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On the Endomorphism Rings of Local Cohomology Modules

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Abstract. Let R be a commutative Noetherian ring and $\mathfrak{a}$ a proper ideal of R. We show that if $n := \operatorname{grade}_R \mathfrak{a}$, then $\operatorname{End}_R(H^n_{\mathfrak{a}}(R)) \cong \operatorname{Ext}^n_R(H^n_{\mathfrak{a}}(R), R)$. We also prove that, for a nonnegative integer n such that $H^i_{\mathfrak{a}}(R) = 0$ for every $i \neq n$, if $\operatorname{Ext}^i_R(R_z, R) = 0$ for all $i > 0$ and $z \in \mathfrak{a}$, then $\operatorname{End}_R(H^n_{\mathfrak{a}}(R))$ is a homomorphic image of R, where R_z is the ring of fractions of R with respect to a multiplicatively closed subset $\{z^j \mid j \geqslant 0\}$ of R. Moreover, if $\operatorname{Hom}_R(R_z, R) = 0$ for all $z \in \mathfrak{a}$, then $\mu_{H^n_{\mathfrak{a}}(R)}$ is an isomorphism, where $\mu_{H^n_{\mathfrak{a}}(R)}$ is the canonical ring homomorphism $R \to \operatorname{End}_R(H^n_{\mathfrak{a}}(R))$.

1 Introduction

Let R be a commutative ring and M be an R-module. There is a canonical map

$$\mu_M : R \longrightarrow \operatorname{End}_R(M)$$

such that for $r \in R$, $\mu_M(r)$ is the multiplication map by r on M. It is easy to see that μ_M is a homomorphism of (associative) R-algebras. In general, μ_M is neither injective nor surjective. So, we consider that it is of interest to determine some conditions on M that ensure that μ_M is bijective.

Let $(R, \mathfrak{m})$ be a Noetherian local ring and $H^n_{\mathfrak{a}}(-)$ be the n-th local cohomology functor with support in an ideal $\mathfrak{a}$ of R. Let $D(-)$ be the Matlis dual functor $\operatorname{Hom}_R(-, E)$, where E is the injective hull of the field $R/\mathfrak{m}$ (cf. [10]). There were some problems related to the module $D(H^n_{\mathfrak{a}}(M))$. (See for example conjecture $(*)$ in [2] and [4].) By using the theory of D-modules of [9], Hellus showed that, in a certain situation, for some positive integer n, $H^n_{\mathfrak{a}}(D(H^n_{\mathfrak{a}}(R)))$ is either E or zero [3]. In [7], the present author obtained a generalization of Hellus' Theorem. By using this generalization in conjunction with the spectral sequences method, Hellus and Stückrad showed that if R is Noetherian local complete and $\mathfrak{a}$ an ideal of R such that $H^i_{\mathfrak{a}}(R) = 0$ for every $i \neq n (= \operatorname{height} \mathfrak{a})$, then $\mu_{H^n_{\mathfrak{a}}(R)}$ is bijective [5].

In this paper, by using a natural generalization of regular sequences, we first prove that $\operatorname{End}_R(H^n_{\mathfrak{a}}(R)) \cong \operatorname{Ext}^n_R(H^n_{\mathfrak{a}}(R), R)$, where n is the grade of a proper ideal $\mathfrak{a}$ of R. Moreover, we show that for a nonnegative integer n such that $H^i_{\mathfrak{a}}(R) = 0$ for every $i \neq n$ if $\operatorname{Ext}^i_R(R_z, R) = 0$ for all $i > 0$ and $z \in \mathfrak{a}$, then $\operatorname{End}_R(H^n_{\mathfrak{a}}(R))$ is a homomorphic image of R. (For an R-module L and an element z in R, we use the

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notation L_z for the module of fractions of L with respect to the multiplicatively closed subset $\{z^u \mid u \geqslant 0\}$ of R.) Also if, in addition, $\mathrm{Hom}_R(R_z, R) = 0$ for all $z \in \mathfrak{a}$, then $\mu_{H^n_{\mathfrak{a}}(R)}$ is bijective. Finally, as a consequence, we deduce the above-mentioned main result of [5].

