NUMERICAL RANGES OF POWERS OF HERMITIAN ELEMENTS

by M. J. CRABB and C. M. McGREGOR

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Introduction An element k of a unital Banach algebra A is said to be Hermitian if its numerical range

$$V(k)=\{\psi(k):\psi\in A',\ \|\psi\|=\psi(1)=1\}$$

is contained in $\mathbb{R}$; equivalently, $\|e^{itk}\|=1 (t\in\mathbb{R})$—see Bonsall and Duncan [**3**] and [**4**]. Here we find the largest possible extent of $V(k^n)$, $n\in\mathbb{N}$, given $V(k)\subseteq[-1,1]$, and so $\|k\|\leq 1$: previous knowledge is in Bollobás [**2**] and Crabb, Duncan and McGregor [**7**]. The largest possible sets all occur in a single example. Surprisingly, they all have straight line segments in their boundaries. The example is in [**2**] and [**7**], but here we give A. Browder's construction from [**5**], partly published in [**6**]. We are grateful to him for a copy of [**5**], and for discussions which led to the present work. We are also grateful to J. Duncan for useful discussions.

Let X be the Banach space of entire functions f such that

$$\|f\|=\sup\{|f(\sigma+it)|e^{-|t|}:\sigma,\ t\text{ real}\}<\infty.$$

For $f\in X$, we have $\|f\|=\sup\{|f(\sigma)|:\sigma\text{ real}\}$—see [**6**] for proofs here. Define h by $h(f)=if'$. Then $h\in B(X)$, h is Hermitian and $\|h\|=1$. Denote $\{x\in X:\|x\|\leq 1\}$ by X_1.

LEMMA 1. *If $f\in X_1$ and $f(0)=1$, then for $T\in B(X)$ we have $(Tf)(0)\in V(T)$.*

Proof. Define the functional ϕ on the Banach algebra $B(X)$ by $\phi(T)=(Tf)(0)$. Then $|\phi(T)|\leq\|Tf\|\leq\|T\|$, so $\|\phi\|\leq 1$. Also, $\phi(I)=1$. Hence $\phi(T)\in V(T)$. *q.e.d.*

Let k be a Hermitian element of a unital Banach algebra A with $\|k\|\leq 1$. Let $\psi\in A'$ with $\|\psi\|=\psi(1)=1$, and let $f(z)=\psi(e^{-izk})$. Then $f\in X_1$ and $f(0)=1$. For ϕ as in the above proof, we have $\phi(h^n)=i^nf^{(n)}(0)=\psi(k^n)$ $(n=0,1,2,\ldots)$. Hence $V(p(k))\subseteq V(p(h))$ for any polynomial p. The same argument with the restrictions $\psi(1)=\phi(I)=1$ removed shows that $\|p(k)\|\leq\|p(h)\|$.

The next two theorems contain the main results of the paper. They are proved in the sequel.

THEOREM 2 (even power case). *Let $\zeta(\theta)$ be the $2n$-th derivative at 0, with respect to x, of $e^{-i\theta}(\cos\rho+i\theta\rho^{-1}\sin\rho)$, where $\rho^2=x^2+\theta^2$. Then the boundary of $V(h^{2n})$ consists of the curves $\zeta(\theta)$ and $\overline{\zeta(\theta)}$, $0\leq\theta\leq\pi$, and the line segment $[\zeta(\pi),\overline{\zeta(\pi)}]$.*

THEOREM 3 (odd power case). *Let $\zeta(\theta)$ be the $(2n+1)$-th derivative at 0, with respect to z, of $i^{2n+1}e^{-iA}(\cos Q+i(A+A^{-1}\alpha z)Q^{-1}\sin Q)$,* where *$A=\sqrt{\theta^2+\alpha^2}$ and $Q^2=(z+\alpha)^2+\theta^2$, α being the first positive zero of $(d^{2n}/dz^{2n})(\sin\rho/\rho)$ with $\rho^2=z^2+\theta^2$. Let θ_0 be the first positive θ for which $A=\pi$. Then the boundary of $V(h^{2n+1})$ consists of the curves $\pm\zeta(\theta)$ and $\pm\overline{\zeta(\theta)}$, $0\leq\theta\leq\theta_0$, and the line segments $[\pm\zeta(\theta_0),\mp\overline{\zeta(\theta_0)}]$.*

