BULL. AUSTRAL. MATH. SOC. 20J06
VOL. 51 (1995) [421–432]

THE COHOMOLOGY RINGS OF FINITE GROUPS WITH SEMI-DIHEDRAL SYLOW 2-SUBGROUPS

ANGELINA CHIN

In this paper we determine the mod-2 cohomology rings and the 2-primary part of the integral cohomology rings of finite groups with semi-dihedral Sylow 2-subgroups. The method used here is algebraic and can be considered as elementary.

INTRODUCTION

In this paper we determine the mod-2 cohomology rings and the 2-primary part of the integral cohomology rings of finite groups with semi-dihedral Sylow 2-subgroups. The mod-2 cohomology rings of such groups have in fact been obtained by Martino [4] via topological methods. The method used here is algebraic and can be considered as elementary.

Let G be a finite group with a semi-dihedral Sylow 2-subgroup. The starting point here is a result of Webb [5] on the Poincaré series of $H^*(G, \mathbb{F}_2)$. For any subgroup H of G, let $A_G(H) = N_G(H)/C_G(H)$ where as usual, $N_G(H)$ denotes the normaliser of H in G and $C_G(H)$ denotes the centraliser of H in G. Using some structural properties of Mackey functors, Webb proved the following:

THEOREM. (Webb, [5]) *The Poincaré series of $H^*(G, \mathbb{F}_2)$ is*

$$P_G(t) = \frac{1+t^5}{(1-t^3)(1-t^4)} + \frac{\lambda t(1+t)}{(1-t^4)} + \frac{\mu t}{(1-t)(1-t^3)}$$

where

$$\lambda = \begin{cases} 0 & \text{if } 3 \mid |A_G(Q_8)| \\ 1 & \text{otherwise} \end{cases} \quad \text{and} \quad \mu = \begin{cases} 0 & \text{if } 3 \mid |A_G(C_2 \times C_2)| \\ 1 & \text{otherwise.} \end{cases}$$

THE METHOD

It is well-known that if A is a G-ring, the p-part of the cohomology $H^*(G, A)_p$ is isomorphic (via the restriction map) to the stable elements of the cohomology ring of its Sylow p-subgroup for each prime p dividing the order of G. Using this fact, the Poincaré series of $H^*(G, \mathbb{F}_2)$, the ring structure of $H^*(SD, \mathbb{F}_2)$ and the action of the Steenrod algebra on $H^*(SD, \mathbb{F}_2)$, we are able to determine the ring structure of $H^*(G, \mathbb{F}_2)$.

Received 28th June, 1994.

1. The ring $H^*(G, \mathbb{F}_2)$

The algebra $H^*(SD, \mathbb{F}_2)$ has been obtained by Priddy and Evens [3] and is as follows:

$H^*(SD, \mathbb{F}_2)$ is generated by the elements

$$w_x,\ w_1,\ w_3,\ c_4$$

where $\deg w_x = \deg w_1 = 1$, $\deg w_3 = 3$, $\deg c_4 = 4$ subject to the relations

$$w_1^2 = w_x w_1, \qquad w_x^2 w_1 = 0,$$
$$w_3^2 = w_x(w_x + w_1)c_4, \qquad w_1 w_3 = 0.$$

The action of the Steenrod algebra on $H^*(SD, \mathbb{F}_2)$ is given as follows:

	Sq^1	Sq^2	Sq^3	Sq^4
w_x	w_x^2	0	0	0
w_1	w_1^2	0	0	0
w_3	0	$(w_x + w_1)(c_4 + w_x w_3)$	w_3^2	0
c_4	0	$w_x(w_x + w_1)c_4$	0	c_4^2

By Webb's theorem, we have four cases to consider.

Case 1. $3 \mid |A_G(Q_8)|$, $3 \mid |A_G(C_2 \times C_2)|$.

In this case, the Poincaré series of $H^*(G, \mathbb{F}_2)$ is

$$P_G(t) = \frac{1 + t^5}{(1 - t^3)(1 - t^4)}.$$

We have

n	1	2	3	4	5	6	7	8	9	10	11	12
$\dim_{\mathbb{F}_2} H^n(G, \mathbb{F}_2)$	0	0	1	1	1	1	1	2	2	1	2	3

Let α, β, μ denote the generators of degrees 3, 4, 5 in $H^*(G, \mathbb{F}_2)$, respectively. The possibilities for the restrictions of these generators to $H^*(SD, \mathbb{F}_2)$ are as follows:

	The possibilities
$\mathrm{Res}_{G,SD}\alpha$	w_x^3, w_3, $w_x^3 + w_3$
$\mathrm{Res}_{G,SD}\beta$	w_x^4, $w_x w_3$, c_4, $w_x^4 + w_x w_3$, $w_x^4 + c_4$, $w_x w_3 + c_4$, $w_x^4 + w_x w_3 + c_4$
$\mathrm{Res}_{G,SD}\mu$	w_x^5, $w_x^2 w_3$, $w_x c_4$, $w_1 c_4$, $w_x^5 + w_x^2 w_3$, $w_x^5 + w_x c_4$, $w_x^5 + w_1 c_4$, $w_x^2 w_3 + w_x c_4$, $w_x^2 w_3 + w_1 c_4$, $w_x c_4 + w_1 c_4$, $w_x^5 + w_x^2 w_3 + w_x c_4$, $w_x^5 + w_x^2 w_3 + w_1 c_4$, $w_x^5 + (w_x + w_1)c_4$, $w_x^2 w_3 + (w_x + w_1)c_4$, $w_x^5 + w_x^2 w_3 + (w_x + w_1)c_4$

For all the above possibilities, we have that $\alpha^2\beta$ and μ^2 are both non-zero. Then since $\dim_{\mathbb{F}_2} H^{10}(G, \mathbb{F}_2) = 1$, we must have that $\mu^2 = \alpha^2\beta$. From this relation it is clear that $\mathrm{Res}_{G,SD}\,\mu \neq w_1 c_4$. Therefore α, β, μ are non-nilpotent elements.

