A GENERALIZATION OF THE RING OF TRIGONULAR MATRICES

STEPHEN U. CHASE

1. Introduction

Let $R$ be a ring with unit, and $e$ be an idempotent in $R$ such that $(1 - e)Re = 0$. In this note we shall explore the relationships between homological properties of $R$ and those of its subring $eRe$.

Examples of such rings are abundant, the most common being perhaps the ring $R$ of all two-by-two upper triangular matrices over a field, where—

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$ 

In fact, it is easy to see that every ring of the type described above is in some sense a ring of upper triangular matrices, an observation which justifies the title of this paper.

We exhibit two applications of our results. First, we construct an example of a left semi-hereditary ring which is not right semi-hereditary, thus providing a negative answer to a question of Cartan and Eilenberg ([2], p. 15). Our second application is related to the work of Jans and Nakayama ([5]) and Nakano ([6]) on a class of semi-primary rings which is a special case of the type of ring considered here (recall that a ring $R$ is semi-primary if its Jacobson radical $N$ is nilpotent and $R/N$ satisfies minimum condition on left ideals). Our systematic treatment of the more general situation described above enables us to easily derive- and in some cases strengthen-several of the results of these authors.

Throughout this note every ring will be assumed to have a unit which acts as the identity on all modules. A ring $R$ will be called semi-simple if it has global dimension zero, or, equivalently, it satisfies minimum condition on left ideals and has trivial Jacobson radical ([2], p. 11). $R$ will be called regular.

Received August 20, 1960.
2. The Triangular Matrix Construction

Let $e$ be an idempotent in a ring $R$ with the property that $e'Re = 0$, where $e' = 1 - e$. Set $S = eRe$, $S' = e'Re'$.

**Proposition 2.1.** The mapping $f: R \to S$ defined by $f(a) = eae$ is a ring epimorphism.

*Proof.* $f$ is clearly onto, and preserves addition. If $a, b \in R$, then $f(ab) = e(ab)e = (ea)(e + e')(be) = (eae)(eb) = f(a)f(b)$, since $e$ is an idempotent, $e + e' = 1$, and $ebe = 0$.

Now let $A$ be a left $S$-module. The epimorphism $f: R \to S$ defines in the usual way a left $R$-module structure on $A$; we shall denote this $R$-module, constructed from $A$, by $T(A)$. On the other hand, if $B$ is a left $R$-module, then $U(B) = eB$ is, in the usual way, a left $S$-module. We have thus defined functors $T: \mathcal{C}(S) \to \mathcal{C}(R)$, $U: \mathcal{C}(R) \to \mathcal{C}(S)$, where $\mathcal{C}(R)$ and $\mathcal{C}(S)$ are the categories of all left modules over $R$ and $S$, respectively.

**Theorem 2.1.** In the situation described above, the functors $T$ and $U$ are additive and exact, and $UT$ is naturally equivalent to the identity functor on $\mathcal{C}(S)$. Furthermore, if $A$ is a left $S$-module, then $\text{hd}_R T(A) = \text{hd}_S A$.

*Proof.* That $T$ and $U$ are additive is clear from the definitions. It is also easy to see that $T$ is exact and $UT$ is naturally equivalent to the identity functor on $\mathcal{C}(S)$. Let $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ be an exact sequence of left $R$-modules, and set $\alpha^* = U(\alpha)$, $\beta^* = U(\beta)$. We have the sequence $U(A) \xrightarrow{\alpha^*} U(B) \xrightarrow{\beta^*} U(C)$, which we must show is exact. Recall that $U(A) = eA$ and $\alpha^*(ea) = e\alpha(a)$ for $a \in A$; $U(B), U(C)$ and $\beta^*$ are similarly defined. If $ea \in U(A)$, then $\beta^* \alpha^*(ea) = \beta^* e\alpha(a) = e(\beta\alpha)(a) = 0$, since $\beta\alpha = 0$. On the other hand, suppose that $\beta^*(eb) = 0$, where $eb \in U(B), b \in B$. Then $0 = \beta^*(eb) = e\beta(b) = \beta(eb)$, and so $eb = \alpha(a)$ for some $a \in A$, since $\ker(\beta) = \alpha(A)$. Then $ea \in U(A)$, and $\alpha^*(ea) = e\alpha(a) = e(eb) = eb$. Hence the sequence $U(A) \xrightarrow{\alpha^*} U(B) \xrightarrow{\beta^*} U(C)$ is exact, completing the proof that $U$ is an exact functor.