Throughout this paper, R will denote a commutative Noetherian ring with non-zero identity and $\mathfrak{a}$ an ideal of R. We shall use $\mathbb{N}_0$ (respectively $\mathbb{N}$) to denote the set of nonnegative (respectively positive) integers. Also M will denote a finitely generated R-module. Our terminology follows the textbook [1] on local cohomology.

2 Endomorphism Ring

In this note, we study the endomorphism ring of local cohomology module $H^n_{\mathfrak{a}}(R)$ for a nonnegative integer n. To do this, we need a natural generalization of regular sequences, called filter regular sequences.

We say that a sequence $x_1, \dots, x_n$ of elements of $\mathfrak{a}$ is an $\mathfrak{a}$-filter regular sequence on M if

$$\mathrm{Supp}_R\left(\frac{(x_1, \dots, x_{i-1})M :_M x_i}{(x_1, \dots, x_{i-1})M}\right) \subseteq V(\mathfrak{a})$$

for all $i = 1, \dots, n$, where $V(\mathfrak{a})$ denotes the set of prime ideals of R containing $\mathfrak{a}$. Also, we say that an element $x \in \mathfrak{a}$ is an $\mathfrak{a}$-filter regular element on M if $\mathrm{Supp}_R(0 :_M x) \subseteq V(\mathfrak{a})$. The concept of an $\mathfrak{a}$-filter regular sequence on M is a generalization of the concept of a filter regular sequence, which has been studied in [6, 8, 11, 12] and has led to some interesting results. Both concepts coincide if $\mathfrak{a}$ is an $\mathfrak{m}$-primary ideal of a local ring with maximal ideal $\mathfrak{m}$. Note that $x_1, \dots, x_n$ is a weak M-sequence if and only if it is an R-filter regular sequence on M. It is easy to see that the analogue of [12, Appendix 2(ii)] holds true whenever R is Noetherian, M is finitely generated and $\mathfrak{m}$ replaced by $\mathfrak{a}$. If $x_1, \dots, x_n$ is an $\mathfrak{a}$-filter regular sequence on M, then there is an element $y \in \mathfrak{a}$ such that $x_1, \dots, x_n, y$ is an $\mathfrak{a}$-filter regular sequence on M. Thus, for a positive integer n, there exists an $\mathfrak{a}$-filter regular sequence on M of length n.

Now, we recall an exact sequence of local cohomology modules.

Proposition 2.1 (See [7, Lemma 2.2]) *For a nonnegative integer n and an $\mathfrak{a}$-filter regular sequence $x_1, \dots, x_{n+1} \in \mathfrak{a}$ on M, there exists an exact sequence*

$$0 \longrightarrow H^n_{\mathfrak{a}}(M) \longrightarrow H^n_{(x_1,\dots,x_n)}(M) \longrightarrow (H^n_{(x_1,\dots,x_n)}(M))_{x_{n+1}} \longrightarrow H^{n+1}_{(x_1,\dots,x_{n+1})}(M) \longrightarrow 0.$$

The following lemma is important to further our investigation in this paper.

Lemma 2.2 *Let T be an $\mathfrak{a}$-torsion R-module and $x \in \mathfrak{a}$. Then, for every R-module L, $\mathrm{Ext}^i_R(T, L_x) = 0$ for all $i \in \mathbb{N}_0$.*

Proof Suppose that f is an arbitrary element of $\mathrm{Hom}_R(T, L_x)$ and $t \in T$. Then $f(t) = \ell/x^u$ for some $\ell \in L$ and $u \in \mathbb{N}_0$. Since T is an $\mathfrak{a}$-torsion R-module, there exists a positive integer v such that $\mathfrak{a}^v t = 0$. Hence $x^v\ell/x^u = 0$ in L_x. This implies that $x^\omega \ell = 0$ for some $\omega \in \mathbb{N}_0$ and so $f(t) = 0$. Thus $\mathrm{Hom}_R(T, L_x) = 0$.