Proposition 1.1. *In case 1,* $\mathrm{Res}_{G,SD}\,\alpha = w_3$, $\mathrm{Res}_{G,SD}\,\beta = c_4 + w_x^4$, $\mathrm{Res}_{G,SD}\,\mu = w_x^2 w_3 + (w_x + w_1)c_4$.

Proof: Suppose that $\mathrm{Res}_{G,SD}\,\alpha = w_x^3$. Then

$$\mathrm{Res}_{G,SD}\,\mathrm{Sq}^1(\alpha) = \mathrm{Sq}^1(w_x^3) = w_x^4 \neq 0.$$

Hence, $\mathrm{Sq}^1(\alpha) \neq 0$ and it follows that $\beta = \mathrm{Sq}^1(\alpha)$. Since

$$\mathrm{Res}_{G,SD}\,\mathrm{Sq}^2(\alpha) = \mathrm{Sq}^2\,\mathrm{Res}_{G,SD}(\alpha) = \mathrm{Sq}^2(w_x^3) = w_x^5 \neq 0,$$

so $\mathrm{Sq}^2(\alpha) \neq 0$ and we must have that $\mu = \mathrm{Sq}^2(\alpha)$. Therefore

$$\mathrm{Res}_{G,SD}\,\beta^2 = w_x^8 = \mathrm{Res}_{G,SD}\,\alpha\mu$$

and hence, $\beta^2 = \alpha\mu$. Then since $\dim_{\mathbb{F}_2} H^8(G, \mathbb{F}_2) = 2$, there must be an element $\zeta \in H^8(G, \mathbb{F}_2)$ such that $H^8(G, \mathbb{F}_2) = \langle \beta^2, \zeta \rangle$. Clearly, $\mathrm{Res}_{G,SD}\,\zeta \neq w_x^8$. Since $\dim_{\mathbb{F}_2} H^{10}(G, \mathbb{F}_2) = 1$, so $\mathrm{Res}_{G,SD}\,\zeta$ must be c_4^2. Then $H^{11}(G, \mathbb{F}_2) = \langle \alpha\beta^2, \alpha\zeta \rangle$ and by inspection, $H^9(G, \mathbb{F}_2) = \langle \alpha^3 \rangle$; contradicting the fact that $\dim_{\mathbb{F}_2} H^9(G, \mathbb{F}_2) = 2$. Hence, $\mathrm{Res}_{G,SD}\,\alpha \neq w_x^3$.

Next suppose that $\mathrm{Res}_{G,SD}\,\alpha = w_x^3 + w_3$. Then

$$\mathrm{Res}_{G,SD}\,\mathrm{Sq}^1(\alpha) = \mathrm{Sq}^1\,\mathrm{Res}_{G,SD}(\alpha) = Sq^1(w_x^3 + w_3) = w_x^4 \neq 0.$$

Hence $\mathrm{Sq}^1(\alpha) \neq 0$ and so $\mathrm{Sq}^1(\alpha) = \beta$. Then $\mathrm{Res}_{G,SD}(\mu^2) = \mathrm{Res}_{G,SD}\,\alpha^2\beta = (w_x^6 + w_3^2)w_x^4 = (w_x^5 + w_3 w_x^2)^2$. Since $u^2 \neq 0$ for any non-zero $u \in H^5(SD, \mathbb{F}_2)$, it follows that $\mathrm{Res}_{G,SD}\,\mu = w_x^5 + w_3 w_x^2$. Therefore $\mathrm{Res}_{G,SD}\,\mathrm{Sq}^1(\mu) = \mathrm{Sq}^1(\mathrm{Res}_{G,SD}(\mu)) = \mathrm{Sq}^1(w_x^5 + w_3 w_x^2) = w_x^6$. But w_x^6 is not in the image of $\mathrm{Res}_{G,SD}$ generated by $w_x^3 + w_3, w_x^4$ and $w_x^5 + w_3 w_x^2$. Hence $\mathrm{Res}_{G,SD}\,\alpha \neq w_x^3 + w_3$.

We must therefore have $\mathrm{Res}_{G,SD}\,\alpha = w_3$. Then $\mathrm{Res}_{G,SD}\,\mathrm{Sq}^2(\alpha) = \mathrm{Sq}^2\,\mathrm{Res}_{G,SD}(\alpha) = \mathrm{Sq}^2 w_3 = (w_x + w_1)(c_4 + w_x w_3) \neq 0$. Therefore $\mathrm{Sq}^2(\alpha) \neq 0$ and hence, $\mu = \mathrm{Sq}^2(\alpha)$. Since $\mu^2 = \alpha^2\beta$, so

$$\begin{aligned}\left(w_x^2 w_3 + (w_x + w_1)c_4\right)^2 &= \mathrm{Res}_{G,SD}\,\mu^2 = \mathrm{Res}_{G,SD}(\alpha^2\beta)\\ &= w_3^2\,\mathrm{Res}_{G,SD}\,\beta.\end{aligned}$$

Hence, we must have that $\mathrm{Res}_{G,SD}\,\beta = c_4 + w_x^4$. □

By Proposition 1.1 and the Poincaré series of $H^*(G, \mathbb{F}_2)$ we see that the elements α, β, μ are sufficient to generate the ring $H^*(G, \mathbb{F}_2)$. We thus have

https://doi.org/10.1017/S0004972700014246 Published online by Cambridge University Press

PROPOSITION 1.2. *In case 1,*

$$H^*(G,\mathbb{F}_2)\cong \mathbb{F}_2[\alpha,\beta,\mu]/(\mu^2+\alpha^2\beta),$$

where deg $\alpha = 3$, deg $\beta = 4$ *and* deg $\mu = 5$.