We claim now that, if $A$ is a left $S$-module, then $T(A)$ is $R$-projective if and only if $A$ is $S$-projective. For observe that $R = Re \oplus Re' = eRe \oplus e'Re' \cong T(S) \oplus T(Re')$ as a left $R$-module, since $e'Re = 0$; i.e., $S$, viewed as a left $R$-module, is a...
direct summand of $R$. It then follows by a direct sum argument that $T(A)$ is $R$-projective if $A$ is $S$-projective, since it is easily seen that $T$ preserves arbitrary direct sums. Conversely, suppose that $T(A)$ is $R$-projective. Consider a diagram of the following type—

$$
\begin{array}{c}
B \longrightarrow B'' \longrightarrow 0 \\
\uparrow^\alpha \\
A
\end{array}
$$

where $B$ and $B''$ are left $S$-modules, and the row is exact. Applying $T$, we get the following diagram of $R$-modules—

$$
\begin{array}{c}
T(B) \longrightarrow T(B'') \longrightarrow 0 \\
\uparrow^{T(\alpha)} \\
T(A)
\end{array}
$$

The row is exact, since $T$ is an exact functor. Since $T(A)$ is $R$-projective, there exists an $R$-homomorphism $\theta : T(A) \rightarrow T(B)$ which, when inserted in the above diagram, renders it commutative. Then, applying $U$ and using the fact that $UT$ is naturally equivalent to the identity functor on $\mathcal{C}(S)$, we get a commutative diagram—

$$
\begin{array}{c}
B \longrightarrow B'' \longrightarrow 0 \\
U(\theta) \downarrow \uparrow^\alpha \\
A
\end{array}
$$

from which it follows that $A$ is $S$-projective.

Now let $A$ be any left $S$-module, and $0 \rightarrow K \rightarrow P \rightarrow A \rightarrow 0$ be exact, where $P$ is a projective left $S$-module. We then get the exact sequence $0 \rightarrow T(K) \rightarrow T(P) \rightarrow T(A) \rightarrow 0$ of left $R$-modules. We have from our previous remarks that $T(P)$ is $R$-projective, and one sequence splits if and only if the other splits. It then follows by an easy induction argument that $hd_R T(A) = hd_S A$, completing the proof of the theorem.

There are analogous statements describing the relationship between $R$ and $S'$. We state these without proof.

**Proposition 2.2.** The mapping $f' : R \rightarrow S'$ defined by $f(a) = e'ae'$ is a ring epimorphism.
THEOREM 2.2. Let $\mathcal{C}'(R)$ and $\mathcal{C}'(S)$ be the categories of all right modules over $R$ and $S'$, respectively. Define functors $T': \mathcal{C}'(S') \to \mathcal{C}'(R)$ and $U': \mathcal{C}'(R) \to \mathcal{C}'(S')$ in entirely similar fashion to the definitions of $T$ and $U$ on $\mathcal{C}(S)$ and $\mathcal{C}(R)$. Then $T'$ and $U'$ are additive and exact, and $U'T'$ is naturally equivalent to the identity functor on $\mathcal{C}'(S')$. Furthermore, if $A$ is a right $S$-module, then $hd_\pi T'(A) = hd_\pi A$.