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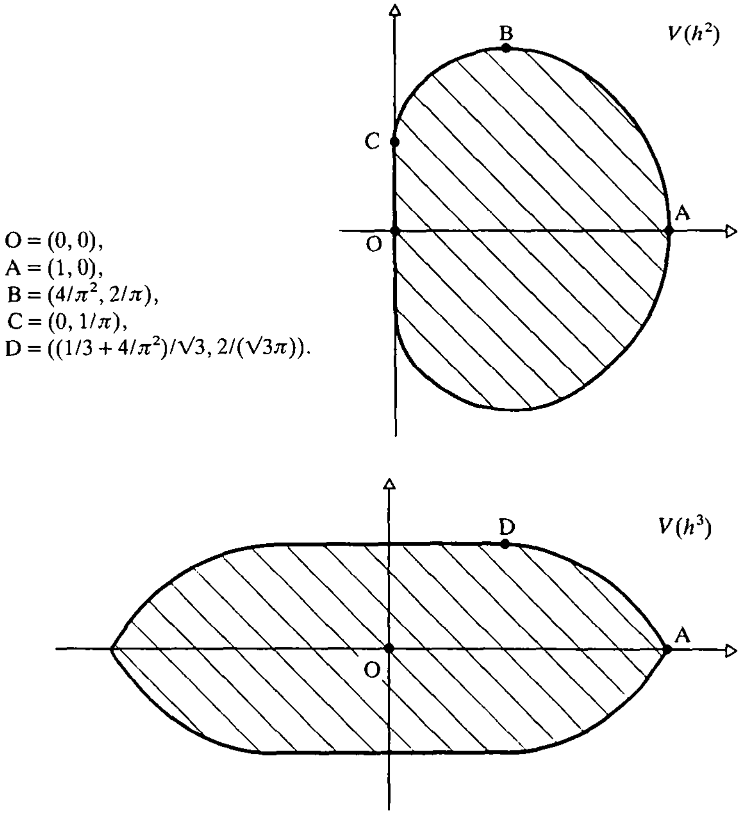

Figure 1

Figure 1 illustrates Theorems 2 and 3 with $n = 1$.

LEMMA 4. *For any polynomial* p, $\|p(h)\| = \sup\{|(p(h)f)(0)| : f \in X_1\}$.

Proof. Write m for the sup. By considering $f(s+u)$ for $f \in X_1$, we obtain $|(p(h)f)(u)| \leq m$. So $\|p(h)\| \leq m$, and the reverse inequality is clear. *q.e.d.*

For any $a \in B(X)$, we have $V(a) \subseteq \{\zeta : |\zeta| \leq \|a\|\}$. Hence if $\zeta \in V(a)$ and $|\zeta| = \|a\|$, then $\zeta \in \partial V(a)$. Also, $V(a+\gamma) = V(a) + \gamma$ $(\gamma \in \mathbb{C})$.

The even power case. Define, for $\theta, z \in \mathbb{C}$, $f_\theta(z) = \cos\rho = \sum_{n=0}^{\infty} (-1)^n (z^2+\theta^2)^n/(2n)!$, where $\rho = \sqrt{z^2+\theta^2}$. We can take either square root and get the same value for f_θ. Observe that $f_\theta'(z) = -z g_\theta(z)$, where $g_\theta(z) = \sin\rho/\rho$. For $\theta \geq 0$, f_θ and g_θ are even functions in X_1. To see this, apply Lemma 3.2 of [7] to the function $\phi(w,z) = f_{iw}(iz)$. This gives $f_\theta \in X_1$ for $\theta \in \mathbb{R}$. Then $\sin\rho/\rho = \int_0^1 \cos(u\rho)\,du \in X_1$.

Let $0 \leq \theta < \pi$, and put $e = f_\theta$. Consider, for f in X, $\int_{\Gamma_j} F(z)\,dz$, where

$$F(z) = p(z)f(z)/(z^{2n}e'(z)) = -p(z)f(z)/(z^{2n+1}g_\theta(z)).$$

Here Γ_j is the square with corners $(j+\frac{1}{2})\pi(\pm 1, \pm i)$, and

$$p(z) = \sum_{k=0}^{2n-2} \frac{g^{(k)}(0)}{k!} z^k.$$

On Γ_j, $|\sin z| > \frac{1}{3}e^{|t|}$, where $z = \sigma + it$. Hence $|\sin \rho| = |\sin(z+w)|$, where $|w| \to 0$ as $|z| \to \infty$, $> |\sin z \cos w| - |\cos z \sin w| > \frac{1}{4}e^{|t|}$ for all large enough $|z|$, since $|\cos z| \leq e^{|t|}$. Since $|z/\rho| \to 1$ as $|z| \to \infty$, we get $|e'(z)| > \frac{1}{5}e^{|t|}$ for all large $|z|$. Hence $\int_{\Gamma_j} F\,dz \to 0$ as $j \to \infty$, and the sum of the residues of F is 0. The function F is meromorphic with poles at 0 and at $\{\alpha_k\} \subset \mathbb{R}$, the zeros of g_θ. Also, $g_\theta(z) = p(z) + z^{2n}q(z)$, where q is entire. So

$$\frac{p(z)}{g_\theta(z)} = \left(1 + \frac{q(z)}{p(z)} z^{2n}\right)^{-1} = 1 - \frac{q(0)}{p(0)} z^{2n} + \ldots = 1 - \frac{g^{(2n)}(0)}{g(0)} \frac{z^{2n}}{(2n)!} + \ldots.$$