Using the properties of the Steenrod operations, the action of the Steenrod algebra on $H^*(SD,\mathbb{F}_2)$ and the fact that Sq^i commutes with $\mathrm{Res}_{G,SD}$, we obtain the following:

PROPOSITION 1.3. *In case 1, the action of the Steenrod algebra on* $H^*(G,\mathbb{F}_2)$ *is given as follows:*

	Sq^1	Sq^2	Sq^3	Sq^4	Sq^5
α	0	μ	α^2	0	0
β	0	α^2	0	β^2	0
μ	α^2	0	0	$\alpha^3+\beta\mu$	μ^2

CASE 2. $3 \nmid |A_G(Q_8)|$, $3 \mid |A_G(C_2\times C_2)|$.

The Poincaré series of $H^*(G,\mathbb{F}_2)$ in this case is

$$\begin{aligned}P_G(t)&=\frac{1+t^5}{(1-t^3)(1-t^4)}+\frac{t(1+t)}{(1-t^4)}\\&=\frac{1+t+t^2-t^4}{(1-t^3)(1-t^4)}.\end{aligned}$$

We have

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\dim_{\mathbb{F}_2} H^n(G,\mathbb{F}_2)$	1	1	1	1	2	2	1	2	3	2	2	3	3	3	3	3

Let ξ denote the generator of degree 1 in $H^*(G,\mathbb{F}_2)$. Then $\mathrm{Res}_{G,SD}\,\xi$ is w_1, w_1+w_x or w_x. Suppose that $\mathrm{Res}_{G,SD}\,\xi = w_x$. Then ξ is a non-nilpotent generator of $H^*(G,\mathbb{F}_2)$. Since $\dim_{\mathbb{F}_2} H^n(G,\mathbb{F}_2) = 1$ for $n = 3,4$, it follows that w_3 and c_4 are non-stable elements of $H^*(SD,\mathbb{F}_2)$. Since $\dim_{\mathbb{F}_2} H^5(G,\mathbb{F}_2)=2$, there must exist an element $\nu\in H^5(G,\mathbb{F}_2)$ such that $H^5(G,\mathbb{F}_2)=\langle\xi^5,\nu\rangle$. By inspection the possibilities for $\mathrm{Res}_{G,SD}\,\nu$ are the same as the possibilities for $\mathrm{Res}_{G,SD}\,\mu$ in Case 1 except for w_x^5. Then $H^6(G,\mathbb{F}_2)=\langle\xi^6,\xi\nu\rangle$. Since $\dim_{\mathbb{F}_2} H^7(G,\mathbb{F}_2)=1$ and $\xi^7\neq 0$, so we must have $\mathrm{Res}_{G,SD}\,\nu = w_1c_4$. Since $\dim_{\mathbb{F}_2} H^8(G,\mathbb{F}_2)=2$ there is some element $\zeta\in H^8(G,\mathbb{F}_2)$ such that $H^8(G,\mathbb{F}_2)=\langle\xi^8,\zeta\rangle$. For all the possibilities of $\mathrm{Res}_{G,SD}\,\zeta$ we have that ξ^{10}, ν^2 and $\xi^2\zeta$ are $\mathbb{F}_2$-linearly independent; contradicting the fact that $\dim_{\mathbb{F}_2} H^{10}(G,\mathbb{F}_2)=2$. By the same argument, $\mathrm{Res}_{G,SD}\,\xi\neq w_1+w_x$. Therefore $\mathrm{Res}_{G,SD}\,\xi = w_1$. Since $w_1^3=0$, so $\xi^3=0$.

Let α and β denote the generators of degrees 3 and 4 in $H^*(G, \mathbb{F}_2)$, respectively. The possibilities for $\mathrm{Res}_{G,SD}\,\alpha$ are w_x^3, w_3 and $w_x^3 + w_3$. Therefore $\mathrm{Res}_{G,SD}\,\xi\alpha = 0$ and hence $\xi\alpha = 0$. It follows that $0 = \mathrm{Sq}^1(\xi\alpha) = \xi^2\alpha + \xi\,\mathrm{Sq}^1(\alpha) = \xi\,\mathrm{Sq}^1(\alpha)$. If $\mathrm{Sq}^1(\alpha) = \beta$, then $\xi\beta = 0$ and the possibilities for $\mathrm{Res}_{G,SD}\,\beta$ are w_x^4, $w_x w_3$ and $w_x^4 + w_x w_3$. But $\mathrm{Sq}^1(\beta) = 0$ implies that $\mathrm{Res}_{G,SD}\,\beta = w_x^4$ and hence, $\mathrm{Res}_{G,SD}\,\alpha = w_x^3$ or $w_x^3 + w_3$.