Now, changing our point of view, let $S$ and $S'$ be arbitrary rings with unit, and $M$ be an $(S, S')$-bimodule. Let $R$ be the ring of all two-by-two matrices of the form

$$
\begin{pmatrix}
    a & u \\
    0 & b
\end{pmatrix} \quad a \in S, b \in S', u \in M
$$

where addition is defined to be component-wise, and multiplication is defined by the rule

$$
\begin{pmatrix}
    a & u \\
    0 & b
\end{pmatrix}
\begin{pmatrix}
    a' & u' \\
    0 & b'
\end{pmatrix} =
\begin{pmatrix}
    aa' + ub' & au' + ub' \\
    0 & bb'
\end{pmatrix}.
$$

It is easy to verify that $R$ is an associative ring with unit. Set

$$
e = \begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}, \quad e' = 1 - e = \begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix}
$$

$e$ and $e'$ are idempotents in $R$, and $e'Re = 0$. Furthermore, $S \approx eRe$, $S' \approx e'Re'$, and $M \approx eRe'$ as an $(S, S')$-bimodule. We shall write $R = \mathcal{I}(S, S', M)$. $R$, $S$, and $S'$ satisfy the conditions of Theorems 2.1 and 2.2. Finally, we have the following result.

THEOREM 2.3. Let $R$ be a ring, and $e$ be an idempotent in $R$ such that $e'Re = 0$, where $e' = 1 - e$. Let $S = eRe$, $S' = e'Re'$, and $M = eRe'$. Then the mapping $g: R \to \mathcal{I}(S, S', M)$ defined by

$$
g(a) = \begin{pmatrix}
    eae & eae' \\
    0 & e'a'e
\end{pmatrix} \quad a \in R
$$

is an isomorphism.

Proof. Routine computation.
3. A Left Semi-Hereditary Ring which is Not Right Semi-hereditary

We are now ready to construct the previously mentioned example of a left semi-hereditary ring which is not right semi-hereditary. Let $S'$ be any commutative ring which is regular but not semi-simple (e.g., the direct product of an infinite number of copies of a field). Let $I$ be an ideal in $S'$ which is not a direct summand of $S'$; such an ideal exists, since $S'$ is not semi-simple. Set $S = S'/I$. $S$ is a regular ring, since it is a residue class ring of the regular ring $S'$. We may view $S$ as an $(S, S')$-bimodule. Observe that $S$ is not projective as a right $S'$-module, since $I$ is not a direct summand of $S'$. Set $R = F(S, S', S)$.

**Proposition 3.1.** $R$ is left semi-hereditary but not right semi-hereditary.

**Proof.** Define $G(R)$, $G(S)$, $T$, $U$, etc., as in Theorems 2.1 and 2.2. Let $J'$ be the set of all elements of $R$ of the form—

$$\begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix} \quad u \in S.$$ 

It is easy to see that $J'$ is a right ideal in $R$, and the mapping $\varphi: T'(S) \to J'$ defined by—

$$\varphi(u) = \begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix} \quad u \in S$$

is an $R$-isomorphism. Here we are viewing $S$ as an element of $G'(S')$; i.e., as a right $S'$-module. Since $S$ is finitely generated (even cyclic) over $S'$, it follows from the definition of $T'$ that $T'(S)$ is finitely $R$-generated, and so $J'$ is a finitely generated right ideal in $R$. But, by Theorem 2.2, we have that $h_{d_R}J' = h_{d_R}T'(S) = h_{d_S}S > 0$. It then follows that $R$ is not right semi-hereditary.

We now show that $R$ is left semi-hereditary. Let $J = (\alpha_1, \ldots, \alpha_n)$ be a finitely generated left ideal in $R$, where—

$$\alpha_i = \begin{pmatrix} a_i & u_i \\ 0 & b_i \end{pmatrix} \quad a_i \in S, \ u_i \in S, \ b_i \in S'.$$

