Nagoya Math. J., **249** (2023), 11–49
DOI 10.1017/nmj.2022.20

ON μ_n-ACTIONS ON K3 SURFACES IN POSITIVE CHARACTERISTIC

YUYA MATSUMOTO

Abstract. In characteristic 0, symplectic automorphisms of K3 surfaces (i.e., automorphisms preserving the global 2-form) and non-symplectic ones behave differently. In this paper, we consider the actions of the group schemes μ_n on K3 surfaces (possibly with rational double point [RDP] singularities) in characteristic p, where n may be divisible by p. We introduce the notion of symplecticness of such actions, and we show that symplectic μ_n-actions have similar properties, such as possible orders, fixed loci, and quotients, to symplectic automorphisms of order n in characteristic 0. We also study local μ_n-actions on RDPs.

Contents

Received July 10, 2021. Revised February 20, 2022. Accepted July 30, 2022.
2020 Mathematics subject classification: Primary 14J28; Secondary 14L15, 14L30.
Keywords: K3 surfaces, finite group schemes, μ_n, symplectic actions
This work was supported by JSPS KAKENHI Grant Numbers 15H05738 and 16K17560.

§1. Introduction

K3 surfaces are proper smooth surfaces X with $\Omega_X^2 \cong \mathcal{O}_X$ and $H^1(X, \mathcal{O}_X) = 0$. The first condition implies that X admits an everywhere nonvanishing 2-form, and such a 2-form is unique up to scalar. An automorphism of a K3 surface is called *symplectic* if it preserves the global 2-form. It is known that symplectic and non-symplectic automorphisms behave very differently.

For example, Nikulin [Ni, §§4 and 5] proved that quotients of K3 surfaces in characteristic 0 by a symplectic action of a finite group G has only rational double points (RDPs) as singularities and that the minimal resolutions of the quotients are again K3 surfaces. Moreover, he determined the number of fixed points (which are always isolated) if G is cyclic. To the contrary, the quotients by non-symplectic actions of finite groups are never birational to K3 surfaces; instead, they are birational to either Enriques surfaces or rational surfaces.

These results hold in characteristic $p > 0$ provided p does not divide the order of G (see Theorem 5.1), but are no longer true for order p automorphisms in characteristic p. In this case, the notion of symplecticness is useless, since any such automorphism is automatically symplectic (since there are no nontrivial pth root of unity in characteristic p) and, for small p, there exist examples of automorphisms with non-K3 quotients (see [DK1], [DK2]).

In this paper, we consider actions of the finite group schemes μ_n (n may be divisible by p) on RDP K3 surfaces, by which we mean surfaces with RDP singularities whose minimal resolutions are K3 surfaces. It is essential to allow RDPs since smooth K3 surfaces never admit actions of μ_p (see Remark 2.2). We introduce the notion of symplecticness and fixed points of such actions (Definitions 2.5 and 2.7). Then we prove the following properties, which are parallel to the properties of automorphisms of order not divisible by the characteristic.

THEOREM 1.1 (Theorems 6.1 and 6.2). *Let X be an RDP K3 surface in characteristic p, equipped with a μ_n-action. If the action is symplectic, then the quotient X/μ_n is an RDP K3 surface. If $n = p$ and the action is non-symplectic, then the quotient X/μ_p is an RDP Enriques surface if the action is fixed-point-free (which is possible only if $p = 2$), and otherwise it is a rational surface.*

THEOREM 1.2 (Theorems 7.1 and 8.2).

- *There exists an RDP K3 surface X in characteristic p equipped with a μ_p-action if and only if $p \leq 19$.*
- *If X is an RDP K3 surface X in characteristic p equipped with a μ_n-action, then $\phi(n) \leq 20$, in particular $n \leq 66$. Moreover, for each p, we determine the set of n for which such an action exists.*
- *For each p, there exists an RDP K3 surface X in characteristic p equipped with a symplectic μ_n-action if and only if $n \leq 8$, and we determine the number of fixed points.*

To prove the main results, we first study (in §§3 and 4) μ_n-actions on local rings of surfaces at smooth points and RDPs. We define the notion of symplecticness of such actions (Definitions 3.1 and 4.1) and prove the following result.

THEOREM 1.3 (Theorem 4.6 and Propositions 4.7 and 4.9). *Let X be the localization at a closed point w of an RDP surface in characteristic p equipped with a μ_p-action. Let $\pi : X \to X/\mu_p$ be the quotient morphism.*

- *If w is not fixed by the action, then $\pi(w)$ is either a smooth point or an RDP.*
- *If w is fixed and the action is symplectic at w, then w is an isolated fixed point and $\pi(w)$ is an RDP.*
- *If w is an isolated fixed point and the action is non-symplectic at w, then $\pi(w)$ is a non-RDP singularity.*

We classify the possible actions in the non-fixed case (Table 3) and the symplectic case (Table 4).

Moreover, we also give a partial classification of local μ_{p^e}- and μ_n-actions (Propositions 4.12 and 4.13) and a complete classification of local symplectic μ_n-actions (Proposition 4.14). We hope that these local results would have applications other than K3 surfaces.

The results on μ_n-quotients, orders of symplectic μ_n-actions, and orders of μ_n-actions on K3 surfaces are discussed in §§6–8, respectively.

In §9, we give several examples of μ_n-actions on K3 surfaces.

Throughout the paper, we work over an algebraically closed field k of $\operatorname{char} k = p \geq 0$. Varieties are separated integral k-schemes of finite type (not necessarily proper or smooth), and surfaces are two-dimensional varieties. We denote the smooth locus of a variety X by X^{sm}.

§2. Preliminaries

2.1 K3 surfaces and rational double points

RDP singularities of surfaces are precisely the canonical surface singularities that are not smooth. They are classified into types A_n $(n \geq 1)$, D_n $(n \geq 4)$, and E_n $(n = 6, 7, 8)$ by their dual graph of the exceptional curves of the minimal resolution, which are Dynkin diagrams of type A_n, D_n, or E_n. The number n is equal to the number of the exceptional curves, and we say that the RDP is of *index n*. The dual graph determines the formal isomorphism class of an RDP except in certain cases in characteristics $2, 3, 5$. For the exceptional cases, we use Artin's notation D_n^r and E_n^r (see [A2]).

We recall the classification, given by Bombieri and Mumford [BM2], of proper smooth surfaces X with numerically trivial canonical divisor K_X: they consist of four classes, with the characterizing properties as reported in Table 1. Here, $b_i = \dim H^i_{\text{ét}}(X, \mathbb{Q}_l)$ is the ith l-adic Betti number for an auxiliary prime $l \neq \operatorname{char} k$. Enriques and (quasi-)hyperelliptic surfaces in characteristics 2 and 3 may have unusual values of $\dim H^1(\mathcal{O}_X)$ and $\operatorname{ord}(K_X)$.

The distinction between hyperelliptic and quasi-hyperelliptic surfaces is not important in this paper. Furthermore, the choice of the origin of an abelian surface is not important.

DEFINITION 2.1. *RDP surfaces* are surfaces that have only RDPs as singularities (if any). Hence, any smooth surface is an RDP surface by definition.

RDP K3 surfaces are proper RDP surfaces whose minimal resolutions are (smooth) K3 surfaces. We similarly define RDP abelian, RDP Enriques, and RDP (quasi-)hyperelliptic surfaces.

However, since abelian surfaces and (quasi-)hyperelliptic surfaces do not admit smooth rational curves, any RDP abelian or RDP (quasi-)hyperelliptic surface is smooth.

Table 1. Surfaces with numerically trivial canonical divisors.

	$\dim H^1(\mathcal{O}_X)$	b_1	b_2	$\mathrm{ord}(K_X)$	char
Abelian	2	4	6	1	Any
K3	0	0	22	1	Any
Enriques	0	0	10	2	Any
Enriques	1	0	10	1	2
(Quasi-)hyperelliptic	1	2	2	2, 3, 4, 6	Any
(Quasi-)hyperelliptic	2	2	2	1	2, 3

Remark 2.2. Smooth K3 surfaces in characteristic $p > 0$ admit no nontrivial global vector fields ([RS, Th. 7], [Ny1, Cor. 3.5]), and hence admit no nontrivial actions of μ_p (or α_p). However, RDP K3 surfaces may admit such actions.

