GENERALISATIONS OF SOME INTEGRALS INVOLVING BESSEL FUNCTIONS AND E-FUNCTIONS

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(Received 30th October, 1950)

§ 1. *Introductory.* In § 3 a generalisation of the formula [MacRobert, *Phil. Mag.*, Ser. 7, XXXI, p. 258]

$$4\int_0^\infty \lambda^{m-1} K_n(2\lambda) E(p\,;\ \alpha_s : q\,;\ \rho_t : x\lambda^{-2})\, d\lambda = E(p+2\,;\ \alpha_s : q\,;\ \rho_t : x), \quad \text{.........(1)}$$

where $\alpha_{p+1} = \frac{1}{2}m + \frac{1}{2}n$, $\alpha_{p+2} = \frac{1}{2}m - \frac{1}{2}n$, $R(m \pm n) > 0$, and x is real and positive, will be established. In the course of the proof Hardy's formula [*Mess. of Maths.*, LVI, (1927), p. 190],

$$\int_0^\infty K_n(x) K_n\left(\frac{b}{x}\right) dx = \pi K_{2n}(2\sqrt{b}), \quad \text{.........(2)}$$

where $R(b) > 0$, will be required. This was originally proved by an application of Mellin's Inversion Formula. An alternative proof is given in § 2, and some related formulae are deduced.

§ 2. *Proof of Hardy's Formula.* Denote the integral on the left of (2) by $F(b)$; then

$$F'(b) = \int_0^\infty K_n(x) K_n'\left(\frac{b}{x}\right) \frac{1}{x}\, dx,$$

and

$$F''(b) = \int_0^\infty K_n(x) K_n''\left(\frac{b}{x}\right) \frac{1}{x^2}\, dx\,;$$

so that

$$b^2 F''(b) = \int_0^\infty K_n(x) \left\{\left(\frac{b^2}{x^2} + n^2\right) K_n\left(\frac{b}{x}\right) - \frac{b}{x} K_n'\left(\frac{b}{x}\right)\right\} dx$$

$$= n^2 F(b) - bF'(b) + b^2 \int_0^\infty K_n(x) K_n\left(\frac{b}{x}\right) \frac{dx}{x^2}.$$

But, on replacing x by b/x in $F(b)$, it is seen that

$$F(b) = b\int_0^\infty K_n(x) K_n\left(\frac{b}{x}\right) \frac{1}{x^2}\, dx.$$

Hence

$$b^2 F''(b) + bF'(b) - (b + n^2) F(b) = 0. \quad \text{.........(3)}$$

Now in the equation

$$x^2 y'' + xy' - (x^2 + 4n^2) y = 0, \quad \text{.........(4)}$$

with solutions $K_{2n}(x)$ and $I_{2n}(x)$, put $x = 2\sqrt{b}$, and it reduces to (3). Therefore

$$\int_0^\infty K_n(x) K_n\left(\frac{b}{x}\right) dx = AK_{2n}(2\sqrt{b}) + BI_{2n}(2\sqrt{b}).$$

Here let $b \to \infty$, and it is seen that B must be zero. Thus

$$\int_0^\infty K_n(x) \left\{I_{-n}\left(\frac{b}{x}\right) - I_n\left(\frac{b}{x}\right)\right\} dx = \frac{A}{2\cos n\pi}\left\{I_{-2n}(2\sqrt{b}) - I_{2n}(2\sqrt{b})\right\}.$$

Now assume that $R(n) > 0$, multiply by b^n and let $b \to 0$; then

$$\frac{2^n}{\Gamma(1-n)} \int_0^\infty K_n(x)\, x^n\, dx = \frac{A}{2\cos n\pi\, \Gamma(1-2n)}.$$

https://doi.org/10.1017/S2040618500035498 Published online by Cambridge University Press

But, if $R(m \pm n) > 0$,

$$\int_0^\infty K_n(x) x^{m-1}\, dx = 2^{m-2} \Gamma\left(\frac{m+n}{2}\right) \Gamma\left(\frac{m-n}{2}\right). \qquad \text{(5)}$$

Therefore

$$\frac{2^{2n-1}}{\Gamma(1-n)} \Gamma(n+\tfrac{1}{2}) \Gamma(\tfrac{1}{2}) = \frac{A}{2 \cos n\pi \Gamma(1-2n)},$$

and from this it follows that $A = \pi$.