Since $S'$ is regular, it follows that $S'b_1 + \ldots + S'b_n = S'e$ for some idempotent $e$ in $S'$. Write $\lambda_1b_1 + \ldots + \lambda_nb_n = e$, where $\lambda_i \in S'$; then—
\[
\begin{pmatrix}
0 & 0 \\
0 & \lambda_i b_i
\end{pmatrix} = \begin{pmatrix}
0 & 0 \\
0 & \lambda_i
\end{pmatrix} \begin{pmatrix}
\alpha_i & u_i \\
0 & b_i
\end{pmatrix} = \begin{pmatrix}
0 & 0 \\
0 & \lambda_i
\end{pmatrix} \alpha_i \in J.
\]

Hence, setting—
\[
\varepsilon = \begin{pmatrix}
0 & 0 \\
0 & e
\end{pmatrix}
\]
we get that \( \varepsilon \) is in \( J \), and so \( Re \subseteq J \). Observe that \( Re \) consists of all elements of \( R \) of the form—
\[
\begin{pmatrix}
0 & ue \\
0 & be
\end{pmatrix} \quad u \in S, \ b \in S'.
\]

Since \( b_i = b_ie \) for all \( i = 1, \ldots, n \), we then get that—
\[
\begin{pmatrix}
\alpha_i & u_i(1-e) \\
0 & 0
\end{pmatrix} = \begin{pmatrix}
\alpha_i & u_i \\
0 & b_i
\end{pmatrix} - \begin{pmatrix}
0 & ue \\
0 & be
\end{pmatrix} \in J.
\]

Let \( L \) be the left \( S \)-submodule of \( S \oplus S \) generated by the elements \((\alpha_i, u_i(1-e))\), \( i = 1, \ldots, n \). Define a mapping \( h: T(L) \oplus Re \to R \) by—
\[
h((u, v) + \alpha \varepsilon) = \begin{pmatrix}
u \\
v + xe
\end{pmatrix} + \alpha \varepsilon = \begin{pmatrix}
u \\
0 + xe
\end{pmatrix}
\]
where—
\[
\alpha = \begin{pmatrix}
a & x \\
0 & b
\end{pmatrix} \quad a \in S, \ x \in S, \ b \in S.
\]

It is easily verified that \( h \) is a left \( R \)-module homomorphism, and \( \text{Image}(h) \subseteq J \). Also—
\[
h(\alpha_i, u_i(1-e)) = \begin{pmatrix}
\alpha_i & u_i(1-e) \\
0 & 0
\end{pmatrix}
\]
and so, since \( Re \subseteq \text{Image}(h) \), it follows from \((*)\) and \((**)\) above that \( \text{Image}(h) = J \).

Suppose now that \( h((u, v) + \alpha \varepsilon) = 0 \), where \((u, v) \in L\) and
\[
\alpha = \begin{pmatrix}
a & x \\
0 & b
\end{pmatrix}.
\]

It follows from the definition of \( L \) that \( v = v(1-e) \), and so we have—
A GENERALIZATION OF THE RING OF TRIANGULAR MATRICES

\[ h((u, v) + \alpha v) = \begin{pmatrix} u & v \\ 0 & 0 \end{pmatrix} + \alpha v = \begin{pmatrix} u & v(1-e) + xe \\ 0 & be \end{pmatrix} = 0. \]

Then \( u = be = v(1-e) + xe = 0 \), from which it follows immediately that \( v = v(1-e) = 0 \) and \( xe = 0 \), since \( e \) is an idempotent in \( S' \). Therefore \( (u, v) = 0 \), and—

\[ \alpha v = \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & e \end{pmatrix} = \begin{pmatrix} 0 & xe \\ 0 & be \end{pmatrix} = 0. \]

Hence \((u, v) + \alpha v = 0\). It then follows that \( h \) is an isomorphism; i.e., \( J \cong T(L) \oplus R \) as a left \( R \)-module. Now, \( L \) is a finitely generated submodule of the free left \( S \)-module \( S \oplus S \); thus, since \( S \) is regular and therefore left semi-hereditary, we have that \( L \) is \( S \)-projective. Then, since \( R \) is a direct summand of \( R \) (\( e \) being an idempotent), it follows from Theorem 2.1 that \( hd_nJ = hd_nT(L) = hd_nL = 0 \). Therefore every finitely generated left ideal in \( R \) is projective, and so \( R \) is left semi-hereditary. This completes the proof of the proposition.