Proposition 2.3. *For any RDP surface X, the pullback by the morphism $X^{\mathrm{sm}} \cong \tilde{X} \setminus E \hookrightarrow \tilde{X}$ to the minimal resolution $\tilde{X}$ of X induces an isomorphism $H^0(X^{\mathrm{sm}}, (\Omega^2_X)^{\otimes n}) \cong H^0(\tilde{X}, (\Omega^2_{\tilde{X}})^{\otimes n})$, where E is the exceptional divisor. Nonvanishing forms on one side correspond to nonvanishing ones on the other side.*

Proof. This follows from the following local version applied repeatedly. ∎

Proposition 2.4. *Let $(A, \mathfrak{m})$ be the localization at a closed point of an RDP surface. Then $H^0(\operatorname{Spec} A \setminus \{\mathfrak{m}\}, \Omega^2_{A/k})$ is a free A-module of rank 1. If A is smooth, then this space is isomorphic to $H^0(\operatorname{Spec} A, \Omega^2_{A/k})$. If A is an RDP and $(A', \mathfrak{m}')$ is the localization at a closed point of $\mathrm{Bl}_{\mathfrak{m}} A$, then any generator of the above space extends to a generator of $H^0(\operatorname{Spec} A' \setminus \{\mathfrak{m}'\}, \Omega^2_{A'/k})$.*

Proof. If A is smooth, then $\Omega^2_{A/k}$ is free of rank 1 and the assertion is clear. Suppose A is an RDP. Then it is a hypersurface isolated singularity, and it is well known that for such singularities, the canonical divisor is trivial, and then the former assertion follows. Since an RDP is a canonical singularity, the pullback of the canonical divisor to $\mathrm{Bl}_{\mathfrak{m}} A$ is also trivial, and hence the latter assertion follows. ∎

2.2 Group schemes of multiplicative type

Recall that we are working over an algebraically closed field k.

We consider finite commutative group schemes G of multiplicative type over k. This means that G is of the form $\prod_j \mu_{n_j}$ for some positive integers n_j. The Cartier dual $G^\vee = \mathcal{H}om(G, \mathbb{G}_m)$ of G is a finite étale group scheme and can be identified with the finite group $G^\vee(k)$ of k-valued points. Using this finite commutative group $G^\vee$, we have the following explicit description: $G = \operatorname{Spec} k[t_i]_{i \in G^\vee} / (t_i t_j - t_{i+j}, t_0 - 1)$, with the group operations $m\colon G \times G \to G$, $e\colon \operatorname{Spec} k \to G$, $i\colon G \to G$ given by $m^*(t_i) = t_i \otimes t_i$, $e^*(t_i) = 1$, $i^*(t_i) = t_{-i}$.

An action $\alpha\colon G \times \operatorname{Spec} B \to \operatorname{Spec} B$ corresponds, via $\alpha^*(b) = \sum_{i \in G^\vee} t_i \otimes \mathrm{pr}_i(b)$, to decompositions $B = \bigoplus_{i \in G^\vee} B_i$ to k-vector subspaces satisfying $B_i B_j \subset B_{i+j}$. We say an element b or a subset of B_i to be *homogeneous of weight* i and we write $\mathrm{wt}(b) = i$.

Such a decomposition $B = \bigoplus_i B_i$ naturally extends to a decomposition $\Omega^*_{B/k} = \bigoplus_i (\Omega^*_{B/k})_i$ satisfying $d(B_i) \subset (\Omega^1_{B/k})_i$ and $(\Omega^*_{B/k})_i (\Omega^*_{B/k})_j \subset (\Omega^*_{B/k})_{i+j}$.

If G acts on a scheme X that is not necessarily affine but admits a covering by G-stable affine open subschemes (which is the case if, e.g., X is quasi-projective or G is local),

https://doi.org/10.1017/nmj.2022.20 Published online by Cambridge University Press

then the G-action admits a quotient $\pi : X \to X/G$, and induces decompositions $\pi_* \mathcal{O}_X = \bigoplus_i (\pi_* \mathcal{O}_X)_i$, $\pi_* \Omega^*_{X/k} = \bigoplus_i (\pi_* \Omega^*_{X/k})_i$, and $H^0(X, (\Omega^*_{X/k})^{\otimes n}) = \bigoplus_i (H^0(X, (\Omega^*_{X/k})^{\otimes n}))_i$, compatible with multiplications.

If $\operatorname{char} k$ does not divide the order of $G^\vee$, then B_i are the eigenspaces for the action of $G(k)$ with eigenvalues $i \in G^\vee(k) = \operatorname{Hom}(G(k), k^*)$.

If $\operatorname{char} k = p > 0$ and $G^\vee$ is cyclic of order p (hence $G \cong \mu_p = \operatorname{Spec} k[t_1]/(t_1^p - 1)$ for a choice of a generator 1 of $G^\vee$), then giving such a decomposition is also equivalent to giving a k-derivation D on B of multiplicative type (i.e., $D^p = D$) under the correspondence $B_i = B^{D=i} = \{b \in B \mid D(b) = ib\}$ (this correspondence depends on the choice of a generator 1 of $G^\vee$). Moreover, D extends to a k-linear endomorphism on $\Omega^*_{B/k}$ satisfying $D(df) = d(D(f))$, $D^p = D$, and the Leibniz rule $D(\omega \wedge \eta) = \omega \wedge D(\eta) + D(\omega) \wedge \eta$.

Now, we generalize the notion of symplecticness of automorphisms to actions of group schemes like μ_n.

Definition 2.5. Let G be a finite group scheme of multiplicative type. Let X be either an abelian surface or an RDP K3 surface, equipped with an action of G. We say that the action is *symplectic* if the one-dimensional space $H^0(X^{\mathrm{sm}}, \Omega^2_{X/k})$ with respect to the action of G is of weight 0.

Remark 2.6. Under the assumptions of Definition 2.5, suppose G is reduced. Equivalently, this means that G is a constant group scheme corresponding to a finite abelian group of order prime to p. Then, by Proposition 2.3, our symplecticness is equivalent to the symplecticness of the induced G-action on the minimal resolution $\tilde{X}$ in the usual sense (i.e., preserving the global 2-form). This suggests that our definition of the symplecticness of μ_n-actions is a natural generalization of that of $\mathbb{Z}/m\mathbb{Z}$-actions (order m automorphisms) for m not divisible by $\operatorname{char} k$.

On the other hand, if $G = \mathbb{Z}/p\mathbb{Z}$ (which does not belong to the class considered in Definition 2.5), then any action of G preserves the global 2-form, since there are no nontrivial pth roots of unity. Thus, the usual definition of symplecticness is useless in this case. We do not know whether there is a useful notion of symplecticness in a larger class of group schemes containing $\mathbb{Z}/p\mathbb{Z}$ or α_p.

2.3 Derivations of multiplicative type

In this section, assume that $\operatorname{char} k = p > 0$.

Recall that, given an action of a group scheme G on a scheme X, the fixed point scheme $X^G \subset X$ is characterized by the property $X^G(T) = \operatorname{Hom}_G(T, X)$ for any k-scheme T equipped with the trivial G-action. If $G = \mu_p$ and D is the corresponding derivation, we write $\operatorname{Fix}(D) = X^G$ and also call it the fixed locus of D.

Definition 2.7. We say that a closed point $w \in X$ is *fixed* by the μ_n-action, or by the corresponding derivation if $n = p$, if $w \in X^{\mu_n}$.

Proposition 2.8. *Let k be an algebraically closed field. Let $X = \operatorname{Spec} B$ be a Noetherian affine k-scheme equipped with a μ_{p^e}-action. For each closed point $w \in X$, the assertions (1)–(4) are equivalent. If $e = 1$ and D is the corresponding derivation, then the assertions (1)–(6) are equivalent, and if moreover X is a smooth variety, then (7) is also equivalent.*

1. *w is a μ_{p^e}-fixed point.*
2. *The maximal ideal $\mathfrak{m}_w$ of $\mathcal{O}_{X,w}$ is generated by homogeneous elements.*

3. *The canonical morphism $B \to B/\mathfrak{m}_w$ is μ_{p^e}-equivariant, where $B/\mathfrak{m}_w$ is equipped with the trivial action (i.e., the decomposition concentrated on $(-)_0$).*
4. $B_i \subset \mathfrak{m}_w$ *for each* $i \neq 0$.
5. $D(\mathfrak{m}_w) \subset \mathfrak{m}_w$.
6. $D(\mathcal{O}_{X,w}) \subset \mathfrak{m}_w$.
7. *D has singularity at w in the sense of [RS, §1].*

Finally, if (1) holds, then the μ_{p^e}-action extends to the blowup $\operatorname{Bl}_w X$.

Proof. Let $B = \bigoplus_{i \in \mathbb{Z}/p^e\mathbb{Z}} B_i$ be the corresponding decomposition.