Hence the residue of F at 0 is $-(1/(2n)!)(f^{(2n)}(0) - \tau f(0))$, where $\tau = g_\theta^{(2n)}(0)/g_\theta(0)$. Therefore, where we are defining $\phi \in X'$,

$$\phi(f) = ((ih)^{2n} - \tau)f(0) = f^{(2n)}(0) - \tau f(0) = (2n)! \sum_k \frac{p(\alpha_k)f(\alpha_k)}{\alpha_k^{2n}e''(\alpha_k)}. \tag{1}$$

At $z = \alpha_k$, $\sin\rho = 0$, so $e = \cos\rho = \pm 1$, and e, e'' have opposite signs: note that $e(\mathbb{R}) = [-1, 1]$. Hence for all k, $e(\alpha_k)/e''(\alpha_k) < 0$. Thus $|\phi(e)| = \max\{|\phi(f)| : f \in X_1\}$, attaining the estimate $(2n)! \sum_k \alpha_k^{-2n} |p(\alpha_k)|/|e''(\alpha_k)|$, if the $p(\alpha_k)$ have constant sign. For $n = 1$ this follows since p is constant. For $n > 1$, it is proved later.

Thus by Lemma 4, $|\phi(e)| = \|(-1)^n h^{2n} - \tau\| = \|h^{2n} - \tau'\|$, where $\tau' = (-1)^n\tau$. Define $k_\theta(z) = k(z) = e^{-i\theta}(\cos\rho + i\theta \sin\rho/\rho) = e^{-i\theta}(f_\theta(z) + i\theta g_\theta(z))$. Then $k \in X$, and $|k| \leq 1$ on $\mathbb{R}$, so $k \in X_1$. Since $k(0) = 1$, $\zeta = \zeta(\theta) = (-1)^n k^{(2n)}(0) \in V(h^{2n})$. By the definition of ϕ and τ, $\phi(g_\theta) = 0$. Thus $(-1)^n(\zeta - \tau') = k^{(2n)}(0) - \tau = \phi(k) = e^{-i\theta}\phi(f_\theta)$, and $|\zeta - \tau'| = |\phi(f_\theta)| = \|h^{2n} - \tau'\|$. Since $\zeta - \tau' \in V(h^{2n} - \tau')$, we get $\zeta \in \partial V(h^{2n})$. Also, $V(h^{2n} - \tau') \subseteq \{z : |z| \leq |\zeta - \tau'|\}$. Hence $V(h^{2n})$ is contained in a circle with centre at τ' and through ζ.

As $\theta \to \pi$, $g_\theta(0) = \sin\theta/\theta \to 0$. We prove below that $g_\theta^{(2n)}(0) \neq 0$ for $0 \leq \theta \leq \pi$. These are continuous in θ, and so $|\tau'| = |\tau| \to \infty$ as $\theta \to \pi$. Also, $\zeta(\theta) \to \zeta(\pi) = \zeta_0$ say, which is also in $V(h^{2n})$ and $\mathrm{Im}(\zeta_0) = -\pi(-1)^n g_\pi^{(2n)}(0) \neq 0$ (below).

The function $\overline{k(\bar{z})}$ gives $\bar{\zeta}$ and $\bar{\zeta}_0$ in $\partial V(h^{2n})$. Hence the line segment $[\zeta_0, \bar{\zeta}_0] \subseteq V$. The discs with centre τ' and through ζ tend, as $\theta \to \pi$, to a half-plane with edge through ζ_0 and $\bar{\zeta}_0$, which also contains V. Thus $[\zeta_0, \bar{\zeta}_0] \subseteq \partial V(h^{2n})$. Since $f_\theta(z)$ and $g_\theta(z)$ are continuous in θ and z, $\zeta(\theta)$ for $0 \leq \theta \leq \pi$ is a continuous curve C in ∂V. For $\theta = 0$, we have $k(z) = \cos z$, and $\zeta = 1$. So C runs from 1 to ζ_0. The curve $\bar{C}$ is continuous from 1 to $\bar{\zeta}_0$, so with C and $[\zeta_0, \bar{\zeta}_0]$, we have a closed curve which must be all of $\partial V(h^{2n})$, since $V(h^{2n})$ is a convex set.

From above, $\zeta - \tau' = (-1)^n e^{-i\theta}\phi(f_\theta)$, and $\phi(f_\theta)$ is real. For $\theta = \dfrac{\pi}{2}$ we get $\zeta = \tau' + i\eta$, with η real. $V(h^{2n})$ is contained in the circle with centre τ' and through ζ. Hence $\max\{|\mathrm{Im}\, z| : z \in V(h^{2n})\} = |\eta|$, and occurs at ζ. Also, if σ is real and $\neq \tau'$, then since

$\zeta-\sigma\in V(h^{2n}-\sigma)$, we get $\|h^{2n}-\sigma\|\geqslant|\zeta-\sigma|>|\zeta-\tau'|=\|h^{2n}-\tau'\|$. Thus $\min\{\|h^{2n}-\sigma\|:\sigma\in\mathbb{R}\}$ occurs at $\sigma=\tau'$.