4. Applications to Semi-Primary Rings

In general it seems to be difficult to express the global dimension of \( R = T(R', S, A) \) in terms of the homological invariants of \( R', S, \) and \( A \). However, it is easy to obtain complete information for the special case in which \( R \) is semi-primary and \( S \) is semi-simple; this information will play a key role in our later results on semi-primary rings.

**Lemma 4.1.** Let \( R' \) be a semi-primary ring with radical \( N' \), \( S \) be a semi-simple ring, and \( A \) be an \((R', S)\)-bimodule. Set \( R = T(R', S, A) \). Then \( R \) is a semi-primary ring with radical \( N \) consisting of all elements of \( R \) of the form—

\[ \begin{pmatrix} a' & x \\ 0 & 0 \end{pmatrix}, \quad a' \in N', \ x \in A. \]

Furthermore, \( \text{gl. dim.} R = \max\{\text{gl. dim.} R', 1 + hd_nA\} \).

**Proof.** Let \( f' : R' \to R'/N' \) be the canonical homomorphism, and define a mapping \( f : R \to (R'/N') \oplus S \) by—

\[ f \left( \begin{pmatrix} a' & x \\ 0 & c \end{pmatrix} \right) = (f'(a'), c) \quad a' \in R', \ x \in A, \ c \in S. \]
It is easily verified that \( f \) is an epimorphism with kernel \( N \). Another routine computation establishes the fact that, if \( (N')' = 0 \), then \( N'^{+1} = 0 \). Since \( (R'/N') \oplus S \) is semi-simple, it then follows that \( R \) is a semi-primary ring with radical \( N \).

Now let \( \mathcal{C}(R') \) and \( \mathcal{C}(R) \) be the categories of left \( R' \)-modules and left \( R \)-modules, respectively, and define the functors \( T: \mathcal{C}(R') \to \mathcal{C}(R) \) and \( U: \mathcal{C}(R) \to \mathcal{C}(R') \) as in Theorem 2.1. Define a mapping \( g: N \to T(N') \oplus T(A) \) by

\[
g \left( \begin{array}{cc} a' & x \\ 0 & 0 \end{array} \right) = (a', x) \quad a' \in R', \ x \in A.
\]

It is easy to see that \( g \) is a left \( R \)-module isomorphism. Hence, using Theorem 2.1, we get that \( \text{hd}_{R}N = \max\{\text{hd}_{R}T(N'), \text{hd}_{R}T(A)\} = \max\{\text{hd}_{R}N', \text{hd}_{R}A\} \). But, since \( R \) and \( R' \) are semi-primary rings, we have that \( \text{gl.dim.} R = 1 + \text{hd}_{R}N \) and \( \text{gl.dim.} R' = 1 + \text{hd}_{R'}N' \) (see [1]). We then get that \( \text{gl.dim.} R = 1 + \text{hd}_{R}N = 1 + \max\{\text{hd}_{R}N', 1 + \text{hd}_{R}A\} = \max\{\text{gl.dim.} R', 1 + \text{hd}_{R}A\} \), completing the proof of the lemma.

**Remark.** It will be noted that in the statement and proof of the above lemma we have made no distinction between the left and right global dimensions of \( R \). This is permissible in view of the fact that, since \( R \) is a semi-primary ring, both the left and right global dimensions of \( R \) are equal to the weak global dimension of \( R \), and are hence equal to each other (see [1]).