($1 \iff 3$) By the definition of X^{μ_p}, a closed point $w \in X$ is a (k-valued) point of X^{μ_p} if and only if $B \to B/\mathfrak{m}_w$ is compatible with the projections pr_i to the ith summand $(-)_i$ for all i, where $B/\mathfrak{m}_w$ is equipped with the trivial decomposition.

($2 \iff 3$) If (3) holds, then we have $\mathrm{pr}_i(\mathfrak{m}_w) \subset \mathfrak{m}_w$ for all i, and then each element x of $\mathfrak{m}_w$ is the sum of homogeneous elements $\mathrm{pr}_i(x) \in \mathfrak{m}_w$. Conversely, if $\mathfrak{m}_w$ is generated by homogeneous elements, then $\mathrm{pr}_i(\mathfrak{m}_w) \subset \mathfrak{m}_w$ for all i, which implies (3).

($3 \iff 4$) Easy.

Assume $e = 1$.

($2 \iff 5$) Assume $D(\mathfrak{m}_w) \subset \mathfrak{m}_w$. Take a system of generators (x_j) of $\mathfrak{m}_w$. For each j, let $x_j = \sum_{i \in \mathbb{F}_p} x_{j,i}$ be the decomposition of x_j in $B = \bigoplus_i B_i$. Then $D^l(x_j) = \sum_i i^l x_{j,i}$ is also in $\mathfrak{m}_w$. Since the matrix $(i^l)_{i,l=0}^{p-1}$ is invertible, this implies that $x_{j,i} \in \mathfrak{m}_w$. Thus $\mathfrak{m}_w$ is generated by eigenvectors. The converse is clear.

($5 \iff 6$) This is clear since $\mathcal{O}_{X,w} = \mathfrak{m}_w + k$ and $D|_k = 0$.

($5 \iff 7$) Take coordinates $x_1, \dots, x_n$ at a point w and write $D = \sum_j f_j \cdot (\partial/\partial x_j)$. Then both of the conditions are equivalent to $(f_j) \subset \mathfrak{m}_w$.

We show the final assertion assuming (2). If the maximal ideal $\mathfrak{m}$ is generated by homogeneous elements $x_j \in B_{i_j}$, then, for each j, we can extend the action on the affine piece $\operatorname{Spec} B[x_h/x_j]_h$ of $\operatorname{Bl}_w X$ by declaring x_h/x_j to be homogeneous of weight $i_h - i_j$. ∎

The next lemma enables us to take useful coordinates at a point not fixed by D.

Lemma 2.9. *If B is a Noetherian local ring, D is a derivation of multiplicative type, and the closed point is not fixed by D, then the maximal ideal $\mathfrak{m}$ of B is generated by elements $x_1, \dots, x_{m-1}, y$ with $\mathrm{wt}(x_j) = 0$ and $\mathrm{wt}(1+y) = 1$. If $\mathfrak{m}$ is generated by n elements, then we can take $m = n$. If $\dim B \geq 2$, then D does not extend to a derivation of the blowup $\operatorname{Bl}_{\mathfrak{m}} B$.*

Proof. Recall that a subset of $\mathfrak{m}$ generates $\mathfrak{m}$ if and only if it generates $\mathfrak{m}/\mathfrak{m}^2$.

Take elements $x'_1, \dots, x'_m$ generating $\mathfrak{m}$, and let $x'_j = \sum_{i \in \mathbb{F}_p} x'_{j,i}$ be the decompositions to eigenvectors. By assumption, there exists a pair (j,i) with $x'_{j,i} \notin \mathfrak{m}$. We take such j_0, i_0, and we may assume $i_0 \neq 0$. We may assume $x'_{j_0,i_0} - 1 \in \mathfrak{m}$. Then $y = x'_{j_0,i_0} - 1$ satisfies $y \in \mathfrak{m}$ and $D(y) = i_0(y+1)$. We have $y \notin \mathfrak{m}^2$, since $D(\mathfrak{m}^2) \subset \mathfrak{m}$. By replacing y with $(y+1)^q - 1$ for an integer q with $qi_0 \equiv 1 (\mathrm{mod}\ p)$, we may assume $i_0 = 1$. For each j, let $x_j = \sum_i (y+1)^{-i} x'_{j,i}$. Then we have $D(x_j) = 0$ and, since $x_j \equiv x'_j (\mathrm{mod}\ (y))$, the elements x_j, y generate $\mathfrak{m}/\mathfrak{m}^2$ and hence generate $\mathfrak{m}$. We can omit one of the x_j's and then the remaining elements satisfies the required conditions (after renumbering).

To show the latter assertion, it suffices to show that D does not extend to $B' := B[x_j/y]_j$. If it extends, then we have $D(x_j/y) = -x_j(y+1)/y^2 \in B'$, hence $x_j/y^2 \in B'$, and then on $\operatorname{Spec} B'$ we have that $y = 0$ implies $x_j/y = 0$, which is impossible since $\dim B' \geq 2$. ∎

https://doi.org/10.1017/nmj.2022.20 Published online by Cambridge University Press

Before stating the next proposition, we recall the following notion from [RS]. Assume X is a smooth irreducible variety and D is a nontrivial derivation. Then $\mathrm{Fix}(D)$ consists of its divisorial part (D) and non-divisorial part $\langle D\rangle$. If we write $D = f\sum_i g_i \frac{\partial}{\partial x_i}$ for some local coordinates $x_1,\dots,x_m$ with g_i having no common factor, then (D) and $\langle D\rangle$ correspond to the ideals (f) and (g_i), respectively. If D is of multiplicative type with $\langle D\rangle = \emptyset$, then it follows from Proposition 2.8 that for suitable coordinates near any fixed point, we have $D = ax_m\cdot(\partial/\partial x_m)$ and that $\mathrm{Fix}(D)$ is a smooth divisor (possibly empty).

Assuming that $\mathrm{Fix}(D)$ is divisorial, in which case the quotient is a smooth variety by [S, Prop. 6], the highest differential forms on smooth loci of X and X^D are related in the following way.

Proposition 2.10. *Let X be a smooth variety of dimension m (not necessarily proper) equipped with a nontrivial derivation D of multiplicative type such that* $\mathrm{Fix}(D)$ *is divisorial. Let Δ be the divisor* $\mathrm{Fix}(D)$*. Then there is a unique collection of isomorphisms*

$$(\pi_*(\Omega^m_{X/k}(\Delta))^{\otimes n})_0 \cong (\Omega^m_{X^D/k}(\pi_*(\Delta)))^{\otimes n}$$

for all integers n, compatible with multiplication, preserving the zero loci, and sending (for $n=1$)

$$f_0\cdot df_1\wedge\cdots\wedge df_{m-1}\wedge d\log(f_m)\mapsto f_0\cdot df_1\wedge\cdots\wedge df_{m-1}\wedge d\log(f_m^p)$$

if $f_0,\dots,f_{m-1}$ are homogeneous of weight 0 and f_m is homogeneous of some weight (not necessarily 0).

In particular, if the action is fixed-point-free, then we have isomorphisms

$$(\pi_*(\Omega^m_{X/k})^{\otimes n})_0 \cong (\Omega^m_{X^D/k})^{\otimes n} \quad \textit{and}$$
$$H^0(X,(\Omega^m_{X/k})^{\otimes n})_0 \cong H^0(X^D,(\Omega^m_{X^D/k})^{\otimes n})$$

with the same properties.

Proof. The isomorphism for $n=0$ is clear. It suffices to construct the isomorphism for $n=1$ that is compatible with multiplication with $n=0$ forms and with restriction to open subschemes.