Since $\dim_{\mathbb{F}_2} H^5(G, \mathbb{F}_2) = 2$, there are elements μ, ν such that $H^5(G, \mathbb{F}_2) = \langle \mu, \nu \rangle$. Then since $\dim_{\mathbb{F}_2} H^{10}(G, \mathbb{F}_2) = 2$, we must have that

$$\mathrm{Res}_{G,SD}\,\alpha = w_x^3,\ \mathrm{Res}_{G,SD}\,\mu = w_x^5 \text{ and } \mathrm{Res}_{G,SD}\,\nu = w_1 c_4.$$

Then

$$H^6(G, \mathbb{F}_2) = \langle \xi\nu, \alpha^2 \rangle \text{ and } H^7(G, \mathbb{F}_2) = \langle \alpha\beta \rangle.$$

Since $\beta^2 = \alpha\mu \in H^8(G, \mathbb{F}_2)$ and $\dim_{\mathbb{F}_2} H^8(G, \mathbb{F}_2) = 2$, there must exist an element $\zeta \in H^8(G, \mathbb{F}_2)$ such that $H^8(G, \mathbb{F}_2) = \langle \alpha\mu, \zeta \rangle$. In order that $\dim_{\mathbb{F}_2} H^{10}(G, \mathbb{F}_2) = 2$, we must have $\mathrm{Res}_{G,SD}\,\zeta = c_4^2$. Then $H^9(G, \mathbb{F}_2) = \langle \xi\zeta, \alpha^3, \eta \rangle$ for some η.

Since $\dim_{\mathbb{F}_2} H^{10}(G, \mathbb{F}_2) = 2$, so $\mathrm{Res}_{G,SD}\,\eta$ must be an $\mathbb{F}_2$-linear combination of $w_x^6 w_3$ and $w_x^2 w_3 c_4$. Then $\mathrm{Res}_{G,SD}\,Sq^2\eta \in \langle w_x^7 c_4, w_x^4 w_3 c_4 + w_x^3 c_4^2 \rangle$. Note that $\mathrm{Res}_{G,SD}\,\alpha\beta^2 = w_x^{11}$ and $\mathrm{Res}_{G,SD}\,\alpha\zeta = w_x^3 c_4^2$. Then $\alpha\beta^2$, $\alpha\zeta$ and $Sq^2\eta$ are $\mathbb{F}_2$-linearly independent in $H^{11}(G, \mathbb{F}_2)$; contradicting the fact that $\dim_{\mathbb{F}_2} H^{11}(G, \mathbb{F}_2) = 2$.

Therefore $\mathrm{Sq}^1(\alpha) = 0$ and we must have $\mathrm{Res}_{G,SD}\,\alpha = w_3$. By an argument similar to that above we have $\xi\beta \neq 0$, so the possibilities for $\mathrm{Res}_{G,SD}\,\beta$ are c_4, $w_x^4 + c_4$, $w_x w_3 + c_4$ and $w_x^4 + w_x w_3 + c_4$. If $\mathrm{Res}_{G,SD}\,\beta$ is $w_x w_3 + c_4$ or $w_x^4 + w_x w_3 + c_4$, then $\mathrm{Res}_{G,SD}\,\mathrm{Sq}^1(\beta) = \mathrm{Sq}^1\,\mathrm{Res}_{G,SD}(\beta) = w_x^2 w_3$. Therefore $H^5(G, \mathbb{F}_2) = \langle \xi\beta, \mu \rangle$ where $\mathrm{Res}_{G,SD}\,\mu = w_x^2 w_3$. Then $H^{10}(G, \mathbb{F}_2) = \langle \mu^2, \xi^2\beta^2, \alpha^2\beta \rangle$; contradicting the fact that $\dim_{\mathbb{F}_2} H^{10}(G, \mathbb{F}_2) = 2$. Suppose $\mathrm{Res}_{G,SD}\,\beta = c_4$. Then $H^5(G, \mathbb{F}_2) = \langle \xi\beta, \mu \rangle$ for some μ such that $\mathrm{Res}_{G,SD}\,\mu \neq w_1 c_4$. Since $\mathrm{Res}_{G,SD}\,\mathrm{Sq}^2(\alpha) = \mathrm{Sq}^2\,\mathrm{Res}_{G,SD}\,\alpha = \mathrm{Sq}^2(w_3) = w_x^2 w_3 + (w_x + w_1)c_4$, so $\mathrm{Res}_{G,SD}\,\mu$ is either $w_x^2 w_3 + (w_x + w_1)c_4$ or $w_x^2 w_3 + w_x c_4$. In either case $H^{10}(G, \mathbb{F}_2) = \langle \xi^2\beta^2, \alpha^2\beta, \mu^2 \rangle$; contradicting the fact that $\dim_{\mathbb{F}_2} H^{10}(G, \mathbb{F}_2) = 2$. Hence, we must have $\mathrm{Res}_{G,SD}\,\beta = c_4 + w_x^4$. By the same reason, there is some element μ such that $H^5(G, \mathbb{F}_2) = \langle \xi\beta, \mu \rangle$ and $\mathrm{Res}_{G,SD}\,\mu = w_x^2 w_3 + (w_x + w_1)c_4$. Then $\xi\mu = 0$ and $\mu^2 = \alpha^2\beta$. It is straightforward to check that ξ, α, β and μ are sufficient to generate the ring $H^*(G, \mathbb{F}_2)$.

We have thus proved the following:

PROPOSITION 1.4. *In case 2,*

$$H^*(G, \mathbb{F}_2) = \mathbb{F}_2[\xi, \alpha, \beta, \mu]/(\xi^3, \xi\alpha, \xi\mu, \mu^2 + \alpha^2\beta)$$

where $\deg\xi = 1$, $\deg\alpha = 3$, $\deg\beta = 4$, $\deg\mu = 5$,

$$\mathrm{Res}_{G,SD}\,\xi = w_1, \quad \mathrm{Res}_{G,SD}\,\alpha = w_3, \quad \mathrm{Res}_{G,SD}\,\beta = c_4 + w_x^4$$

and $\mathrm{Res}_{G,SD}\,\mu = w_x^2 w_3 + (w_x + w_1)c_4$.