We can prove that $\sup\{\operatorname{Re} e^{-i\theta}z:z\in V(h^{2n})\}=(-1)^n f_\theta^{(2n)}(0)$ $(0\leqslant\theta\leqslant\pi)$. For $V(h^2)$, this is $\sin\theta/\theta$. This was found first by J. Duncan, who also pointed out that $\frac{1}{2}V(h^2)=W(T)$, the numerical range of the Volterra operator on $L^2(0,1)$—see Halmos [**8**, p. 109].

The following completes the proof of Theorem 2.

LEMMA 5. *For $0\leqslant\theta<\pi$, g_θ has the following property, for degrees of polynomial $\geqslant 2$.*

A partial sum (polynomial) of the power series at 0 is, on $\mathbb{R}$, either always $\geqslant$ the function, or always $\leqslant$ the function. (2)

Hence, at the zeros of g_θ the polynomial has constant sign.

Proof. The functions $\cos x$ and $\sin x/x$ have property (2) for all degrees (e.g. Hardy [**9**, ExxXLVI, 5]). This gives (2) for $\theta=0$, so assume now that $\theta>0$.

We have $g'(x)=-xk(x)$, where $k(x)=(\sin\rho-\rho\cos\rho)/\rho^3=\sqrt{\pi/2}\,\rho^{-3/2}J_{3/2}(\rho)$, and J_n is the usual Bessel function. From Luke [**10**, p. 299, Eqn. (26)], for $\operatorname{Re}\mu>-1$, $\operatorname{Re}\nu>-1$,

$$\int_0^{\pi/2} J_\mu(\theta\sin t)J_\nu(x\cos t)\sin^{\mu+1}t\cos^{\nu+1}t\,dt=\theta^\mu x^\nu J_{\mu+\nu+1}(\rho)/\rho^{\mu+\nu+1}. \tag{3}$$

If we put $\mu=1$, $\nu=-\frac{1}{2}$, we get $k(x)=\theta^{-1}\int_0^{\pi/2}\cos(x\cos t)J_1(\theta\sin t)\sin^2 t\,dt$. It is enough to prove (2) for k and its polynomials of degree $\geqslant 0$: we multiply by x and integrate to establish (2) for g. As k is an integral of functions $x\to\alpha\cos(\beta x)$ with $\alpha>0$, each of which satisfies (2) in the same direction for any degree, it follows that k satisfies (2). *q.e.d.*

The above also gives $g_\theta^{(2n)}(0)=-(2n-1)k^{(2n-2)}(0)$, and

$$(-1)^k k^{(2k)}(0)=\theta^{-1}\int_0^{\pi/2}\cos^{2k}t\sin^2 t J_1(\theta\sin t)\,dt>0,$$

since $J_1>0$ on $]0,\pi]$. Hence $g_\theta^{(2n)}(0)\neq 0$, for $n\in\mathbb{N}$ and $0<\theta\leqslant\pi$.

REMARKS. To see that $\sup\operatorname{Re} e^{-i\theta}V(h^{2n})=(-1)^n f_\theta^{(2n)}(0)$, note that with the above notation, in the disc centred at τ' which contains $V(h^{2n})$ and has ζ in $V(h^{2n})$ on its boundary, we have that the segment $[\tau',\zeta]$ makes an angle θ with the real axis. Hence the tangent to the circle at ζ is a support line of $V(h^{2n})$.

We can prove that

$$f_\theta(x)=\cos\rho=\cos x-\int_0^{\pi/2}\theta\cos(x\cos t)J_1(\theta\sin t)\,dt.$$

This gives $(-1)^n f_\theta^{(2n)}(0)=1-\theta\int_0^{\pi/2}\cos^{2n}t J_1(\theta\sin t)\,dt$. Since $J_1(\theta\sin t)>0$, $(-1)^n f^{(2n)}(0)$ increases monotonically to 1 as $n\to\infty$. So the $V(h^{2n})$ expand up to the unit disc.

For $V(h^2)$, the line segment in the boundary is $[-i/\pi, i/\pi]$. The point $4/\pi^2+2i/\pi$ gives $\max\{|\operatorname{Im} z|:z\in V(h^2)\}$.

Functions $e(z)=\cos\rho$ for $\theta\geqslant\pi$ are also "extremal functions". For instance, if for

$\pi < \theta < 2\pi$ we integrate $(z^2+\theta^2-\pi^2)f(z)/(z^4e'(z))$ as above, we find the norm in X of $h^4+\xi h^2+\eta$ for certain ξ, η.