From (2) other formulae of the same type can be derived as follows.

In (2) let amp b decrease by $\frac{1}{2}\pi$, finally writing b/i in place of b; then, since

$$K_n(t) = i^n G_n(it), \qquad \text{(6)}$$

the formula becomes

$$\int_0^\infty K_n(x) G_n\left(\frac{b}{x}\right) dx = \pi i^{-n} K_{2n}(2\sqrt{b} \,.\, e^{-i\pi/4}), \qquad \text{(7)}$$

provided that $-\frac{5}{2} < R(n) < \frac{5}{2}$.

Similarly, on replacing b by ib, it is seen that

$$\int_0^\infty K_n(x) G_n\left(\frac{b}{x} e^{i\pi}\right) dx = \pi i^{-n} K_{2n}(2\sqrt{b} \,.\, e^{i\pi/4}), \qquad \text{(8)}$$

where $-\frac{5}{2} < R(n) < \frac{5}{2}$.

Hence, using the formula

$$\pi i J_n(t) = G_n(t) - i^{2n} G_n(te^{i\pi}), \qquad \text{(9)}$$

it follows that

$$i \int_0^\infty K_n(x) J_n\left(\frac{b}{x}\right) dx = i^{-n} K_{2n}(2\sqrt{b} \,.\, e^{-i\pi/4}) - i^n K_{2n}(2\sqrt{b} \,.\, e^{i\pi/4}), \qquad \text{(10)}$$

where $-\frac{5}{2} < R(n) < \frac{5}{2}$. This formula also is given in Hardy's paper.

Again, let amp x and amp b increase simultaneously by $\frac{1}{2}\pi$, so that x becomes ix and b becomes ib; then

$$\int_0^\infty G_n(xe^{i\pi})\, J_n\left(\frac{b}{x}\right) dx = K_{2n}(2\sqrt{b} \,.\, e^{i\pi/2}) - i^{-2n} K_{2n}(2\sqrt{b}), \qquad \text{(11)}$$

where $-\frac{1}{2} < R(n) < \frac{5}{2}$.

Similarly, if amp x and amp b decrease simultaneously by $\frac{1}{2}\pi$, (10) becomes

$$\int_0^\infty G_n(x) J_n\left(\frac{b}{x}\right) dx = i^{-2n} K_{2n}(2\sqrt{b} \,.\, e^{-i\pi/2}) - K_{2n}(2\sqrt{b}), \qquad \text{(12)}$$

where $-\frac{1}{2} < R(n) < \frac{5}{2}$.

Finally, from (9), (11) and (12) it follows that

$$\begin{aligned} \pi i \int_0^\infty J_n(x) J_n\left(\frac{b}{x}\right) dx &= i^{-2n} K_{2n}(2\sqrt{b} \,.\, e^{-i\pi/2}) - i^{2n} K_{2n}(2\sqrt{b} \,.\, e^{i\pi/2}) \\ &= G_{2n}(2\sqrt{b}) - i^{4n} G_{2n}(2\sqrt{b} \,.\, e^{i\pi}) \\ &= \pi i J_{2n}(2\sqrt{b}), \end{aligned}$$

so that

$$\int_0^\infty J_n(x) J_n\left(\frac{b}{x}\right) dx = J_{2n}(2\sqrt{b}), \qquad \text{(13)}$$

where $R(n) > -\frac{1}{2}$. This formula was given by Bateman (*Proc. Camb. Phil. Soc.*, XXI, (1908), p. 186).

https://doi.org/10.1017/S2040618500035498 Published online by Cambridge University Press