Using the Steenrod algebra axioms, we obtain

Proposition 1.5. *In case 2, the action of the Steenrod algebra on $H^*(G, \mathbb{F}_2)$ is given as follows:*

	Sq^1	Sq^2	Sq^3	Sq^4	Sq^5
ξ	ξ^2	0	0	0	0
α	0	μ	α^2	0	0
β	0	α^2	0	β^2	0
μ	α^2	0	0	$\alpha^3+\beta\mu$	μ^2

Case 3. $3 \mid |A_G(Q_8)|$, $3 \nmid |A_G(C_2\times C_2)|$.

The Poincaré series of $H^*(G, \mathbb{F}_2)$ in this case is

$$\begin{aligned} P_G(t) &= \frac{1+t^5}{(1-t^3)(1-t^4)} + \frac{t}{(1-t)(1-t^3)} \\ &= \frac{1+t^3}{(1-t)(1-t^4)}. \end{aligned}$$

We have

n	1	2	3	4	5	6	7	8	9	10	11	12
$\dim_{\mathbb{F}_2} H^n(G,\mathbb{F}_2)$	1	1	2	3	3	3	4	5	5	5	6	7

Let ξ denote the generator of degree 1 in $H^*(G, \mathbb{F}_2)$. Then $\mathrm{Res}_{G,SD}\,\xi$ is one of w_1, w_x and $w_1 + w_x$. Suppose that $\mathrm{Res}_{G,SD}\,\xi = w_1$. Then

$$\begin{aligned} H^2(G, \mathbb{F}_2) &= \langle \xi^2 \rangle, \\ H^3(G, \mathbb{F}_2) &= \langle a_3, b_3 \rangle \end{aligned}$$

where $\mathrm{Res}_{G,SD}\,a_3 = w_3$, $\mathrm{Res}_{G,SD}\,b_3 = w_x^3$,

$$H^4(G, \mathbb{F}_2) = \langle a_4, b_4, d_4 \rangle$$

where $\mathrm{Res}_{G,SD}\,a_4 = c_4$, $\mathrm{Res}_{G,SD}\,b_4 = w_x^4$, $\mathrm{Res}_{G,SD}\,d_4 = w_x w_3$,

$$H^5(G, \mathbb{F}_2) = \langle \xi a_4, a_5, b_5, c_5 \rangle$$

where $\mathrm{Res}_{G,SD}\,a_5 = w_x^5$, $\mathrm{Res}_{G,SD}\,b_5 = w_x^2 w_3$, $\mathrm{Res}_{G,SD}\,c_5 = w_x c_4$; contradicting the fact that $\dim_{\mathbb{F}_2} H^5(G, \mathbb{F}_2) = 3$.

Now suppose that $\mathrm{Res}_{G,SD}\,\xi = w_x$. Then

$$\begin{aligned}
&H^2(G, \mathbb{F}_2) = \langle \xi^2 \rangle, \\
&H^3(G, \mathbb{F}_2) = \langle \xi^3, a_3 \rangle \quad \text{where} \quad \mathrm{Res}_{G,SD}\, a_3 = w_3, \\
&H^4(G, \mathbb{F}_2) = \langle \xi^4, \xi a_3, a_4 \rangle \text{where} \quad \mathrm{Res}_{G,SD}\, a_4 = c_4, \\
&H^5(G, \mathbb{F}_2) = \langle \xi^5, \xi^2 a_3, \xi a_4 \rangle, \\
&H^6(G, \mathbb{F}_2) = \langle \xi^6, \xi^3 a_3, \xi^2 a_4, a_3^2 \rangle;
\end{aligned}$$

contradicting the fact that $\dim_{\mathbb{F}_2} H^6(G, \mathbb{F}_2) = 3$. Hence, we must have $\mathrm{Res}_{G,SD}\,\xi = w_x + w_1$. Then

$$\begin{aligned}
&H^3(G, \mathbb{F}_2) = \langle \xi^3, \mu \rangle \quad \text{where} \quad \mathrm{Res}_{G,SD}\, \mu = w_3, \\
&H^4(G, \mathbb{F}_2) = \langle \xi^4, \xi\mu, \beta \rangle \quad \text{where} \quad \mathrm{Res}_{G,SD}\, \beta = c_4, \\
&H^5(G, \mathbb{F}_2) = \langle \xi^5, \xi^2\mu, \xi\beta \rangle.
\end{aligned}$$

Since $\mathrm{Res}_{G,SD}\,\mu^2 = w_3^2 = (w_x^2 + w_1^2)c_4 = \mathrm{Res}_{G,SD}\,\xi^2\beta$, so $\mu^2 = \xi^2\beta$. By inspection, we have that ξ, μ, β are sufficient to generate the ring. We have thus proved the following:

PROPOSITION 1.6. *In case 3,*

$$H^*(G, \mathbb{F}_2) = \mathbb{F}_2[\xi, \mu, \beta]/(\mu^2 + \xi^2\beta)$$

where $\deg\xi = 1$, $\deg\mu = 3$, $\deg\beta = 4$,

$$\mathrm{Res}_{G,SD}\,\xi = w_x + w_1, \quad \mathrm{Res}_{G,SD}\,\mu = w_3, \quad \mathrm{Res}_{G,SD}\,\beta = c_4.$$

By the Steenrod algebra axioms, we have

PROPOSITION 1.7. *In case 3, the action of the Steenrod algebra on* $H^*(G, \mathbb{F}_2)$ *is given as follows:*

	Sq^1	Sq^2	Sq^3	Sq^4
ξ	ξ^2	0	0	0
μ	0	$\xi^2\mu + \xi\beta$	μ^2	0
β	0	μ^2	0	β^2

CASE 4. $3 \nmid |A_G(Q_8)|$, $3 \nmid |A_G(C_2 \times C_2)|$.