The odd power case. For f in X, consider

$$\int_{\Delta_j} \frac{f(z)p(z)}{z^{2n+2}e'(z)}\,dz,$$

with $e(z)=\cos\sqrt{(z+\alpha)^2+\theta^2}=f_\theta(z+\alpha)$ for certain $\alpha>0$, $0\leqslant\theta<\pi$, and p the $2n$-degree polynomial which starts the power series of e'. Take $\Delta_j=\Gamma_j-\alpha$, Γ_j as before, and let $j\to\infty$. This gives

$$\phi(f)=f^{(2n+1)}(0)-\tau f(0)=-(2n+1)!\sum_k \frac{p(\alpha_k)f(\alpha_k)}{\alpha_k^{2n+2}e''(\alpha_k)} \tag{4}$$

where $\{\alpha_k\}$ are the zeros of e', $\tau=e^{(2n+2)}(0)/e'(0)$, and we are defining $\phi\in X'$. Then $|\phi(e)|=\max\{|\phi(f)|:f\in X_1\}$ if $p(\alpha_k)=0$ when $|e(\alpha_k)|\neq 1$, i.e. for $\alpha_k=-\alpha$, and

$$p(\alpha_k) \text{ has the same sign at all } \alpha_k\neq-\alpha. \tag{5}$$

For then $|\phi(e)|$ attains the estimate

$$(2n+1)!\sum \alpha_k^{-2n-2}\,|p(\alpha_k)/e''(\alpha_k)|.$$

Since $-e'(x)=(x+\alpha)g_\theta(x+\alpha)=(x+\alpha)\sum_{k=0}^{\infty}a_kx^k$ (say), we have

$$-p(x)=(x+\alpha)(a_0+\ldots+a_{2n-1}x^{2n-1})+\alpha a_{2n}x^{2n}.$$

We require $p(-\alpha)=0$, i.e. $g_\theta^{(2n)}(\alpha)=a_{2n}=0$. Then $-p(x)=(x+\alpha)(a_0+\ldots+a_{2n-1}x^{2n-1})$. This is to have constant sign at the zeros of $g_\theta(x+\alpha)$. Put $t=x+\alpha$. We require $tr_{2n-1}(t)$ to have constant sign at the zeros of $g_\theta(t)$, where

$$r_{2n-1}(t)=\sum_{k=0}^{2n-1}\frac{g^{(k)}(\alpha)}{k!}(t-\alpha)^k.$$

Let $\beta=\sqrt{\pi^2-\theta^2}$, the first positive zero of g_θ. We prove the following, for $n\geqslant 2$ and certain θ. There exists α, $0<\alpha<\beta$, such that $g_\theta^{(2n)}(\alpha)=0$ and $(r_{2n-1}-g_\theta)(t)$ has one sign for $t<\alpha$, the opposite sign for $t>\alpha$: we say that r_{2n-1} crosses g_θ at α. Therefore $t(r_{2n-1}-g_\theta)(t)$ has the same sign for $t\in\,]-\infty,0[\,\cup\,]\alpha,\infty[$, which set contains the zeros of g_θ. Hence at these zeros, $tr_{2n-1}(t)=t(r_{2n-1}-g_\theta)(t)$ has constant sign. The case $n=1$ is considered later.

In (3) we put $\mu=0$ and $\nu=-\frac{1}{2}$. This gives, after substitution for $\cos t$,

$$g_\theta(x)=\sin\rho/\rho=\int_0^1\cos(xt)J_0(\theta\sqrt{1-t^2})\,dt.$$

Fix $n\in\mathbb{N}$. Let $G_\theta(x)=(-1)^ng_\theta^{(2n)}(x)$. Hence $G_\theta(x)=\int_0^1\cos(xt)w_n(t)\,dt$, where $w_n(t)=t^{2n}J_0(\theta\sqrt{1-t^2})$. Our method is as follows. We find $0<\alpha<\pi$ such that $g_\theta^{(2n+2)}(\alpha)=G_\theta''(\alpha)=0$. Define $T(x)=G(\alpha)+(x-\alpha)G'(\alpha)$. We prove that T crosses G at α. Integrating this $2n$ times, we find that r_{2n+1} crosses g_θ at α.

Since J_0 decreases on $[0, \pi]$, $w_0(t) = J_0(\theta\sqrt{1-t^2})$ increases on $[0, 1]$. Let z_1 be the first zero of J_0, so $z_1 \doteq 2{\cdot}4$. When $\theta > z_1$, let $a = \sqrt{1-(z_1/\theta)^2}$. Then $w_n(t) < 0$ $(0 \leq t < a)$, and

$$\textit{on }]a, 1],\ w_n \textit{ is positive and increasing.} \tag{6}$$

When $\theta \leq z_1$, take $a = 0$, so (6) is still valid.