**Lemma 4.2.** Let \( R \) be a semi-primary ring with radical \( N \), and \( e_1, \ldots, e_r \) be a complete set of mutually orthogonal primitive idempotents of \( R \). Suppose that there exists \( s < r \) such that \( e_iN = 0 \) for \( i > s \) but \( e_jN \neq 0 \) for \( j \leq s \). Set \( e = e_{s+1} + \cdots + e_r \), \( e' = 1 - e \), \( R' = e' Re' \), \( S = eRe \), and \( A = e' Re \). Then \( eRe' = 0 \), and \( R \simeq \mathcal{F}(R', S, A) \). Furthermore, \( S \) is semi-simple.

**Proof.** Clearly \( eNe \subseteq eN = 0 \), and so \( S = eRe \) is semi-simple. If \( j \leq s \) and \( i > s \), then \( e_iN = 0 \), \( e_jN \neq 0 \), and so \( e_iR \not\subseteq e_jR \) as right \( R \)-modules. Since \( e_iR \simeq \text{Hom}_R(Re_i, R) \) and \( e_jR \simeq \text{Hom}_R(Re_j, R) \), it then follows that \( Re_i \not\subseteq Re_j \), and thus \( Re_i/Ne_i \not\subseteq Re_j/Ne_j \). We then obtain that \( e_iRe_j \subseteq N \), since \( R/N \) is semi-simple. Hence \( eRe' \subseteq N \), and so \( eRe' \subseteq eN = 0 \). We then get from Theorem 2.3 that \( R \simeq \mathcal{F}(R', S, A) \), completing the proof of the lemma.
Definition 4.1. Let $R$ be a semi-primary ring with radical $N$. $R$ will be called triangular if any complete set $e_1, \ldots, e_r$ of mutually orthogonal primitive idempotents of $R$ can be indexed so that $e_iN e_j = 0$ whenever $i \geq j$.

Our principal results concerning triangular rings, which parallel the work of Jans and Nakayama ([5]), are summarized in the theorems which follow.

Theorem 4.1. Let $R$ be a semi-primary ring with radical $N$. Then the following statements are equivalent—

(a) $R$ is triangular.

(b) There exists a complete set $e_1, \ldots, e_n$ of mutually orthogonal primitive idempotents of $R$ such that $e_iN e_j = 0$ if $i \geq j$.

(c) $\text{gl. dim.} (R/I) < \infty$ for any two-sided ideal $I$ in $R$.

(d) $\text{gl. dim.} (R/N^2) < \infty$.

If any (and hence all) of these conditions hold, then $\text{gl. dim.} R < r$, where $r$ is the number of isomorphism classes of simple left $R$-modules.

Proof. (a) $\Rightarrow$ (b): Obvious.

(b) $\Rightarrow$ (c): Let $R$ satisfy (b). It is then clear that (b) also holds for any epimorphic image of $R$; hence to establish (c) we need only show that $R$ itself has finite global dimension. We shall show that $\text{gl. dim.} R < r$. If $r = 1$, it follows easily from (b) that $N = 0$, in which case $R$ is semi-simple and so $\text{gl. dim.} R = 0$. Proceed by induction on $r$; assume the statement true for $r' < r$. By hypothesis, there exists a complete set $e_1, \ldots, e_n$ of mutually orthogonal primitive idempotents of $R$ such that $e_iN e_j = 0$ if $i \geq j$. Then clearly $e_n N = 0$. We may assume that, for some integer $k < n$, $e_{k+1} N = e_{k+2} N = \cdots = e_n N = 0$, but $e_i N \neq 0$ for $i \leq k$. Let $e = e_{k+1} + \cdots + e_n$ and $e' = 1 - e$. It then follows from Lemma 4.2 that $eRe' = 0$, $S = eRe$ is semi-simple, and $R \cong S(R', S, A)$, where $R' = e' Re'$ and $A = e' Re'$. Observe now that $R'$ also satisfies (b) and has fewer than $r$ simple components; hence, by the induction assumption, $R'$ is a semi-primary ring and $\text{gl. dim.} R' < r - 1$. That $\text{gl. dim.} R < r$ then follows immediately from Lemma 4.1.

(c) $\Rightarrow$ (d): Obvious.