Take a closed point $w\in X$. Let $\varepsilon=1$ (resp. $\varepsilon=0$) if $w\notin\Delta$ (resp. $w\in\Delta$). By Lemma 2.9 (resp. by [RS, Th. 1]), there are coordinates $x_1,\dots,x_m$ on a neighborhood of w with $D(x_j)=0$ for $j<m$ and $D(x_m)=a(\varepsilon+x_m)$ for some $a\in\mathbb{F}_p^*$. We define

$$\phi\colon (\pi_*(\Omega^m_{X/k}(\Delta)))_0 \to \Omega^m_{X^D/k}(\pi_*(\Delta))$$
$$f\cdot dx_1\wedge\cdots\wedge dx_{m-1}\wedge d\log(\varepsilon+x_m)\mapsto f\cdot dx_1\wedge\cdots\wedge dx_{m-1}\wedge d\log(\varepsilon+x_m^p)$$

for f of weight 0 (note that $dx_1\wedge\cdots\wedge dx_{m-1}\wedge d\log(\varepsilon+x_m)$ is a local generator of the left-hand side). We show that then ϕ sends

$$f_0\cdot df_1\wedge\cdots\wedge df_{m-1}\wedge d\log(f_m)\mapsto f_0\cdot df_1\wedge\cdots\wedge df_{m-1}\wedge d\log(f_m^p)$$

for any $f_0,\dots,f_{m-1}$ and f_m as in the statement. This implies that ϕ does not depend on the choice of the coordinates and hence that ϕ induces a well-defined morphism of sheaves. Then since $dx_1\wedge\cdots\wedge dx_{m-1}\wedge d\log(\varepsilon+x_m)$ (resp. $dx_1\wedge\cdots\wedge dx_{m-1}\wedge d\log(\varepsilon+x_m^p)$) is a local generator of $(\Omega^m_{X/k}(\Delta))_0$ (resp. $\Omega^m_{X^D/k}(\pi_*(\Delta))$), it follows that ϕ is an isomorphism and $\phi^{\otimes n}$ are well-defined isomorphisms.

https://doi.org/10.1017/nmj.2022.20 Published online by Cambridge University Press

We may pass to the completion, so consider $f_h \in k[[x_1, \dots, x_m]]$. By the assumption on the weight, we have $f_h \in k[[x_1, \dots, x_{m-1}, x_m^p]]$ for $h < m$ and $f_m \in (\varepsilon + x_m)^b k[[x_1, \dots, x_{m-1}, x_m^p]]$ for some $0 \le b < p$. Then we have $\partial f_h / \partial x_m = 0$ for $h < m$ and $\partial f_m / \partial x_m = b f_m / (\varepsilon + x_m)$. Hence, we have

$$\begin{aligned}
&f_0 \cdot df_1 \wedge \cdots \wedge df_{m-1} \wedge d\log(f_m) \\
&= f_0((\varepsilon + x_m)/f_m) \det(\partial f_h / \partial x_j)_{1 \le h,j \le m} \cdot dx_1 \wedge \cdots \wedge dx_{m-1} \wedge d\log(\varepsilon + x_m) \\
&= b f_0 \det(\partial f_h / \partial x_j)_{1 \le h,j \le m-1} \cdot dx_1 \wedge \cdots \wedge dx_{m-1} \wedge d\log(\varepsilon + x_m) \\
&\overset{\phi}{\mapsto} b f_0 \det(\partial f_h / \partial x_j)_{1 \le h,j \le m-1} \cdot dx_1 \wedge \cdots \wedge dx_{m-1} \wedge d\log(\varepsilon + x_m^p).
\end{aligned}$$

On the other hand, in the invariant subalgebra $k[[x_1, \dots, x_{m-1}, x_m^p]]$, we have $\partial f_m^p / \partial x_j = 0$ for $j < m$ and $\partial f_m^p / \partial x_m^p = b f_m^p / (\varepsilon + x_m^p)$. Hence, we have

$$\begin{aligned}
&f_0 \cdot df_1 \wedge \cdots \wedge df_{m-1} \wedge d\log(f_m^p) \\
&= \cdots = b f_0 \det(\partial f_h / \partial x_j)_{1 \le h,j \le m-1} \cdot dx_1 \wedge \cdots \wedge dx_{m-1} \wedge d\log(\varepsilon + x_m^p).
\end{aligned}$$

The assertion follows. ∎

We will give another abstract proof of Proposition 2.10 in [Mat3, Prop. 2.12].

2.4 Global properties of derivations

LEMMA 2.11. *Let $C \subset \mathbb{P}^2$ be a quadratic curve (not necessarily irreducible nor reduced) in characteristic p, and let D be a p-closed derivation. Then* $\mathrm{Fix}(D) \neq \emptyset$.

Proof. Suppose C is integral. Then $C \cong \mathbb{P}^1$ and the result is well known (indeed, $T_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(2)$).

Suppose C is reducible. We may assume $C = (xy = 0)$. Let $U = \operatorname{Spec} k[x,y]/(xy)$. Then $T_U = (x\frac{d}{dx} - y\frac{d}{dy}) \cdot \mathcal{O}_U$ and hence the origin belongs to $\mathrm{Fix}(D)$.

Suppose C is non-reduced. We may assume $C = (X_3^2 = 0)$. If $p \neq 2$, then D induces a derivation D_{red} on $C_{\mathrm{red}} \cong \mathbb{P}^1$, and we have $\mathrm{Fix}(D) \approx \mathrm{Fix}(D_{\mathrm{red}}) \neq \emptyset$. Suppose $p = 2$. It is easy to see that $H^0(C, T_C) \overset{\sim}{\to} H^0(C, T_{\mathbb{P}^2}|_C) \overset{\sim}{\leftarrow} H^0(\mathbb{P}^2, T_{\mathbb{P}^2})$. Hence, there exist $f_1, f_2, f_3 \in H^0(\mathbb{P}^2, \mathcal{O}(1)) = \bigoplus_{i=1}^3 kX_i$ such that $D(\frac{X_i}{X_j}) = \frac{f_i}{X_j} - \frac{X_i f_j}{X_j^2}$. If $f_3 \in kX_3$, then D induces a derivation D_{red} on $C_{\mathrm{red}} \cong \mathbb{P}^1$, and we conclude as above. Suppose $f_3 \notin kX_3$. By a coordinate change, we may assume $f_3 - X_2 \in kX_3$. Letting $x_i = X_i/X_1$ $(i = 2,3)$ and restricting to $\operatorname{Spec} k[x_2, x_3]/(x_3^2) = (X_1 \neq 0) \subset C$, we have $D(x_3) - x_2 \in (x_3)$, in particular $D(x_3) \in \mathfrak{m} := (x_2, x_3)$. If $D(x_2) \in \mathfrak{m}$, then the origin is a fixed point. Suppose $D(x_2) \notin \mathfrak{m}$ and $D^2 = hD$. Then $h = D^2(x_2)/D(x_2) \in \mathcal{O}_{\mathfrak{m}}$, and hence $D(x_2) \equiv D^2(x_3) = hD(x_3) \equiv 0 \pmod{\mathfrak{m}}$, contradiction. ∎

COROLLARY 2.12. *Suppose μ_p acts on an RDP surface X and fixes an RDP w. Then the action extends to the blowup $\mathrm{Bl}_w X$ and there exists a fixed point above w.*

Proof. The action extends to the blowup by Proposition 2.8. Let D' be the induced derivation on $\mathrm{Bl}_w X$. Let $C \subset \mathrm{Bl}_w X$ be the (possibly non-reduced) exceptional divisor, which is a quadratic curve in $\mathbb{P}^2$ since w is an RDP. Since $D'(\mathcal{I}_C) \subset \mathcal{I}_C$, D' induces a derivation D'_C (of multiplicative type) on C. By Lemma 2.11, D'_C has at least one fixed point, and that point is also a fixed point of D'. ∎

Later we will also need the following Katsura–Takeda formula on *rational* vector fields (i.e., derivations on the fraction field $k(X)$). For a rational derivation D locally of the form

https://doi.org/10.1017/nmj.2022.20 Published online by Cambridge University Press

$f^{-1}D'$ for some regular function f and (regular) derivation D', we define the divisorial and nondivisorial parts by $(D) = (D') - \operatorname{div}(f)$ and $\langle D \rangle = \langle D' \rangle$.

PROPOSITION 2.13 [KT, Prop. 2.1]. *Let X be a smooth proper surface, and let D be a nonzero rational vector field. Then we have*

$$\deg c_2(X) = \deg\langle D\rangle - K_X \cdot (D) - (D)^2.$$

§3. Tame symplectic actions on RDPs

Hereafter, all action of groups and group schemes on schemes are assumed faithful.

Throughout this section, we work under the following setting. $B = \mathcal{O}_{X,w}$ is the localization of an RDP surface X over an algebraically closed field k at a closed point w (either a smooth point or an RDP), $\mathfrak{m} \subset B$ is the maximal ideal, G is a finite group acting on X, and the action restricts to $\operatorname{Spec} B$. Assume that the order of G is not divisible by $p = \operatorname{char} k$.

DEFINITION 3.1. We say that the G-action on B is *symplectic* if it acts on the one-dimensional k-vector space $H^0(\operatorname{Spec} B \setminus \{\mathfrak{m}\}, \Omega^2_{B/k}) \otimes_B (B/\mathfrak{m})$ trivially.

If $G = \mathbb{Z}/p\mathbb{Z}$, then any action is symplectic (cf. Remark 2.6), and hence the notion is useless in this case.

REMARK 3.2. If B is as above and the G-action is symplectic, then the rank-1 free B-module $H^0(\operatorname{Spec} B \setminus \{\mathfrak{m}\}, \Omega^2_{B/k})$ admits a generator ω that is G-invariant. Indeed, take a generator ω', then $\omega := (1/|G|)\sum_{g\in G} g^*\omega'$ is clearly G-invariant and it is nonvanishing, since it is nonvanishing after $\otimes(B/\mathfrak{m})$.