The Poincaré series of $H^*(G, \mathbb{F}_2)$ in this case is

$$\begin{aligned}
P_G(t) &= \frac{1+t^5}{(1-t^3)(1-t^4)} + \frac{t(1+t)}{1-t^4} + \frac{t}{(1-t)(1-t^3)} \\
&= \frac{1+t}{(1-t)(1-t^4)}.
\end{aligned}$$

This is the same as the Poincaré series of $H^*(SD, \mathbb{F}_2)$. We thus have

PROPOSITION 1.8. *In case 4,* $H^*(G, \mathbb{F}_2) \cong H^*(SD, \mathbb{F}_2)$.

The action of the Steenrod algebra on $H^*(G, \mathbb{F}_2)$ follows from the action on $H^*(SD, \mathbb{F}_2)$.

REMARK.

This method can clearly be used to obtain the cohomology ring of a finite group once its additive structure and the cohomology of its Sylow p-subgroups are known.

In [**1**], Asai and Sasaki obtained the mod-2 cohomology rings of finite groups with dihedral or quaternion Sylow 2-subgroups algebraically. The method used here is different from theirs. We note that one of the criteria for the method of Asai and Sasaki to work is that $\dim_{\mathbb{F}_2} H^{n+1}(G, \mathbb{F}_2) \geqslant \dim_{\mathbb{F}_2} H^n(G, \mathbb{F}_2)$ for all $n \geqslant 1$. (see [**1**, Theorem 2.5]). We thus see that their method would not work in general for finite groups with semi-dihedral Sylow 2-subgroups.

2. THE 2-PART $H^*(G, \mathbb{Z})_2$

Let $\Delta : H^i(G, \mathbb{F}_2) \to H^{i+1}(G, \mathbb{F}_2)$ and $\delta : H^i(G, \mathbb{F}_2) \to H^{i+1}(G, \mathbb{Z})$ denote the Bockstein homomorphisms. Then $\Delta = \pi_* \cdot \delta$ where $\pi_* : H^i(G, \mathbb{Z}) \to H^i(G, \mathbb{F}_2)$ is induced from $\pi : \mathbb{Z} \to \mathbb{F}_2$.

PROPOSITION 2.1. (Cárdenas and Lluis, [**2**]) *The Poincaré series of* $H^*(G, \mathbb{Z})_2 \otimes \mathbb{F}_2$ *is*

$$P_G(H^*(G, \mathbb{Z})_2 \otimes \mathbb{F}_2, t) = \frac{t}{1+t} P_G(H^*(G, \mathbb{F}_2), t) + \frac{1}{1+t}.$$

The integral cohomology ring of a semi-dihedral group has been obtained by Priddy and Evens [**3**]:

If SD is of order 2^{n+1}, then

$$H^*(SD, \mathbb{Z}) = \mathbb{Z}[\alpha, \beta, \eta, \gamma]$$

where $\deg \eta = \deg \beta = 2$, $\deg \alpha = 4$, $\deg \gamma = 5$ subject to the relations

$$2^n \alpha = 2\beta = 2\eta = 2\gamma = 0,$$
$$\eta^2 = \eta\beta = \eta\gamma = 0, \quad \gamma^2 = \beta^3 \alpha \quad (n \geqslant 3).$$

CASE (1). $3 \mid |A_G(Q_8)|$, $3 \mid |A_G(C_2 \times C_2)|$.

By Proposition 2.1,

$$P_G(H^*(G, \mathbb{Z})_2 \otimes \mathbb{F}_2, t) = \frac{1+t^5}{(1-t^3)(1-t^4)} \cdot \frac{t}{1+t} + \frac{1}{1+t}$$
$$= \frac{1 - t^3 + t^6}{(1-t^3)(1-t^4)}.$$

https://doi.org/10.1017/S0004972700014246 Published online by Cambridge University Press

We have

n	1	2	3	4	5	6	7	8	9	10	11	12
$\dim_{\mathbb{F}_2} H^n(G,\mathbb{Z})_2 \otimes \mathbb{F}_2$	0	0	0	1	0	1	0	1	1	1	0	2

Let ξ and ζ denote the generators of degrees 4 and 6 in $H^*(G, \mathbb{Z})_2$, respectively. We have shown that $H^*(G, \mathbb{F}_2) = \mathbb{F}_2[a_3, b_4, c_5]/(c_5^2 + a_3^2 b_4)$ where the subscript gives the degree of the generator. Since $\Delta(a_3) = \mathrm{Sq}^1(a_3) = 0$, so $\delta(a_3) \in \operatorname{Ker} \pi_* = \operatorname{Im} 2$. Then since $\mathrm{Res}_{G,SD}\, a_3 = w_3$, so $\delta(a_3) = 2^{n-1}\xi$. Since $\Delta(c_5) = a_3^2 \neq 0$, so $\delta(c_5) \neq 0$. We may then take $\zeta = \delta(c_5)$. Let ν denote the generator of degree 9 in $H^*(G, \mathbb{Z})_2$. Since $\Delta(a_3 c_5) = a_3^3 \neq 0$, so $\delta(a_3 c_5) \neq 0$. Hence, we may take $\nu = \delta(a_3 c_5)$. Note that

$$\nu^2 = \delta(a_3 c_5)\delta(a_3 c_5) = \delta((a_3 c_5)\Delta(a_3 c_5)) = \delta(a_3^4 c_5)$$

and

$$\zeta^3 = \delta(c_5)\delta(c_5)^2 = \delta(c_5\Delta(c_5))\delta(c_5) = \delta(a_3^2 c_5)\delta(c_5) = \delta(a_3^4 c_5).$$