§ 3. *Generalisation of the Integral.* The formula to be proved is

$$2^{2^r+r+1}\pi^{2^r-1}\int_0^\infty \lambda^{2^r m-1} K_{2^r n}(2^{r+1}\lambda) E(p\,;\ \alpha_s : q\,;\ \rho_t : x\lambda^{-2^{r+1}})\, d\lambda$$
$$=E(p+2^{r+1}\,;\ \alpha_s : q\,;\ \rho_t : x), \quad\ldots\ldots(14)$$

where $r=0, 1, 2, \ldots$, and

$$\left.\begin{aligned}\alpha_{p+2k+1}&=\tfrac{1}{2}m+\tfrac{1}{2}n+\frac{k}{2^r}\\ \alpha_{p+2k+2}&=\tfrac{1}{2}m-\tfrac{1}{2}n+\frac{k}{2^r}\end{aligned}\right\} k=0, 1, 2, \ldots, 2^r-1, \quad\ldots\ldots(14')$$

$R(m\pm n)>0$ and x is real and positive.

It can be proved by induction ; for, assuming that it is valid, it follows that

$$E(p+2^{r+2}\,;\ \alpha_s : q\,;\ \rho_t : x)$$
$$=2^{2^r+r+1}\pi^{2^r-1}\int_0^\infty \lambda^{2^r l-1} K_{2^r n}(2^{r+1}\lambda) E(p+2^{r+1}\,;\ \alpha_s : q\,;\ \rho_t : x\lambda^{-2^{r+1}})\, d\lambda,$$

where

$$\left.\begin{aligned}\alpha_{p+2^{r+1}+2k+1}&=\tfrac{1}{2}l+\tfrac{1}{2}n+\frac{k}{2^r}\\ \alpha_{p+2^{r+1}+2k+2}&=\tfrac{1}{2}l-\tfrac{1}{2}n+\frac{k}{2^r}\end{aligned}\right\} k=0, 1, 2, \ldots, 2^r-1,$$

$$=2^{2^{r+1}+2r+2}\pi^{2^{r+1}-2}\int_0^\infty \lambda^{2^r l-1} K_{2^r n}(2^{r+1}\lambda)\, d\lambda$$
$$\times\int_0^\infty \mu^{2^r m-1} K_{2^r n}(2^{r+1}\mu) E(p\,;\ \alpha_s : q\,;\ \rho_t : x(\lambda\mu)^{-2^{r+1}})\, d\mu.$$

Here replace μ by μ/λ and get

$$2^{2^{r+1}+2r+2}\pi^{2^{r+1}-2}\int_0^\infty \lambda^{2^r(l-m)-1} K_{2^r n}(2^{r+1}\lambda)\, d\lambda$$
$$\times\int_0^\infty \mu^{2^r m-1} K_{2^r n}\left(2^{r+1}\frac{\mu}{\lambda}\right) E(p\,;\ \alpha_s : q\,;\ \rho_t : x\mu^{-2^{r+1}})\, d\mu.$$

Next put $l=m+2^{-r}$ and change the order of integration, so getting

$$2^{2^{r+1}+2r+2}\pi^{2^{r+1}-2}\int_0^\infty \mu^{2^r m-1} E(p\,;\ \alpha_s : q\,;\ \rho_t : x\mu^{-2^{r+1}})\, d\mu$$
$$\times\int_0^\infty K_{2^r n}(2^{r+1}\lambda)\, K_{2^r n}\left(2^{r+1}\frac{\mu}{\lambda}\right) d\lambda,$$

where

$$\left.\begin{aligned}\alpha_{p+2^{r+1}+2k+1}&=\tfrac{1}{2}m+\tfrac{1}{2}n+\frac{2k+1}{2^{r+1}}\\ \alpha_{p+2^{r+1}+2k+2}&=\tfrac{1}{2}m-\tfrac{1}{2}n+\frac{2k+1}{2^{r+1}}\end{aligned}\right\} \begin{aligned}&k=0, 1, 2, \ldots, 2^r-1,\\ &\text{or}\\ &2k+1=1, 3, 5, \ldots, 2^{r+1}-1.\end{aligned}$$