REMARK 3.3. If X is an RDP K3 surface and $w \in X$ is a fixed closed point, then this is consistent with the usual notion of symplecticness, since a generator of $H^0(X^{\mathrm{sm}}, \Omega^2) \cong H^0(\tilde{X}, \Omega^2)$ (Proposition 2.3) restricts to a generator of this one-dimensional space. Thus, the symplecticness of an automorphism of an RDP K3 surface can be checked locally at any fixed point (if there exists any). The same for abelian surfaces.

PROPOSITION 3.4. *Let B and G be as above (in particular, the order of G is not divisible by $p = \operatorname{char} k$). Then the invariant ring B^G is again the localization at a closed point of an RDP surface.*

Let $\tilde{X} \to X$ be the minimal resolution at w. Then $\tilde{X}/G \to X/G$ is crepant.

Proof. Let ω be a generator of the rank-1 free B-module $H^0(\operatorname{Spec} B \setminus \{\mathfrak{m}\}, \Omega^2_{B/k})$. By Remark 3.2, we may assume ω is G-invariant. The action of G on X induces an action on $\tilde{X}$, and ω extends to a regular nonvanishing 2-form on $\tilde{X}$. At each closed point $w' \in \tilde{X}$, the stabilizer $G_{w'} \subset G$ acts on $T_{w'}\tilde{X}$ via $\mathrm{SL}_2(k)$ since G preserves ω. Hence, the quotient $\tilde{X}/G$ has only RDPs as singularities. Since ω is preserved by G, it induces a regular nonvanishing 2-form on $(\tilde{X}/G)^{\mathrm{sm}}$, and since RDPs are canonical singularities, it extends to a regular nonvanishing 2-form on $\widetilde{\tilde{X}/G}$, the minimal resolution of $\tilde{X}/G$ above w. Thus, B^G is a canonical singularity, that is, either a smooth point or an RDP. ∎

REMARK 3.5. We [Mat1, Prop. 3.8] described possible symplectic actions of finite tame groups on RDPs. For actions of cyclic groups $G = \mathbb{Z}/n\mathbb{Z}$ $(n > 1)$, we have a complete classification: possible n and the types of X and X/G are listed in Table 2.

REMARK 3.6. Singularities of quotients by order p automorphisms in characteristic $p > 0$ tends to be worse than those in characteristic $\neq p$. For example, the quotient of a

Table 2. Tame symplectic cyclic actions on RDPs.

$n=\lvert G\rvert$	X		X/G
Any	A_{m-1}		A_{mn-1}
2	A_{m-1}	$(m \geq 4$ even$)$	$D_{m/2+2}$
4	A_{m-1}	$(m \geq 3$ odd$)$	D_{m+2}
3	D_4	$(p \neq 2)$	E_6
3	D_4^r	$(p=2,\ r=0,1)$	E_6^r
2	D_{m+2}		D_{2m+2}
2	E_6	$(p \neq 3)$	E_7
2	E_6^r	$(p=3,\ r=0,1)$	E_7^r

supersingular abelian surface in characteristic 2 by the automorphism $x \mapsto -x$ is a rational surface with an elliptic singularity [Ka, Th. C].

§4. $\boldsymbol{\mu_n}$-actions on RDPs and quotients

Throughout this section, we work under the following setting. $B = \mathcal{O}_{X,w}$ is the localization of an RDP surface X over an algebraically closed field k of characteristic $p \geq 0$ at a closed point w (either a smooth point or an RDP), $\mathfrak{m} \subset B$ is the maximal ideal, n is a positive integer possibly divisible by p, μ_n acts on X, and the action restricts to $\operatorname{Spec} B$. (Note that w is not necessarily fixed by μ_n.)

If $n = p > 0$, then the corresponding derivation of multiplicative type is denoted by D.

4.1 Symplecticness of $\boldsymbol{\mu_n}$-actions

Assume w is fixed by the μ_n-action. Then the action on B induces an action on $V := H^0(\operatorname{Spec} B \setminus \{\mathfrak{m}\}, \Omega^2_{B/k}) \otimes_B (B/\mathfrak{m})$, that is, a decomposition $V = \bigoplus_{i \in \mathbb{Z}/n\mathbb{Z}} V_i$ of k-vector spaces. Since $\dim_k V = 1$, V is equal to one of the summands. In other words, V is of some weight $i_0 \in \mathbb{Z}/n\mathbb{Z}$.

Definition 4.1. We say that the μ_n-action, or the corresponding derivation if $n = p$, on B is *symplectic* if V is of weight 0.

We say that a μ_n-action, or a derivation D of multiplicative type, on an RDP surface X is *symplectic* at a fixed closed point w if the induced action or derivation on $\mathcal{O}_{X,w}$ is symplectic in the above sense.

Remark 4.2. If $p \nmid n$, then μ_n is (noncanonically) isomorphic to $\mathbb{Z}/n\mathbb{Z}$ and this definition is consistent with Definition 3.1.

Remark 4.3 (cf. Remark 3.2). If B is as above and V is of weight i_0, then the rank-1 free B-module $H^0(\operatorname{Spec} B \setminus \{\mathfrak{m}\}, \Omega^2_{B/k})$ admits a generator ω of weight i_0. Indeed, take a generator ω', let $\omega' = \sum_i \omega'_i$ be its decomposition, and write $\omega'_i = f_i \omega'$ with $f_i \in B$. Since $\sum f_i = 1$, there exists $i_1 \in \mathbb{Z}/n\mathbb{Z}$ with $f_{i_1} \in B^*$. Then $i_0 = i_1$ and hence we can take $\omega = \omega'_{i_1}$. If $n = p$, then this means $D(\omega) = i_0 \omega$.

From this, it follows that if μ_n acts on an RDP surface, then the weight i_0 is a locally constant function on the fixed locus.

Remark 4.4 (cf. Remark 3.3). If X is an RDP K3 surface and $w \in X$ is a fixed closed point, then the action is symplectic in the sense of Definition 2.5 if and only if action is symplectic at w, since a generator of $H^0(X^{\mathrm{sm}}, \Omega^2)$ restricts to a generator of this one-

https://doi.org/10.1017/nmj.2022.20 Published online by Cambridge University Press

dimensional space. Thus, the symplecticness of a μ_n-action on an RDP K3 surface can be checked locally at any fixed point (if there exists any). The same for abelian surfaces.

LEMMA 4.5. *Suppose the closed point w of B is fixed under the μ_n-action. Then B is generated by 2 or 3 homogeneous elements, respectively, if B is smooth or an RDP. Moreover:*

1. *If B is smooth and generated by elements x,y of respective weights a,b, then the action is symplectic if and only if $a+b=0$ (in $\mathbb{Z}/n\mathbb{Z}$).*
2. *If B is an RDP and generated by x,y,z of respective weights a,b,c, then there is $d\in\mathbb{Z}/n\mathbb{Z}$ and a homogeneous power series $F\in k[[x,y,z]]$ of weight d such that $\hat{B}\cong k[[x,y,z]]/(F)$. The action is symplectic if and only if $a+b+c=d$.*

Proof. The first assertion follows from Proposition 2.8.

(1) $H^0(\operatorname{Spec} B\setminus\{\mathfrak{m}\},\Omega^2_{B/k})$ is generated by $dx\wedge dy$, which is of weight $a+b$.

(2) Take an element $H\in k[[x,y,z]]$ such that $\hat{B}=k[[x,y,z]]/(H)$, and let $H=\sum_{i\in\mathbb{Z}/n\mathbb{Z}}H_i$ be the decomposition with respect to the μ_n-action. Since $H=0$ in $\hat{B}$, we have $H_i=0$ in $\hat{B}$, and hence there are $f_i\in k[[x,y,z]]$ such that $H_i=f_iH$. Since $\sum f_i=1$, there exists $d\in\mathbb{Z}/n\mathbb{Z}$ with $f_d\in k[[x,y,z]]^*$. We can take $F=H_d$, which is of weight d.