Now, since T is $\mathfrak{a}$-torsion, by [1, Exercise 2.1.8], there exists an injective resolution of T in which each term is an $\mathfrak{a}$-torsion R-module. Hence, in view of the above paragraph, $\mathrm{Ext}^i_R(T, L_x) = 0$ for all $i \in \mathbb{N}_0$. ∎

Remark 2.3 Let L be an R-module. Consider the map $\mu_L\colon R \to \mathrm{End}_R(L)$ that maps r to the homomorphism given by multiplication by r on L. It is easy to see that μ_L is an R-algebra homomorphism. Let n be a non-negative integer such that $\mathrm{End}_R(H^n_{\mathfrak{a}}(R)) \cong R$. Then the argument used in the last part of proof of [5, Theorem 2.2], showed that the R-algebra homomorphism $\mu_{H^n_{\mathfrak{a}}(R)}$ is bijective.

Proposition 2.4 *Let n be a nonnegative integer and $x_1, \dots, x_n$ be an $\mathfrak{a}$-filter regular sequence on M. Let T be an $\mathfrak{a}$-torsion R-module. Then*

$$\mathrm{Hom}_R(T, H^n_{\mathfrak{a}}(M)) \cong \mathrm{Hom}_R(T, H^n_{(x_1,\dots,x_n)}(M)).$$

In particular, $\mathrm{End}_R(H^n_{\mathfrak{a}}(M)) \cong \mathrm{Hom}_R(H^n_{\mathfrak{a}}(M), H^n_{(x_1,\dots,x_n)}(M))$.

Proof Let x_{n+1} be an element in $\mathfrak{a}$ such that $x_1, \dots, x_n, x_{n+1}$is an $\mathfrak{a}$-filter regular sequence on M. (Note that the existence of such an element is explained in the beginning of this section.) By Proposition 2.1, there exists an exact sequence

$$0 \longrightarrow H^n_{\mathfrak{a}}(M) \longrightarrow H^n_{(x_1,\dots,x_n)}(M) \longrightarrow (H^n_{(x_1,\dots,x_n)}(M))_{x_{n+1}}.$$

Now, by applying the functor $\mathrm{Hom}_R(T, -)$ to the above exact sequence in conjunction with Lemma 2.2, we have the following isomorphism:

$$\mathrm{Hom}_R(T, H^n_{\mathfrak{a}}(M)) \cong \mathrm{Hom}_R(T, H^n_{(x_1,\dots,x_n)}(M)). \qquad \blacksquare$$

In the rest of the paper, we need the Čech complex of R with respect to a sequence of elements of R, so we mention the following notations.

Notations 2.5 Let $\underline{y} := y_1, \dots, y_n$ be a sequence of elements of R. Set $\mathfrak{b} := (y_1, \dots, y_n)$. Recall that the Čech complex $C(\underline{y}, R)^\bullet$ of R with respect to $\underline{y}$ is the complex

$$0 \longrightarrow C^0 \longrightarrow C^1 \longrightarrow \dots \longrightarrow C^i \longrightarrow C^{i+1} \longrightarrow \dots \longrightarrow C^n \longrightarrow 0,$$

where $C^0 = R$ and for $1 \leqslant i \leqslant n$, C^i is a direct sum of some copies of $R_{y_{k(1)}\cdots y_{k(i)}}$, where $1 \leqslant k(1) < k(2) < \cdots < k(i) \leqslant n$ (cf. [1, Proposition and Definition 5.1.5]). Also note that, by [1, Theorem 5.1.19], $H^i(C(\underline{y}, R)^\bullet) \cong H^i_{\mathfrak{b}}(R)$ for all $i \in \mathbb{N}_0$.

In the following theorem we study the endomorphism ring $\mathrm{End}_R(H^{\mathrm{grade}_R \mathfrak{a}}_{\mathfrak{a}}(R))$ as we promised in the introduction.