LEMMA 6. *Let $k: [0, 1] \to \mathbb{R}^+$ be continuous, $k \neq 0$. Then, for $m, j \in \mathbb{Z}$, $0 \leq j < m$,*

$$\int_0^1 k(t)t^m w_0(t)\,dt > a^{m-j}\int_0^1 k(t)t^j w_0(t)\,dt.$$

Proof. We have $k(t)w_0(t)(t^m - a^{m-j}t^j) \geq 0$, with strict inequality for some t. *q.e.d.*

For $\theta = 0$, we shall see in (8), (12) that G'' has a zero α with $0 < \alpha < \beta = \pi$.

THEOREM 7 (Laguerre [1] p. 23). *Let f be an entire function, real on $\mathbb{R}$, with $e^{-|z|}f(z)$ bounded and all the zeros of f real and simple. Then the zeros of f' are real and simple, and interlace the zeros of f.*

Hence this also applies to $f^{(n)}$ in place of f. Note that g_θ satisfies the conditions of Theorem 7, and hence we can apply it to G'. Since $G'(0) = 0$, we get a unique zero $\alpha(\theta)$ of G'' between 0 and the first positive zero of G'. By Hurwitz's theorem, $\alpha(\theta)$ is continuous. Define $A = \sqrt{\alpha^2 + \theta^2}$; $A(\theta)$ is continuous. If $A < \pi$, then $\alpha < \sqrt{\pi^2 - \theta^2} = \beta$. We shall see in (7) that for $\theta > (\sqrt{3}/2)\pi$, we have $\alpha > \beta$ and $A > \pi$. We let θ increase from 0 till the first value θ_0 with $A = \pi$. We shall prove that for each $0 \leq \theta < \theta_0$, the function T crosses G at α, and since also $\alpha < \beta$, these values of θ, α give that $|\phi(e)|$ is the maximum of $|\phi(f)|$ for f in X_1.

Suppose that $(\sqrt{3}/2)\pi < \theta < \pi$. If $0 \leq x \leq \beta$, then $x < \pi/2$. Since

$$g_\theta(x) = \int_0^1 \cos(xt)w_0(t)\,dt > 0,$$

and $\cos(xt) > 0$, here, Lemma 6 gives $G(x) > a^{2n}g_\theta(x) \geq 0$. This inequality for n replaced by $n+1$ is $-G''(x) > 0$. Thus G'' has no zero for $0 \leq x \leq \beta$, and so $\alpha > \beta$. Hence

$$\theta \leq \frac{\sqrt{3}}{2}\pi \quad \textit{if} \quad \alpha < \beta. \tag{7}$$

This argument also shows that for $0 \leq \theta \leq (\sqrt{3}/2)\pi$ and $0 \leq x \leq \pi/2$, since $g_\theta(x) \geq 0$ we have

$$G(x) > 0 > G''(x) \qquad \left(0 \leq x \leq \frac{\pi}{2}\right). \tag{8}$$

Henceforth we assume that $\theta \leq (\sqrt{3}/2)\pi$. Therefore $(\sqrt{3}/2)\theta \leq \frac{3}{4}\pi < z_1$, and this gives $a < \frac{1}{2}$. For $0 < x < \pi$, $g_\theta'(x) = (\rho\cos\rho - \sin\rho)x/\rho^3 < 0$ since $0 < \rho < \sqrt{7}\,\pi/2 <$ the first positive root of $\tan\rho = \rho$. Hence by Lemma 6,

$$-G'(x) = \int_0^1 \sin(xt)t^{2n+1}w_0(t)\,dt > a^{2n}\int_0^1 \sin(xt)tw_0(t)\,dt = -a^{2n}g_\theta'(x) \geq 0.$$

For n replaced by $n+1$ we get $G^{(3)}(x)>0$. Thus we have

$$G'(x)<0<G^{(3)}(x) \quad (0<x<\pi). \tag{9}$$

Let $0\leqslant\theta<\theta_0$, so that $0<\alpha<\beta$ with $G''(\alpha)=0$. Since $G^{(3)}>0$ on $]0,\pi[$, we have

$$G'(\alpha)=\min\{G'(x):0\leqslant x\leqslant\pi\}. \tag{10}$$

Hence

$$T(x)>G(x) \quad (0\leqslant x<\alpha), \qquad T(x)<G(x) \quad (\alpha<x\leqslant\pi). \tag{11}$$

Now put $y=\pi/(1+a)$. For $0<t<\frac{1}{2}(1+a)$, we have $\cos(yt)>0$. The substitution $s=1+a-t$ gives

$$\int_{(1+a)/2}^{1}\cos(yt)w_n(t)\,dt=-\int_{a}^{(1+a)/2}\cos(ys)w_n(1+a-s)\,ds.$$