Then $H^0(\operatorname{Spec}\hat{B}\setminus\{\mathfrak{m}\},\Omega^2_{\hat{B}/k})$ is generated by $\omega=F_x^{-1}dy\wedge dz=F_y^{-1}dz\wedge dx=F_z^{-1}dx\wedge dy$ (this means that the restriction of ω to the open subscheme $\operatorname{Spec}\hat{B}[F_x^{-1}]$ is equal to $F_x^{-1}dy\wedge dz$, and so on), and we have $\operatorname{wt}(\omega)=a+b+c-d$ since $\operatorname{wt}(F_x^{-1})=-(d-a)$ and $\operatorname{wt}(dy\wedge dz)=b+c$, and so on. ∎

4.2 μ_p-actions on RDPs

As noted in §2.3, we know by [S, Prop. 6] (see also [RS, Th. 1 and Corollary]) that the quotient of a smooth variety by a μ_p-action with no isolated fixed point is smooth. We need to consider, more generally, the quotients of surfaces with RDP singularities and with isolated fixed points.

Let $\mathcal{O}_{X,w}$ and μ_n be as in the beginning of §4, and suppose $n=p$. Let $\pi\colon X\to Y=X/\mu_p$ be the quotient morphism.

THEOREM 4.6.

1. *Assume w is non-fixed. If w is a smooth point, then $\pi(w)\in Y$ is also a smooth point. If w is an RDP, then $\pi(w)$ is either a smooth point or an RDP. In either case, $X\times_Y\tilde{Y}\to X$ is crepant, where $\tilde{Y}\to Y$ is the minimal resolution at $\pi(w)$.*
2. *If w is fixed and the action is symplectic at w, then w is an isolated fixed point and $\pi(w)$ is an RDP.*
3. *If w is an isolated fixed point and the action is non-symplectic at w, then $\pi(w)$ is a non-RDP singularity.*

First, we consider non-symplectic actions on isolated fixed points.

Proof of Theorem 4.6(3). By Proposition 2.10, we have an isomorphism

$$(H^0(\operatorname{Spec}\mathcal{O}_{X,w}\setminus\{w\},\Omega^2))_0\cong H^0(\operatorname{Spec}\mathcal{O}_{Y,\pi(w)}\setminus\{\pi(w)\},\Omega^2)$$

https://doi.org/10.1017/nmj.2022.20 Published online by Cambridge University Press

Table 3. Non-fixed μ_p-actions on RDPs.

p	equation		X	$Y = X^D$	$X \times_Y \tilde{Y}$
Any	$xy + z^{mp}$	$(m \geq 2)$	A_{mp-1}	A_{m-1}	mA_{p-1}
Any	$xy + z^p$		A_{p-1}	Smooth	—
5	$x^2 + y^3 + z^5$		E_8^0	Smooth	—
3	$x^2 + z^3 + y^4$		E_6^0	Smooth	—
3	$x^2 + y^3 + yz^3$		E_7^0	A_1	E_6^0
3	$x^2 + z^3 + y^5$		E_8^0	Smooth	—
2	$z^2 + x^2y + xy^m$	$(m \geq 2)$	D_{2m}^0	Smooth	—
2	$x^2 + yz^2 + xy^m$	$(m \geq 2)$	D_{2m+1}^0	A_1	D_{2m}^0
2	$x^2 + xz^2 + y^3$		E_6^0	A_2	D_4^0
2	$z^2 + x^3 + xy^3$		E_7^0	Smooth	—
2	$z^2 + x^3 + y^5$		E_8^0	Smooth	—

preserving the zero loci of 2-forms. If $\pi(w)$ is either a smooth point or an RDP, then the right-hand side has a nonvanishing 2-form and hence there is a nonvanishing form ω on $\operatorname{Spec}\mathcal{O}_{X,w} \setminus \{w\}$ of weight 0. Being nonvanishing, ω is a generator of $H^0(\operatorname{Spec}\mathcal{O}_{X,w} \setminus \{w\}, \Omega^2)$. However, this contradicts the non-symplecticness assumption. ∎

Next, we consider non-fixed points. In fact, we can classify all possible actions and give explicit equations.

PROPOSITION 4.7. *Assume w is not fixed.*

- *If w is a smooth point, then there are coordinates x, y of $\mathcal{O}_{X,w}$ satisfying $D(x) = 0$ and $D(y) \neq 0$, and hence $\mathcal{O}_{Y,\pi(w)}$ has x, y^p as coordinates and in particular $\pi(w)$ is a smooth point.*
- *If w is an RDP, then there is an element $F \in k[[x, y, z^p]]$ and an isomorphism $\hat{\mathcal{O}}_{X,w} \cong k[[x, y, z]]/(F)$ with $D(x) = D(y) = 0$ and $D(z) \neq 0$, and hence $\hat{\mathcal{O}}_{Y,\pi(w)} \cong k[[x, y, z^p]]/(F)$. Moreover, we can take F to be one in Table 3.*

Proof of Theorem 4.6(1) and Proposition 4.7. If w is a smooth point, then taking coordinates x, y as in Lemma 2.9 (i.e., $D(x) = 0$ and $D(y) = 1 + y$), we have $\hat{\mathcal{O}}_{Y,\pi(w)} \cong k[[x, y^p]]$, and hence $\mathcal{O}_{Y,\pi(w)}$ is smooth.

Assume w is an RDP. By Lemma 2.9, we have coordinates x, y, z satisfying $D(x) = D(y) = 0$ and $D(z) \neq 0$. We have $\hat{\mathcal{O}}_{X,w} \cong k[[x, y, z]]/(F)$ for some $F \in k[[x, y, z]]$ such that $D(F) \in (F)$, and we may assume $F \in k[[x, y, z^p]]$. We show that, after replacing F with a multiple by a unit, and after a coordinate change of $k[[x, y, z]]$ that preserves the subring $k[[x, y, z^p]]$, F coincides with one in Table 3. (Such coordinate changes are given by $x', y', z' \in \mathfrak{m}$ that are linearly independent in $\mathfrak{m}/\mathfrak{m}^2$ and satisfy $x', y' \in \mathfrak{m} \cap k[[x, y, z^p]]$.) A similar classification is given in [EH+, Prop. 3.8], but they missed the case of E_7^0 in characteristic 3.

Assume the classification for the moment. Then, in each case, we observe that $\pi(w)$ is either a smooth point or an RDP, and it is straightforward to check that $X \times_Y \tilde{Y}$ is an RDP surface crepant over X. (In Table 3, the entries of the singularities of $X \times_Y \tilde{Y}$ are omitted if Y is already smooth.) For example, consider $X = \operatorname{Spec} k[x, y, z]/(F)$, $F = xy + z^{mp}$ with $m \geq 2$. Then $X' := X \times_Y \operatorname{Bl}_{\pi(w)} Y$ is covered by three affine pieces

$$
\begin{aligned}
X'_1 &= \operatorname{Spec} k[x,y_1,v_1,z]/(y_1+x^{m-2}v_1^m, xv_1-z^p), & y_1 &= y/x, & v_1 &= z^p/x,\\
X'_2 &= \operatorname{Spec} k[x_2,y,v_2,z]/(x_2+y^{m-2}v_2^m, yv_2-z^p), & x_2 &= x/y, & v_2 &= z^p/y,\\
X'_3 &= \operatorname{Spec} k[x_3,y_3,z]/(x_3y_3+z^{(m-2)p}), & x_3 &= x/z^p, & y_3 &= y/z^p.
\end{aligned}
$$

One observes that $\operatorname{Sing}(X')$ consists of two RDPs of type A_{p-1} at the origins of X'_1 and X'_2 and, if $m \geq 3$, one RDP of type $A_{(m-2)p-1}$ at the origin of X'_3. Repeating this, we observe that $X \times_Y \tilde{Y}$ has mA_{p-1}.

Now, we show the classification. We say that F *has* a monomial if the coefficient of that monomial is nonzero. We also write $F = \sum_{h,i,j} a_{hij}x^hy^iz^j$.

First, assume $p > 2$. We may assume that the degree 2 part F_2 is either xy or x^2. Assume $F_2 = xy$. We may assume that F has no xz^j and yz^j. F must have z^j, $j = mp$, and then it is A_{mp-1}. Then, by replacing x with $x + a_{0ij}y^{i-1}z^j$ and y with $y + a_{h0j}x^{h-1}z^j$, and so on, we may assume that F has no y^iz^j with $i > 0$ and no x^hz^j with $h > 0$. Thus, $F = u_1xy + u_2z^{mp}$ for some units u_1, u_2, and then by replacing x, y, F by suitable multiples, we obtain $F = xy + z^{mp}$.

Assume $p > 3$ and $F_2 = x^2$. We may assume that the degree 3 part F_3 is y^3. If $p \geq 7$, it cannot be an RDP. If $p = 5$, then F must have z^5, and then it is E_8^0. We have $F = u_1x^2 + u_2y^3 + u_3z^5$, and then by replacing x, y, F by suitable multiples, we obtain $F = x^2 + y^3 + z^5$. (For example, we let $F = u_3F'$, $x = (u_3u_1^{-1})^{1/2}x'$, $y = (u_3u_2^{-1})^{1/3}y'$. Note that we can take nth roots of units provided $p \nmid n$.)