Theorem 2.6 *Let $\mathfrak{a}$ be a proper ideal of R and $n :=$ $\mathrm{grade}_R\, \mathfrak{a}$. Then, for every $\mathfrak{a}$-torsion R-module T, we have the following isomorphism:*

$$\mathrm{Hom}_R(T, H^n_{\mathfrak{a}}(R)) \cong \mathrm{Ext}^n_R(T, R).$$

In particular, $\mathrm{End}_R(H^n_{\mathfrak{a}}(R)) \cong \mathrm{Ext}^n_R(H^n_{\mathfrak{a}}(R), R)$.

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Proof In view of the case $n = 0$ of Proposition 2.4, we may assume that $n > 0$. Let $\underline{x} := x_1, \dots, x_n$ be a regular sequence on R contained in $\mathfrak{a}$ and

$$C(\underline{x}, R)^\bullet : 0 \longrightarrow C^0 \xrightarrow{d^0} C^1 \longrightarrow \cdots \longrightarrow C^i \xrightarrow{d^i} C^{i+1} \longrightarrow \cdots \xrightarrow{d^{n-1}} C^n \longrightarrow 0$$

denote the Čech complex of R with respect to $\underline{x}$. Since $H^i(C(\underline{x}, R)^\bullet) \cong H^i_{(x_1,\dots,x_n)}(R) = 0$ for all i with $0 \leqslant i \leqslant n-1$ and $H^n_{(x_1,\dots,x_n)}(R) = C^n/\operatorname{Im} d^{n-1}$, we have the following exact sequence

$$0 \longrightarrow C^0 \xrightarrow{d^0} C^1 \longrightarrow \cdots \longrightarrow C^{n-1} \xrightarrow{d^{n-1}} C^n \xrightarrow{\varepsilon} H^n_{(x_1,\dots,x_n)}(R) \longrightarrow 0,$$

where ε is the natural homomorphism. Set $L^i := \operatorname{Im} d^i$ for all i with $0 \leqslant i \leqslant n-1$. Hence, we have the exact sequences

$$0 \longrightarrow L^{n-1} \longrightarrow C^n \longrightarrow H^n_{(x_1,\dots,x_n)}(R) \longrightarrow 0,$$
$$0 \longrightarrow L^{i-1} \longrightarrow C^i \longrightarrow L^i \longrightarrow 0,$$

for all i with $1 \leqslant i \leqslant n-1$. Now, in view of Lemma 2.2, we obtain the following isomorphisms:

$$\operatorname{Hom}_R(T, H^n_{(x_1,\dots,x_n)}(R)) \cong \operatorname{Ext}^1_R(T, L^{n-1}) \cong \operatorname{Ext}^2_R(T, L^{n-2}) \cong \cdots$$
$$\cdots \cong \operatorname{Ext}^{n-1}_R(T, L^1) \cong \operatorname{Ext}^n_R(T, C^0) \cong \operatorname{Ext}^n_R(T, R).$$

Since $x_1, \dots, x_n$ is also an $\mathfrak{a}$-filter regular sequence on R, the result now follows from Proposition 2.4. ∎

For an R-module M, the cohomological dimension of M with respect to $\mathfrak{a}$ is defined as

$$\operatorname{cd}(\mathfrak{a}, M) := \max\{i \in \mathbb{Z} \mid H^i_{\mathfrak{a}}(M) \neq 0\}.$$

Theorem 2.7 *Let $\mathfrak{a}$ be a proper ideal of R such that $n := \operatorname{grade}_R \mathfrak{a} = \operatorname{cd}(\mathfrak{a}, R)$. Let $\operatorname{Ext}^i_R(R_z, R) = 0$ for all $i \in \mathbb{N}$ and $z \in \mathfrak{a}$.*

(i) *$\operatorname{End}_R(H^n_{\mathfrak{a}}(R))$ is a homomorphic image of R.*
(ii) *If, moreover, $\operatorname{Hom}_R(R_z, R) = 0$ for all $z \in \mathfrak{a}$, then $\operatorname{End}_R(H^n_{\mathfrak{a}}(R)) \cong R$ and so $\mu_{H^n_{\mathfrak{a}}(R)}$ is bijective.*

