR 88h:13011.

[SV2] J. Stückrad and W. Vogel, *Castelnuovo bounds for certain subvarieties in* $\mathbf{P}^n$, Math. Ann., **276** (1987), 341–352; MR 88e:13013.

Michael Hellus
NWF I-Mathematik
Universität Regensburg
D-93040 Regensburg
Germany
michael.hellus@mathematik.uni-regensburg.de

Lê Tuân Hoa
Institute of Mathematics Hanoi
18 Hoang Quoc Viet Road
10307 Hanoi
Vietnam
lthoa@math.ac.vn

Jürgen Stückrad
Universität Leipzig
Fakultät für Mathematik und Informatik
Augustusplatz 10/11
D-04109 Leipzig
Germany
stueckrad@math.uni-leipzig.de

https://doi.org/10.1017/S002776300000979X Published online by Cambridge University Press