Assume $p = 3$ and $F_2 = x^2$. We may assume $F_3 = y^3$ or $F_3 = z^3$. If $F_3 = z^3$, then F must have y^4 or y^5, and then it is E_6^0 or E_8^0. We may assume $a_{130} = a_{140} = 0$ by replacing x with $x + (1/2)(a_{130}y^3 + a_{140}y^4)$, and then we transform F as above. If $F_3 = y^3$, then F must have yz^3 and then it is E_7^0. We eliminate a_{1ij} as above, then we have $F = u_1x^2 + u_2y^3 + u_3yz^3 + z^6g(z^3)$ for some power series $g \in k[[z^3]]$. We may assume $u_i \equiv 1 \pmod{\mathfrak{m}}$. We eliminate g by replacing y with $y + z^3g$, and then we transform F as above.

Now, consider $p = 2$. We may assume F_2 is one of $xy + z^2$ (if irreducible), xy (if reducible but not a square), z^2, or x^2 (square, of a linear factor containing z or not). If $F_2 = xy + z^2$ or $F_2 = xy$, then as above, it is A_{mp-1} and F becomes $xy + z^{mp}$.

Assume $p = 2$ and $F_2 = x^2$. If F_3 has yz^2, then F must have xy^m and then it is D_{2m+1}^0. We obtain $F = u_1x^2 + u_2yz^2 + u_3xy^m + z^4g(z) + f(y) + y^{2m}g'(y)$, where $f(y) = f_0(y)^2 + yf_1(y)^2$ is a polynomial of degree $< 2m$, $g(z) \in k[[z]]$, and $g'(y) \in k[[y]]$. We may assume $u_i \equiv 1 \pmod{\mathfrak{m}}$. We eliminate f by replacing x with $x + f_0(y)$ and z with $z + f_1(y)$ and so on. Then we eliminate g and g' by replacing y and x suitably, and take multiples by units as above.

If F_3 has no yz^2, then F must have y^3 and xz^2 and then it is E_6^0. We obtain $F = u_1x^2 + u_2y^3 + u_3xz^2 + ay^2z^2 + z^4g$, $g = g_0(z^2) + yg_1(z^2) + y^2g_2(z^2)$. We may assume $u_i \equiv 1 \pmod{\mathfrak{m}}$. We eliminate a and g by replacing y and x suitably, and then we transform F as above.

Assume $p = 2$ and $F_2 = z^2$. Let $\overline{F_3} = (F_3 \bmod (z)) \in k[[x,y]]$. If $\overline{F_3}$ has three distinct roots, then we may assume $\overline{F_3} = x^3 + y^3$ and then it is D_4^0. We can transform F to $z^2 + x^3 + y^3$ as above, and then to $z^2 + x^2y + xy^2$. If $\overline{F_3}$ has two distinct roots, then we may assume $\overline{F_3} = x^2y$ and F must have xy^m and then it is D_{2m}^0. We obtain $F = u_1z^2 + u_2x^2y + u_3xy^m + g(x) + f(y) + y^{2m-1}g'(y)$, where $f(y) = f_0(y)^2 + yf_1(y)^2$ is a polynomial of degree $< 2m-1$ and $g \in k[[x]]$ and $g' \in k[[y]]$. We argue as in the case of D_{2m+1}^0. If $\overline{F_3}$ has one (triple) root, then we may assume $\overline{F_3} = x^3$ and F must have xy^3 or y^5 and then it is E_7^0 or E_8^0. We transform F as above. ∎

https://doi.org/10.1017/nmj.2022.20 Published online by Cambridge University Press

Next, we consider symplectic actions on fixed points.

Lemma 4.8. *Assume w is fixed, and the action is symplectic at w.*

1. *Assume w is a smooth point. Then w is an isolated fixed point and $\pi(w)$ is an RDP of type A_{p-1}. The eigenvalues of D on the cotangent space $\mathfrak{m}_w/\mathfrak{m}_w^2$ are of the form $a,-a$ for some $a \in \mathbb{F}_p^*$.*
2. *Assume w is an RDP. Let $f : X' = \mathrm{Bl}_w X \to X$. Then X' is an RDP surface, D uniquely extends to a derivation D' on X' which is symplectic at every fixed point above w, and $g\colon Y' = (X')^{D'} \to Y$ is crepant.*

Proof. (1) By Lemma 4.5(1), we have $D = ax\cdot(\partial/\partial x) - ay\cdot(\partial/\partial y)$ with $a \in \mathbb{F}_p$ for some coordinates x,y, and $a \neq 0$ since D is nontrivial. Hence, w is an isolated fixed point of D. We observe that $a,-a$ are the eigenvalues of the action on the cotangent space. We have $\hat{\mathcal{O}}_{X,w}^D = k[[x^p, xy, y^p]]$, and it is an RDP of type A_{p-1}.

(2) By Remark 4.3 and assertion (1), w is an isolated fixed point. By Proposition 2.8, D uniquely extends to D' on X'. Let ω be a generator of $H^0(\operatorname{Spec}\mathcal{O}_{X,w} \setminus \{w\}, \Omega^2)$ with $D(\omega) = 0$. Since w is an RDP, X' is again an RDP surface, and it follows from Proposition 2.3 that ω extends to ω' on $(X')^{\mathrm{sm}}$, which generates $H^0(\operatorname{Spec}\mathcal{O}_{X',w'} \setminus \{w'\}, \Omega^2)$ at any closed point $w' \in X'$ above w, and that $D'(\omega') = 0$. Hence, D' is symplectic at every fixed point above w. Since as above such fixed points are isolated, Y' is smooth outside finitely many isolated points. Applying Proposition 2.10 to ω on $X \setminus \{w\}$ and ω' on $X' \setminus (\mathrm{Sing}(X') \cup \mathrm{Fix}(D'))$, we obtain 2-forms ψ on $Y \setminus \{\pi(w)\}$ and ψ' on $Y' \setminus \pi((\mathrm{Sing}(X') \cup \mathrm{Fix}(D')))$, which are nonvanishing. Comparing ψ and ψ', we observe that g is crepant. ∎

Proof of Theorem 4.6(2). By Remark 4.3 and Lemma 4.8(1), w is an isolated fixed point. By shrinking X, we may assume that D has no fixed point except w.

We construct a finite sequence $(X_j, D_j)_{0 \le j \le n}$ $(n \ge 0)$ of RDP surfaces X_j and derivations D_j on X_j of multiplicative type that is symplectic at each fixed point. Let $(X_0, D_0) = (X, D)$. If X_j has no fixed RDP, then we terminate the sequence at $n = j$. If X_j has at least one fixed RDP, let X_{j+1} be the blowup of X_j at the fixed RDPs and D_{j+1} the extension of D_j to X_{j+1}. Since any RDP becomes smooth after a finite number of blowups at RDPs, this sequence terminates at some $n \ge 0$. By Lemma 4.8(2), D_{j+1} on X_{j+1} is symplectic at each fixed point, and $(X_{j+1})^{D_{j+1}} \to (X_j)^{D_j}$ is crepant. By Theorem 4.6(1) and Lemma 4.8(1), $Y_n = (X_n)^{D_n}$ has canonical singularity (i.e., has no singularity other than RDPs), and since $Y_n \to Y = X^D$ is crepant, also Y has canonical singularity. If $n > 0$, then $\pi(w)$ is not a smooth point since $Y_n \to Y$ is a crepant morphism non-isomorphic at that point, and if $n = 0$, then $\pi(w)$ is not a smooth point by Lemma 4.8(1). Hence, in either case, $\pi(w)$ is an RDP. ∎

Moreover, we can classify all possible symplectic μ_p-actions on RDPs.