Proof First, suppose that $n > 0$ and that $\underline{y} := y_1, \dots, y_t$ is a generating set of $\mathfrak{a}$. Then, by [1, Corollary 3.3.3], we have that $t \geqslant n$. Consider the Čech complex of R with respect to $\underline{y}$ as follows:

$$C(\underline{y}, R)^\bullet : 0 \longrightarrow C^0 \xrightarrow{d^0} C^1 \longrightarrow \cdots \longrightarrow C^n \xrightarrow{d^n} C^{n+1} \longrightarrow \cdots \xrightarrow{d^{t-1}} C^t \longrightarrow 0.$$

For every $i \in \mathbb{N}_0$ with $0 \leqslant i \leqslant t-1$, we put $L^i := \operatorname{Im} d^i$. Since $H^i_{\mathfrak{a}}(R) = 0$ for all i with $i \neq n$, we have the exact sequences

$$0 \longrightarrow C^0 \xrightarrow{d^0} C^1 \longrightarrow \cdots \longrightarrow C^{n-1} \xrightarrow{d^{n-1}} L^{n-1} \longrightarrow 0,$$
$$0 \longrightarrow \operatorname{Ker} d^n \longrightarrow C^n \xrightarrow{d^n} C^{n+1} \longrightarrow \cdots \longrightarrow C^t \longrightarrow 0.$$

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Note that L^n is not defined in the case $n = t$ (similarly for L^1 in the case $n = 0$). Hence we have the exact sequences

$$0 \longrightarrow \operatorname{Ker} d^n \longrightarrow C^n \longrightarrow L^n \longrightarrow 0, \tag{2.1}$$

$$0 \longrightarrow L^{i-1} \longrightarrow C^i \longrightarrow L^i \longrightarrow 0, \tag{2.2}$$

for all i with $1 \leqslant i \leqslant t-1$ and $i \neq n$. Now, by our assumption $\operatorname{Ext}^j_R(C^i, R) = 0$ for all $i, j \in \mathbb{N}$. Thus, by using the exact sequences (2.1) and (2.2), we have the following isomorphisms for $i \in \mathbb{N}$:

$$\operatorname{Ext}^i_R(\operatorname{Ker} d^n, R) \cong \operatorname{Ext}^{i+1}_R(L^n, R) \cong \operatorname{Ext}^{i+2}_R(L^{n+1}, R) \cong \cdots$$
$$\cdots \cong \operatorname{Ext}^{i+t-n}_R(L^{t-1}, R) = \operatorname{Ext}^{i+t-n}_R(C^t, R) = 0,$$

and so

$$\operatorname{Ext}^i_R(\operatorname{Ker} d^n, R) = 0 \text{ for all } i \in \mathbb{N}. \tag{2.3}$$

Since $H^n_{\mathfrak{a}}(R) = \operatorname{Ker} d^n / L^{n-1}$, we have the following exact sequence

$$0 \longrightarrow L^{n-1} \longrightarrow \operatorname{Ker} d^n \longrightarrow H^n_{\mathfrak{a}}(R) \longrightarrow 0. \tag{2.4}$$

Whenever $n = 1$, by considering the R-module $\operatorname{Hom}_R(L^0, R)$, the result immediately follows from Theorem 2.6 and the exact sequence (2.4). So we may also assume that $n \geqslant 2$. Now, Theorem 2.6, in conjunction with (2.2),(2.3), and (2.4), induces the following isomorphisms.

$$\operatorname{End}_R(H^n_{\mathfrak{a}}(R)) \cong \operatorname{Ext}^n_R(H^n_{\mathfrak{a}}(R), R) \cong \operatorname{Ext}^{n-1}_R(L^{n-1}, R) \cong \cdots \cong \operatorname{Ext}^1_R(L^1, R)$$