Therefore, $\nu^2 = \zeta^3$. By inspection, the elements ξ, ζ and ν are sufficient to generate $H^*(G, \mathbb{Z})_2$. We have thus shown

PROPOSITION 2.2. *In case 1, $H^*(G, \mathbb{Z})_2$ is generated by the elements*

$$\xi, \zeta, \nu$$

where $\deg\xi = 4$, $\deg\zeta = 6$, $\deg\nu = 9$ subject to the relations

$$2^n\xi = 2\zeta = 2\nu = 0$$

and

$$\nu^2 = \zeta^3.$$

Further, $\mathrm{Res}_{G,SD}\,\xi = \alpha$, $\mathrm{Res}_{G,SD}\,\zeta = \alpha\beta$ and $\mathrm{Res}_{G,SD}\,\nu = \gamma\alpha$.

CASE (2). $3 \nmid |A_G(Q_8)|$, $3 \mid |A_G(C_2 \times C_2)|$.

By Proposition 2.1,

$$P_G(H^*(G, \mathbb{Z})_2 \otimes \mathbb{F}_2, t) = \frac{1 + t + t^2 - t^4}{(1 - t^3)(1 - t^4)} \cdot \frac{t}{1+t} + \frac{1}{1+t} = \frac{1 + t^2 - t^3 - t^5 + t^6}{(1 - t^3)(1 - t^4)}.$$

We have

n	1	2	3	4	5	6	7	8	9	10	11	12
$\dim_{\mathbb{F}_2} H^n(G,\mathbb{Z})_2 \otimes \mathbb{F}_2$	0	1	0	1	0	2	0	1	1	2	0	2

https://doi.org/10.1017/S0004972700014246 Published online by Cambridge University Press

For convenience, let $\xi = a_1$, $\alpha = b_3$, $\beta = d_4$, $\mu = e_5$ in Proposition 1.4. Let ξ denote the generator of degree 2 in $H^*(G, \mathbb{Z})_2$. Since $\Delta(a_1) = a_1^2 \neq 0$, so $\delta(a_1) \neq 0$. We may then take $\xi = \delta(a_1)$. Since

$$\xi^2 = \delta(a_1)\delta(a_1) = \delta(a_1\Delta(a_1)) = \delta(a_1^3) = 0,$$

so $\mathrm{Res}_{G,SD}\,\xi = \eta$. Let ζ denote the generator of degree 4 in $H^*(G, \mathbb{Z})_2$. Since $\Delta(b_3) = \mathrm{Sq}^1(b_3) = 0$, so $\delta(b_3) \in \mathrm{Ker}\,\pi_* = \mathrm{Im}\,2$. Therefore $\delta(b_3) = 2^{n-1}\zeta$. Then $H^6(G, \mathbb{Z})_2 = \langle \xi\zeta, \nu\rangle$ for some ν. Since $\Delta(e_5) = \mathrm{Sq}^1(e_5) = b_3^2 \neq 0$, so $\delta(e_5) \neq 0$. Hence we may take $\nu = \delta(e_5)$. Note that

$$\xi\nu = \delta(a_1)\delta(e_5) = \delta(a_1\Delta(e_5)) = \delta(a_1 b_3^2) = 0.$$

Let μ denote the generator of degree 9 in $H^*(G, \mathbb{Z})_2$. Since $\Delta(b_3 e_5) = b_3(b_3^2) \neq 0$, so $\delta(b_3 e_5) \neq 0$. Hence we may take $\mu = \delta(b_3 e_5)$. Then

$$\mu^2 = \delta(b_3 e_5)\delta(b_3 e_5) = \delta(b_3 e_5 \Delta(b_3 e_5)) = \delta(b_3 e_5 (b_3^3)) = \delta(b_3^4 e_5)$$

and

$$\nu^3 = \delta(e_5)\delta(e_5)\delta(e_5) = \delta(e_5 b_3^2)\delta(e_5) = \delta(b_3^4 e_5).$$

Therefore $\mu^2 = \nu^3$. It follows that $\mathrm{Res}_{G,SD}\,\nu$ must $\alpha\beta$ and $\mathrm{Res}_{G,SD}\,\mu$ must be $\alpha\gamma$. Note that

$$\xi\mu = \delta(a_1)\delta(b_3 e_5) = \delta(a_1^2 b_3 e_5) = 0.$$

By inspection, the elements ξ, ζ, ν, μ are sufficient to generate $H^*(G, \mathbb{Z})_2$. We thus have

PROPOSITION 2.3. *In case 2, $H^*(G, \mathbb{Z})_2$ is generated by the elements*

$$\xi, \zeta, \nu, \mu$$

where $\deg\xi = 2$, $\deg\zeta = 4$, $\deg\nu = 6$, $\deg\mu = 9$ subject to the relations

$$2\xi = 2^n\zeta = 2\nu = 2\mu = 0$$
$$\xi^2 = \xi\nu = \xi\mu = 0, \quad \mu^2 = \nu^3.$$

Further, $\mathrm{Res}_{G,SD}\,\xi = \eta$, $\mathrm{Res}_{G,SD}\,\zeta = \alpha$, $\mathrm{Res}_{G,SD}\,\nu = \alpha\beta$, $\mathrm{Res}_{G,SD}\,\mu = \alpha\gamma$.