But, from (14′),

$$\left.\begin{aligned}\alpha_{p+2k+1}&=\tfrac{1}{2}m+\tfrac{1}{2}n+\frac{2k}{2^{r+1}}\\ \alpha_{p+2k+2}&=\tfrac{1}{2}m-\tfrac{1}{2}n+\frac{2k}{2^{r+1}}\end{aligned}\right\} \begin{aligned}&k=0, 1, 2, \ldots, 2^r-1,\\ &\text{or}\\ &2k=0, 2, 4, \ldots, 2^{r+1}-2.\end{aligned}$$

Therefore

$$\left.\begin{aligned}\alpha_{p+2k+1}&=\tfrac{1}{2}m+\tfrac{1}{2}n+\frac{k}{2^{r+1}}\\ \alpha_{p+2k+2}&=\tfrac{1}{2}m-\tfrac{1}{2}n+\frac{k}{2^{r+1}}\end{aligned}\right\}\ k=0,\ 1,\ 2,\ \ldots,\ 2^{r+1}-1.$$

Now, from (2), the last integral is equal to

$$\frac{1}{2^{r+1}}\,\pi K_{2^{r+1}n}(2^{r+2}\surd\mu).$$

Hence, on replacing μ by λ^2, we have

$$E(p+2^{r+2}\ ;\ \alpha_s:\ q\ ;\ \rho_t:\ x)=2^{2^{r+1}+r+2}\,\pi^{2^{r+1}-1}$$
$$\times\int_0^\infty \lambda^{2^{r+1}m-1}\,K_{2^{r+1}n}(2^{r+2}\lambda)\,E(p\ ;\ \alpha_s:\ q\ ;\ \rho_t:\ x\lambda^{-2^{r+2}})\,d\lambda,$$

which is (14) with $r+1$ in place of r.

But the formula holds when $r=0$: hence it holds for all positive integral values of r.

If in (14) amp λ is decreased by $\frac{1}{2}\pi$ and amp x by $2^r\pi$, it becomes, by (6),

$$2^{2^r+r+1}\,\pi^{2^r-1}\,i^{2^r(n-m)}\int_0^\infty \lambda^{2^r m-1}\,G_{2^r n}(2^{r+1}\lambda)\,E(p\ ;\ \alpha_s:\ q\ ;\ \rho_t:\ x\lambda^{-2^{r+1}})\,d\lambda$$
$$=E(p+2^{r+1}\ ;\ \alpha_s:\ q\ ;\ \rho_t:\ xe^{-i\pi 2^r}),\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(15)$$

where $R(m\pm n)>0$, $R(\frac{3}{2}-2^r m+2^{r+1}\alpha_s)>0$, $s=1,\ 2,\ \ldots,\ p$, and x is real and positive.

Similarly, and subject to the same conditions,

$$2^{2^r+r+1}\,\pi^{2^r-1}\,i^{2^r(n+m)}\int_0^\infty \lambda^{2^r m-1}\,G_{2^r n}(2^{r+1}\lambda e^{i\pi})\,E(p\ ;\ \alpha_s:\ q\ ;\ \rho_t:\ x\lambda^{-2^{r+1}})\,d\lambda$$
$$=E(p+2^{r+1}\ ;\ \alpha_s:\ q\ ;\ \rho_t:\ xe^{i\pi 2^r}),\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(16)$$

and, from (9),

$$2^{2^r+r+1}\,\pi^{2^r}\,i\int_0^\infty \lambda^{2^r m-1}\,J_{2^r n}(2^{r+1}\lambda)\,E(p\ ;\ \alpha_s:\ q\ ;\ \rho_t:\ x\lambda^{-2^{r+1}})\,d\lambda$$
$$=i^{2^r(m-n)}\,E(p+2^{r+1}\ ;\ \alpha_s:\ q\ ;\ \rho_t:\ xe^{-i\pi 2^r})$$
$$-\,i^{-2^r(m-n)}\,E(p+2^{r+1}\ ;\ \alpha_s:\ q\ ;\ \rho_t:\ xe^{i\pi 2^r}).\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(17)$$

where $R(m+n)>0$, $R(\frac{3}{2}-2^r m+2^{r+1}\alpha_s)>0$, $s=1,\ 2,\ \ldots,\ p$, and x is real and positive.

The case in which $r=0$ was given in a previous paper (see page 7).

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