Also, (2.2) implies the exact sequence

$$\operatorname{Hom}_R(C^1, R) \longrightarrow \operatorname{Hom}_R(L^0, R) \longrightarrow \operatorname{Ext}^1_R(L^1, R) \longrightarrow 0.$$

Therefore, $\operatorname{End}_R(H^n_{\mathfrak{a}}(R))$ is a homomorphic image of $\operatorname{Hom}_R(L^0, R)$ and the last module is R because d^0 is injective. If, moreover, $\operatorname{Hom}_R(R_z, R) = 0$ for all $z \in \mathfrak{a}$, then $\operatorname{Hom}_R(C^1, R) = 0$ and so $\operatorname{End}_R(H^n_{\mathfrak{a}}(R)) \cong R$.

In the case that $n = 0$, by slight modifications in the first part of the above arguments, we conclude that there exist the exact sequences (2.2) and

$$0 \longrightarrow \Gamma_{\mathfrak{a}}(R) \longrightarrow R \longrightarrow L^0 \longrightarrow 0.$$

So we have the exact sequence

$$0 \longrightarrow \operatorname{Hom}_R(L^0, R) \longrightarrow R \longrightarrow \operatorname{Hom}_R(\Gamma_{\mathfrak{a}}(R), R) \longrightarrow \operatorname{Ext}^1_R(L^0, R) \tag{2.5}$$

and the isomorphisms

$$\operatorname{Ext}^1_R(L^0, R) \cong \operatorname{Ext}^2_R(L^1, R) \cong \ldots \cong \operatorname{Ext}^t_R(L^{t-1}, R) \cong \operatorname{Ext}^t_R(C^t, R).$$

But, by our assumption, $\mathrm{Ext}^t_R(C^t, R) = 0$. Thus, in view of Theorem 2.6, $\mathrm{End}_R(\Gamma_{\mathfrak{a}}(R))$ is a homomorphic image of R.

For the last assertion, in light of (2.2), we have the exact sequence

$$\mathrm{Hom}_R(C^1, R) \longrightarrow \mathrm{Hom}_R(L^0, R) \longrightarrow \mathrm{Ext}^1_R(L^1, R) \tag{2.6}$$

and the isomorphisms

$$\mathrm{Ext}^1_R(L^1, R) \cong \mathrm{Ext}^2_R(L^2, R) \cong \ldots \cong \mathrm{Ext}^{t-1}_R(L^{t-1}, R) = \mathrm{Ext}^{t-1}_R(C^t, R) = 0.$$

(Note that if $t = 1$, then $L^1 = 0$.) By our assumption, $\mathrm{Hom}_R(C^1, R) = 0$ and so, by (2.6), $\mathrm{Hom}_R(L^0, R) = 0$. Now, (2.5) completes the proof. ∎

The following corollary is an immediate consequence of Theorem 2.7, which is a main result of [5].

Corollary 2.8 (See [5, Theorem 2.2]) *Let $(R, \mathfrak{m})$ be a Noetherian local complete ring and $\mathfrak{a}$ an ideal of R such that $n := \mathrm{grade}_R\, \mathfrak{a} = \mathrm{cd}(\mathfrak{a}, R)$. Set $H := H^n_{\mathfrak{a}}(R)$. Then*

$$\mu_H \colon R \longrightarrow \mathrm{End}_R(H)$$

is an isomorphism of R-algebras.

Proof Let $i \in \mathbb{N}_0$. Since R is complete and $\mathfrak{a} \subseteq \mathfrak{m}$, for every $z \in \mathfrak{a}$ we have the following isomorphisms

$$\mathrm{Ext}^i_R(R_z, R) \cong \mathrm{Ext}^i_R(R_z, \mathrm{Hom}_R(E, E)) \cong \mathrm{Hom}_R(\mathrm{Tor}^R_i(E, R_z), E) = 0,$$

where E is the injective hull of $R/\mathfrak{m}$. Thus, by Theorem 2.7, $\mathrm{End}_R(H) \cong R$. Now the result follows from Remark 2.3. ∎

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