(d) $\Rightarrow$ (a): Assume that $\text{gl. dim.} (R/N^2) < \infty$. We first prove that $R$ is triangular on the hypothesis that $N^2 = 0$. Let $e_1, \ldots, e_n$ be any complete set of mutually orthogonal primitive idempotents of $R$, and set $A_i = Re_i / Ne_i$. $A_i$
is a simple left \(R\)-module. \(Ne_i\) is the kernel of the obvious epimorphism of \(Re_i\) onto \(A_i\), and is a semi-simple left \(R\)-module, since \(N^2 = 0\). If \(e_i Ne_j \neq 0\), then \(A_i\) is a direct summand of \(Ne_j\), and so \(hd_R(A_i) < hd_R(A_j)\), since both numbers must be finite. We may assume that the \(<e_i>\) are indexed so that \(hd_R(A_i) \geq hd_R(A_j)\) if \(i \geq j\). It is then clear that \(e_i Ne_j=0\) if \(i \geq j\), and hence \(R\) is triangular in this case. This part of the proof is essentially the same as in [5].

Suppose now that \(N^2\) is not necessarily zero. We have from the above paragraph that any complete set \(e_1, \ldots, e_n\) of mutually orthogonal primitive idempotents of \(R\) may be indexed so that \(e_i Ne_j \subseteq N^s\) if \(i \geq j\). Assume that \(e_i Ne_j \subseteq N^s\) whenever \(i \geq j\), where \(s\) is an integer greater than one. Then \(e_i Ne_j = e_i N^s e_j\), and we have—

\[
e_i Ne_j = e_i N^s e_j = (e_i N^{s-1}) (\sum_{k=1}^{n} e_k) (Ne_j) = \sum_{k=1}^{n} (e_i N^{s-1} e_k) (e_k Ne_j).
\]

But if \(i \geq k\), then \(e_i N^{s-1} e_k \subseteq N^s\). On the other hand, if \(k \geq i\), then also \(k \geq j\), and then \(e_k Ne_j \subseteq N^s\). It then follows that \(e_i Ne_j \subseteq N^{s+1}\). Hence, by induction, we see that, if \(i \geq j\), then \(e_i Ne_j\) is contained in every power of \(\ast N\), and hence \(e_i Ne_j = 0\), since \(N\) is nilpotent. Therefore \(R\) is triangular.

The final contention of the theorem follows immediately from inspection of the above arguments. Thus the proof of the theorem is complete.

**Remarks.** 1. Let \(R\) satisfy the conditions of Theorem 4.1. By a somewhat more careful analysis of the situation described in that theorem, utilizing the basic properties of the triangular matrix construction, it is possible to derive the inequality \(\text{gl. dim. } R \leq \text{gl. dim. } (R/N^2) < r\), thus improving the estimate of the theorem on the global dimension of \(R\). Furthermore, if \(\text{gl. dim. } (R/N^2) = m\), then \(N^{m+1} = 0\) ([3], p. 55). These results were proved by Eilenberg, Nagao, and Nakayama ([4]) for residue class rings of hereditary semi-primary rings.

2. Theorem 4.1 was essentially proved, using somewhat different methods, by Jans and Nakayama ([5]) for rings which, in addition to being semi-primary, satisfy a sort of “splitting” condition tantamount to separability of the residue class ring modulo the radical. Their proofs utilized the above-mentioned results of Eilenberg, Nagao, and Nakayama concerning residue class rings of hereditary semi-primary rings, and were based upon their very interesting observation.
that a triangular ring satisfying the above-described splitting condition is a
residue class ring, in a particularly nice way, of a unique hereditary semi-pri-
mary ring. The existence of this hereditary "covering" ring is, of course, the
most important result of the theory; however, it seems to be the only result
for which the extra splitting hypothesis is really necessary.

Next we discuss semi-primary rings with the property that every principal
right ideal is projective. Our observations will culminate in the theorem that
such rings are triangular, a result which was essentially proved by Nakano
([6]). First, a couple of almost obvious lemmas.