CASE (3). $3 \mid |A_G(Q_8)|$, $3 \nmid |A_G(C_2 \times C_2)|$.

By Proposition 2.1,

$$P_G(H^*(G, \mathbb{Z})_2 \otimes \mathbb{F}_2, t) = \frac{1+t^3}{(1-t)(1-t^4)} \cdot \frac{t}{1+t} + \frac{1}{1+t}$$
$$= \frac{1+t^5}{(1-t^2)(1-t^4)}.$$

We have

n	1	2	3	4	5	6	7	8	9	10	11	12
$\dim_{\mathbb{F}_2} H^n(G,\mathbb{Z})_2 \otimes \mathbb{F}_2$	0	1	0	2	1	2	1	3	2	3	2	4

For convenience, let $a_1 = \xi$, $b_3 = \mu$ and $d_4 = \beta$ in Proposition 1.6. Since $\Delta(a_1) = a_1^2 \neq 0$, so $\delta(a_1) \neq 0$. Then $H^2(G, \mathbb{Z})_2 = \langle \xi \rangle$ where $\xi = \delta(a_1)$. Since $\xi^2 \neq 0$, so $H^4(G, \mathbb{Z})_2 = \langle \xi^2, \zeta \rangle$ for some ζ. Let ν denote the generator of degree 5 in $H^*(G, \mathbb{Z})_2$. Since $\Delta(a_1 b_3) = a_1^2 b_3 \neq 0$, so $\delta(a_1 b_3) \neq 0$. We may then take $\nu = \delta(a_1 b_3)$. Clearly, $\mathrm{Res}_{G,SD}\, \xi = \eta + \beta$, $\mathrm{Res}_{G,SD}\, \zeta = \alpha$ and $\mathrm{Res}_{G,SD}\, \nu = \gamma$. Since $\gamma^2 = \beta^3\alpha$, so we must have $\nu^2 = \xi^3\zeta$. By inspection, the elements ξ, ζ and ν are sufficient to generate $H^*(G, \mathbb{Z})_2$. We thus have

PROPOSITION 2.4. *In case 3, $H^*(G, \mathbb{Z})_2$ is generated by the elements*

$$\xi, \zeta, \nu$$

where $\deg \xi = 2$, $\deg \zeta = 4$, $\deg \nu = 5$ subject to the relations

$$2\xi = 2^n\zeta = 2\nu = 0,$$
$$\nu^2 = \xi^3\zeta.$$

Further, $\mathrm{Res}_{G,SD}\, \xi = \eta + \beta$, $\mathrm{Res}_{G,SD}\, \zeta = \alpha$, $\mathrm{Res}_{G,SD}\, \nu = \gamma$.

CASE (4). $3 \nmid |A_G(Q_8)|$, $3 \nmid |A_G(C_2 \times C_2)|$.

The Poincaré series of $H^*(G, \mathbb{Z})_2 \otimes \mathbb{F}_2$ is

$$P_G(H^*(G, \mathbb{Z})_2 \otimes \mathbb{F}_2, t) = \frac{1+t}{(1-t)(1-t^4)} \cdot \frac{t}{1+t} + \frac{1}{1+t}$$
$$= \frac{1+t^2-t^4+t^5}{(1-t^2)(1-t^4)}.$$

We have

n	1	2	3	4	5	6	7	8	9	10	11	12
$\dim_{\mathbb{F}_2} H^n(G,\mathbb{Z})_2 \otimes \mathbb{F}_2$	0	2	0	2	1	3	1	3	2	4	2	4

Since $\dim_{\mathbb{F}_2} H^n(G, \mathbb{Z})_2 \otimes \mathbb{F}_2 = \dim_{\mathbb{F}_2} H^n(SD, \mathbb{Z}) \otimes \mathbb{F}_2$ for $n \leqslant 5$ and $H^*(SD, \mathbb{Z})$ is generated by elements of degrees $\leqslant 5$, we have

PROPOSITION 2.5. *In case 4, $H^*(G, \mathbb{Z})_2 \cong H^*(SD, \mathbb{Z})$.*

References

[1] T. Asai and H. Sasaki, 'The mod 2 cohomology algebras of finite groups with dihedral Sylow 2-subgroups', *Comm. Algebra* **21** (1993), 2771–2790.

[2] H. Cárdenas and E. Lluis, 'On the integral cohomology of a Sylow subgroup of the symmetric group', *Comm. Algebra* **18** (1990), 105–134.

[3] L. Evens and S. Priddy, 'The cohomology of the semi-dihedral group', in *Conference on Algebraic Topology in Honour of P. Hilton*, (R. Piccinini and D. Sjerve, Editors), Contemporary Mathematics Vol. 37 (American Mathematical Society, Providence, RI, 1985), pp. 61–72.

[4] J. Martino, *Stable splittings of the Sylow 2-subgroups of $SL_3(\mathbb{F}_q)$, q odd*, Thesis (Northwestern University, 1988).

[5] P. Webb, 'Two classifications of simple Mackey functors with applications to group cohomology and the decomposition of classifying spaces', *J. Pure Appl. Algebra* **88** (1993), 265–304.

Department of Mathematics
The University of Queensland
Queensland 4072
Australia