Hence

$$\int_a^1\cos(yt)w_n(t)\,dt=\int_a^{(1+a)/2}\cos(yt)(w_n(t)-w_n(1+a-t))\,dt<0$$

by (6), since $a\leqslant t\leqslant 1+a-t$ in the last integral. $\int_0^a\cos(yt)w_n(t)\,dt\leqslant 0$ since here $w_n(t)\leqslant 0$. We add these inequalities to get $G(y)<0$. By (9), we deduce that

$$G(x)<0<G''(x) \quad (\pi/(1+a)\leqslant x\leqslant\pi), \tag{12}$$

since n replaced by $(n+1)$ gives the G'' inequality. Now by (8) and (12), $\alpha<y$. Since $G(y)<0<G(0)$, we have by (10), $-G'(\alpha)>y^{-1}G(0)$. By (11), $T(y)<G(y)<0$. Since T has slope $G'(\alpha)$, we deduce that

$$T(x)<-(1+2a)G(0) \quad (x\geqslant 2\pi). \tag{13}$$

We claim the following.

For each $\pi<x<2\pi$, *there exists* $0<w\leqslant\pi$ *such that* $G'(w)<G'(x)$. (14)

To prove this let $c=\pi/x$ and $b=\pi/(\pi+x)$ so $a<\frac{1}{2}<c<1$. We have $-G'(x)=\int_0^1\sin(xt)v(t)\,dt$, where $v(t)=tw_n(t)$. Consider first the case $a\leqslant b$. Since v increases on $[a,1]$, $\int_a^c\sin(xt)v(t)\,dt=c\int_{a/c}^1\sin(\pi s)v(cs)\,ds<\int_a^1\sin(\pi s)v(s)\,ds$. For $0<t<a$, $\frac{1}{2}(x+\pi)t<\pi a/(2b)\leqslant\pi/2$, and so $\sin(xt)-\sin(\pi t)=2\sin\dfrac{x-\pi}{2}t\cos\dfrac{x+\pi}{2}t>0$. Since v is negative on $[0,a[$, we get $\int_0^a(\sin(xt)-\sin(\pi t))v(t)\,dt\leqslant 0$. For $c<t<1$, we have $v(t)>0$ and $\sin(xt)<0$. Hence $-G'(x)<\int_0^c\sin(xt)v(t)\,dt<\int_0^1\sin(\pi s)v(s)\,ds$ (add the above inequalities) $=-G'(\pi)$. Thus we can take $w=\pi$.

Now suppose that $a>b$. Let $w=\pi a^{-1}-x$. Then $w>0$ since $ax<\pi$, and $\pi-w>\pi+x-\pi b^{-1}=0$. Since $(x+w)a=\pi$ and $x+w<3\pi$, we deduce that $\sin(xt)-\sin(wt)>0$ if $0<t<a$, and <0 if $a<t<1$. Hence $(\sin(xt)-\sin(wt))v(t)\leqslant 0$ for $0<t<1$, and $-G'(x)+G'(w)<0$. Thus (14) is established.

Now by (10) and (14), $G'(x)>G'(\alpha)$ for $\pi<x<2\pi$. Since $T(\pi)<G(\pi)$ by (11), we have $T(x)<G(x)$ $(\pi\leqslant x\leqslant 2\pi)$.

Now consider the case $n=1$, i.e.

$$G(x) = -g_\theta''(x) = \int_0^1 \cos(xt)w_1(t)\,dt; \qquad w_1(t) = t^2 w_0(t).$$

Suppose that $a>0$, i.e. $\theta > z_1$. Since $|J_0| \leq 1$, $|w_0| \leq 1$. Let $A = \int_0^a |w_1|$ and $B = \int_a^1 w_1$. Then $A < \int_0^a t^2\,dt = a^3/3 < a/12$ and

$$B - A = \int_0^1 w_1 = G(0) = -\cos\theta/\theta^2 + \sin\theta/\theta^3 > -4\cos z_1/(3\pi^2) > 1/12.$$

Hence $B/A > 1 + a^{-1}$, and $(B+A)/(B-A) < 1 + 2a$. For all real x,

$$|G(x)| \leq \int_0^1 |w_1| = B + A = G(0)(B+A)/(B-A) < (1+2a)G(0).$$

Hence by (13), $T(x) < G(x)$ $(x > 2\pi)$.

Since G is an even function, (9) shows that G increases on $[-\pi, 0]$. Since $T(0) > G(0)$, we have $T(x) > G(x)$ $(-\pi \leq x \leq 0)$. Since $-\pi G'(\alpha) > (1+a)G(0)$, for $x < -\pi$ we have $T(x) > (2+a)G(0) > (1+2a)G(0) > G(x)$. This completes the proof that T crosses G if $a>0$ and $n=1$. For $n>1$ and $a>0$, the corresponding ratio B/A is larger, $(B+A)/(B-A)$ smaller, and the above still shows that T crosses G at α.