**Lemma 4.3.** Let $R$ be any ring, and $x \in R$. Then the following conditions
are equivalent—

(a) The right ideal $xR$ is projective.

(b) The right annihilator of $x$ is a direct summand of $R$.

(c) There exists an idempotent $e$ in $R$ such that $xe = x$, and if $xa = 0$ then
$ea = 0$.

**Proof.** The lemma may be established by routine computations, which we
omit.

**Lemma 4.4.** Let $R$ be a ring, and suppose that every principal right ideal
in $R$ is projective. If $e$ is an idempotent in $R$, then every principal right ideal
in $S = eRe$ is projective.

**Proof.** We have from our hypotheses and Lemma 4.3 that the right an-
nihilator of every element of $R$ is a direct summand of $R$. It is then a straig-
tforward matter to verify that the same condition holds in $S$. Hence, by Lemma
4.3, every principal right ideal in $S$ is projective. This completes the proof.

**Theorem 4.2.** Let $R$ be a semi-primary ring with the property that every
principal right ideal in $R$ is projective. Then $R$ is triangular.

**Proof.** Let $N$ be the radical of $R$, and $r$ be the number of isomorphism
classes of simple left $R$-modules. If $r = 1$, then $R/N$ is isomorphic as a right
$R$-module to a direct sum of simple right ideals in $R$, each of which is neces-
sarily principal; hence, by our hypothesis, $R/N$ is a projective right $R$-module.
Thus $N$, viewed as a right ideal in $R$, is a direct summand of $R$, which is impossible unless $N = 0$. Therefore $N = 0$, in which case $R$ is a simple ring and the theorem is trivially true.

Proceed by induction on $r$; assume the theorem true for $r' < r$. If $R$ is semi-simple we are done; otherwise $N \neq 0$ and there exists an element $x \in R$ such that $xN = 0$. By hypothesis $xR$ is projective, and so, by Lemma 4.3, there exists an idempotent $e_0 \in R$ such that $xe_0 = x$, and if $xa = 0$ then $e_0a = 0$. Then $xN = 0$, and so $e_0N = 0$. We may write $e_0 = \bar{e} + e_1$, where $\bar{e}$ and $e_1$ are orthogonal idempotents and $\bar{e}$ is primitive. Then $\bar{e} = \bar{e}e_0$, and so $\bar{e}N = \bar{e}e_0N = 0$.

Since $R$ is semi-primary, there exists a complete set $e_1, \ldots, e_n$ of mutually orthogonal primitive idempotents of $R$ such that $e_n = \bar{e}$. Since $e_0N = \bar{e}N = 0$, we may assume that, for some integer $k < n$, $e_{k+1}N = \cdots = e_nN = 0$, but $e_iN \neq 0$ for $i \leq k$. Let $e = e_{k+1} + \cdots + e_n$, $e' = 1 - e$; we then get from Lemma 4.2 that $S = eRe$ is semi-simple and $R \simeq \mathcal{S}(R', S, A)$, where $R' = e'Ra'$ and $A = e'Re$. Observe that $R'$ has less than $r$ isomorphism classes of simple left modules. By Lemma 4.4, every principal right ideal in $R'$ is projective; hence, by the induction assumption, $R'$ is a triangular semi-primary ring. It then follows immediately that $R$ itself is triangular, completing the proof of the theorem.

Nakano ([6]) has proved a result which is essentially equivalent to Theorem 4.2, although he replaced the hypothesis that $R$ be semi-primary by the following assumption: $R = I_1 \oplus \cdots \oplus I_n$, where $I_k$ is a left ideal in $R$ and $\text{Hom}_R(I_k, I_k)$ is a field. However, it is easy to see that such a ring is actually semi-primary; hence Nakano's result is contained in (and is, in fact, equivalent to) Theorem 4.2. Nakano proved also a sort of converse of Theorem 4.2, which we shall not consider here.

References


University of Chicago and
Princeton University