Consider the case $a=0$, i.e. $\theta \leq z_1$. By the above, T crosses G on $[-\pi, 2\pi]$, and $|T(x)| > (1+2a)G(0) = G(0)$ for $x \in \mathbb{R}\backslash[-\pi, 2\pi]$. Since $w_n \geq 0$ now, for real x, $|G(x)| \leq \int_0^1 w_n = G(0)$. Thus $|T(x)| > |G(x)|$ $(x \in \mathbb{R}\backslash[-\pi, 2\pi])$, and T crosses G (on $\mathbb{R}$).

Having now established the required property of $r_3, r_5, \ldots$ to make (5) hold, we return to the case of r_1, i.e. $n=1$ in (4). Note that $r_1 = T$ is linear. By calculation, $g_\theta''(\beta) = \pi^{-4}(2\pi^2 - 3\theta^2)$, and $g_\theta''(0) < 0$. Hence for $0 \leq \theta < \sqrt{2/3}\pi = \theta_0$, g_θ'' has a zero at α, $0 < \alpha < \beta$. Since $g_\theta'(x) < 0$ $(0 < x < \beta)$ and $g_\theta'(0) = 0$, α is unique by Laguerre's theorem applied to g_θ'. Since $g_\theta''(x) < 0$ $(0 < x < \alpha)$, we have $T(0) > g_\theta(0) > 0$, and similarly $T(\beta) < g_\theta(\beta) = 0$. Since T has negative slope, $T > 0$ at all negative zeros of g_θ, and $T < 0$ at all positive zeros of g_θ. Therefore $tT(t) = tr_1(t) < 0$ at all these zeros. This proves (5).

Let $n \in \mathbb{N}$, $0 \leq \theta < \theta_0$ and $Q^2 = (z+\alpha)^2 + \theta^2$. We know that

$$e(z) = f_\theta(z+\alpha) = \cos Q$$

satisfies $|\phi(e)| = \sup\{|\phi(f)| : f \in X_1\}$. Hence by Lemma 4, $|\phi(e)| = \|h^{2n+1} - i^{2n+1}\tau\|$. Define $k(z) = e^{-iA}(\cos Q + i(A + \alpha z/A)\sin Q/Q)$. As in the even case, $k \in X$. For real x, $Q^2 = x^2 + 2\alpha x + A^2 > (A + \alpha x/A)^2$, which gives $|k| \leq 1$ on $\mathbb{R}$ and so $k \in X_1$. Since $k(0) = 1$, $\zeta = i^{2n+1}k^{(2n+1)}(0) \in V(h^{2n+1})$. Since $\sin Q = 0$ at each $\alpha_k \neq -\alpha$, (4) gives $\phi((A + \alpha z/A)\sin Q/Q) = 0$. Hence $|\zeta - \tau'| = |\phi(k)| = |\phi(e)| = \|h^{2n+1} - \tau'\|$, where we put $\tau' = i^{2n+1}\tau$. Thus $\zeta \in \partial V(h^{2n+1})$, and $V(h^{2n+1}) \subseteq$ the circle with centre τ' and through ζ.

We prove that $|\tau| \to \infty$ as $\theta \to \theta_0$ and so $A \to \pi$. We have $e'(0) = -\alpha \sin A/A \to 0$ and $-e^{(2n+2)}(0) = D_x^{2n+1}[(x+\alpha)\sin Q/Q](0) = \alpha D_x^{2n+1}\sin Q/Q(0) = \alpha g_\theta^{(2n+1)}(\alpha) = \alpha(-1)^n G'(\alpha)$ $\neq 0$, by (9), and since $D_x^{2n}\sin Q/Q(0) = 0$. This remains non-zero at θ_0, and so

$|\tau| \to \infty$. As $A \to \pi$, $\zeta \to \zeta_0$ where

$$\pm\mathrm{Re}(\zeta_0) = D_x^{2n+1}[(\pi + \alpha x/\pi)\sin Q/Q](0) = \pi D_x^{2n+1} \sin Q/Q(0) \neq 0.$$

Since $\zeta \in V(h^{2n+1})$, so does ζ_0. The function $\overline{k(-\bar{z})}$ gives $-\bar{\zeta}$ and $-\bar{\zeta}_0$ in V. The above circle centred at τ' has $|\tau'| \to \infty$, and since $\tau' \in i\mathbb{R}$, we get $[\zeta_0, -\bar{\zeta}_0] \subseteq \partial V$, as before. Also, using the functions $\overline{k(\bar{z})}$, we get $[\bar{\zeta}_0, \zeta_0] \subseteq \partial V$. Note that $\zeta_0 \neq -\bar{\zeta}_0$.

When $\theta = 0$, $k(z) = e^{iz}$ and $\zeta = -1$. As θ varies from 0 to θ_0, τ traces a continuous curve from -1 to ζ_0 in ∂V, since A and α are continuous in θ. The reflections of this arc in the axes and the origin are also in ∂V. With the two line segments they give a closed curve, which must be all of ∂V.

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