Proposition 4.9. *Assume w is a fixed RDP, and the action is symplectic at w. Then there is a μ_p-equivariant isomorphism $\hat{\mathcal{O}}_{X,w} \cong k[[x,y,z]]/(F)$ with F equal to one in Table 4 and μ_p acts on x,y,z by respective weights $a,-a,0$ for some $a \in \mathbb{F}_p^*$. The singularities of X, $X' = \mathrm{Bl}_w X$, X/μ_p, and X'/μ_p are displayed in the table.*

Remark 4.10. A polynomial $f \in k[x_1, \dots, x_m]$ is called *quasi-homogeneous* if, for some $a_1, \dots, a_m \in \mathbb{Z}_{\ge 1}$, the monomials appearing in f have the same degree with respect to a (i.e., degree of the monomial $x_1^{i_1} \dots x_m^{i_m}$ is $i_1 a_1 + \dots + i_m a_m$). RDPs whose completions are *not*

Table 4. Symplectic μ_p-actions on RDPs.

p	Equation		X	X/μ_p
Any	— (Smooth point)		A_0	A_{p-1}
Any	$xy+z^m$	$(m \geq 2)$	A_{m-1}	A_{mp-1}
3	$z^2+x^3+y^3+x^2y^2$		E_6^1	E_6^1
3	$z^2+x^3+y^3+x^4y$		E_8^1	E_6^0
2	$x^2+y^2z+xyz^k+z^l$	$(k \geq 1,\ l \geq 2)$	$D_{2k+l}^{\lfloor l/2 \rfloor}$	$D_{2l}^{(l-k)^+}$
2	$x^2+z^2+xyz+y^{2n-2}$	$(n \geq 3)$	$D_{2n}^{n-1}[*]$	$D_{n+2}^{\lfloor n/2 \rfloor}$
2	$x^2+z^2+z^3+xy^3$		E_7^2	D_5^0
2	$x^2+z^3+y^4+xyz$		E_7^3	E_7^3
2	$x^2+z^3+y^4+xy^3$		E_8^3	E_7^2

p	X	X'	X/μ_p	X'/μ_p
Any	A_0	—	A_{p-1}	—
Any	A_{m-1} $(m \geq 3)$	$A_{m-3}+2A_0$	A_{mp-1}	$A_{(m-2)p-1}+2A_{p-1}$
Any	A_1	$2A_0$	A_{2p-1}	$2A_{p-1}$
3	E_6^1	$A_5[\mathsf{n}]+2A_0$	E_6^1	A_1+2A_2
3	E_8^1	$E_7^0[\mathsf{n}]+2A_0$	E_6^0	A_1+2A_2
2	$D_{2k+l}^{\lfloor l/2 \rfloor}$ $(l \geq 4)$	$D_{2k+l-2}^{\lfloor l/2 \rfloor -1}+A_1$	$D_{2l}^{(l-k)^+}$	$D_{2(l-2)}^{(l-2-k)^+}+A_3$
	$(l=3,\ k \geq 2)$	$D_{2k+1}^0[\mathsf{n}]+A_0+A_1$	$D_6^{(3-k)^+}$	$A_1+A_1+A_3$
	$(l=3,\ k=1)$	$A_3[\mathsf{n}]+A_0+A_1$	D_6^2	$A_1+A_1+A_3$
	$(l=2,\ k \geq 2)$	$D_{2k}^0[\mathsf{n}]+A_1$	D_4^0	A_3
	$(l=2,\ k=1)$	$2A_1[\mathsf{n}]+A_1$	D_4^1	A_3
2	$D_{2n}^{n-1}[*]$ $(n \geq 3)$	$D_{2n-2}^{n-2}[*]+A_1[\mathsf{n}]$	$D_{n+2}^{\lfloor n/2 \rfloor}$	$D_{n+1}^{\lfloor (n-1)/2 \rfloor}$
2	E_7^2	D_6^1	D_5^0	D_4^0
2	E_7^3	$D_6^2[*]+A_0$	E_7^3	$D_5^1+A_1$
2	E_8^3	$E_7^2+A_0$	E_7^2	$D_5^0+A_1$

- A_0 is a smooth point that is an isolated fixed point of D.
- $[\mathsf{n}]$ means that the RDP is not fixed by D.
- $\lfloor q \rfloor := \max\{n \in \mathbb{Z} \mid n \leq q\}$ denotes the integer part of a real q.
- $q^+ := \max\{q, 0\}$ denotes the positive part of a real q.
- $[*]$: It follows from the classification that for each (formal) isomorphism class of RDP, there exists only one fixed symplectic μ_p-action up to isomorphism, except for the case of D_{2n}^{n-1} $(n \geq 3)$ in $p=2$, in which case there are two and they are distinguished by the degree 2 part F_2 being a square of a homogeneous element or not. We distinguish them by notation D_{2n}^{n-1} and $D_{2n}^{n-1}[*]$. We use the convention that $D_4^1[*] = D_4^1$.

defined by quasi-homogeneous polynomials, which exist only if $p = 2,3,5$, are precisely D_n^r and E_n^r with $r \neq 0$. It follows from the classification given in Proposition 4.9 (resp. given in Proposition 4.7, resp. which is omitted) that if an RDP of type D_n or E_n admits a fixed symplectic (resp. non-fixed, resp. fixed non-symplectic) μ_p-action, then the singularity is not defined (resp. is defined, resp. is defined) by a quasi-homogeneous polynomial. We do not know any explanation of this phenomenon.

Proof of Proposition 4.9. We consider tuples (a,b,c,F) with $a,b,c \in \mathbb{F}_p$, not all 0, and $F \in k[[x,y,z]]$ such that $F = 0$ defines an RDP and only monomials of weight $a+b+c$ $(\in \mathbb{F}_p)$ appear in F, where x,y,z have respective weights a,b,c. By Lemma 4.5(2), it suffices to consider $k[[x,y,z]]/(F)$ of this form. We show that there exist a μ_p-equivariant isomorphism $k[[x,y,z]]/(F) \cong k[[x',y',z']]/(F')$ with F' in Table 4 and $\mathrm{wt}(x',y',z') = (1,-1,0)$ up to $\mathrm{Aut}(\mu_p) = \mathbb{F}_p^*$ (which amounts to replacing (a,b,c) with (ta,tb,tc) for some $t \in \mathbb{F}_p^*$).

We write $F = \sum_{h,i,j} a_{hij} x^h y^i z^j$, and we say that a polynomial or a formal power series *has* a monomial if its coefficient is nonzero.

First, assume the degree 2 part F_2 is a non-square. Then we may assume that F_2 contains a non-square monomial, say xy. (Indeed, if this is not the case, then $p \neq 2$ and F_2 contains at least two square monomials, say x^2 and y^2, then x and y has the same weight, and then after a linear coordinate change, we may assume F_2 contains xy.) Then we have $c = 0$. If $a+b \notin \{0,a,b\}$, then $F \in (x,y)^2$, which implies that $F = 0$ is not an RDP. If $a+b = a \neq 0$, then $F \in (x)$, again not an RDP. The same if $a+b = b \neq 0$. So we have $a+b = 0$ and hence $F \in k[[x^p, xy, y^p, z]]$. Since F cannot belong to (x,y), there exists an integer m such that F has the monomial z^m. Let m be the smallest such integer. We have $F = u_1 z^m + u_2 xy + g_1(x^p) + g_2(y^p)$ for some units $u_1, u_2 \in k[[x^p, xy, y^p, z]]^*$ and power series g_1, g_2. We may assume $u_1, u_2 \equiv 1 (\bmod \mathfrak{m})$. We eliminate g_1, g_2 by replacing x with $x + g_2/y$ and y with $y + g_1/x$ (and repeating this), and we obtain $F = u_1 z^m + u_2 xy$. By replacing x, y, z, F with suitable multiples, we obtain $F = z^m + xy$.

Next, assume $p \geq 3$ and F_2 is square. We may assume $F_2 = z^2$. We may assume $F_3 \not\equiv 0$ (mod z). If F_3 has $x^2 y$, then by $2c = 2a + b = a + b + c$, we have $b = 0$ and $a = c$, and hence $F \in (x,z)^2$, which is absurd. Hence, we may assume F_3 has y^3. By $2c = 3b = a+b+c$, we have $(a,b,c) = (a, 2a, 3a)$. If F does not have x^3, then $F \in (z^2, x^3 z, xyz, x^6, x^4 y, x^2 y^2, y^3)$, and $F = 0$ cannot define an RDP. Hence, F has x^3, hence $p = 3$, and then $F \in k[[x^3, xy, y^3, z]]$. We may assume that F does not have xyz. To define an RDP, F must have one of x^2y^2, x^4y, xy^4.

If it has $x^2 y^2$, then it is E_6^1. We can eliminate $x^h y^i z$, and we have

$$F = z^2 u_1 + x^3 + y^3 + x^2 y^2 u_2 + \sum_{(h,i,j) \in S_1} a_{hij} x^h y^i z^j + \sum_{(h,i) \in S_2} b_{hi} x^h y^i + \sum_{(h,i) \in S_3} c_{hi} x^h y^i,$$
$$S_1 = \{(4,1,0), (1,4,0)\}, \qquad S_2 = \{(6,0), (7,1)\}, \qquad S_3 = \{(0,6), (1,7)\},$$

where $a_{hij} \in k$, $b_{hi} \in k[[x^3]]$, $c_{hi} \in k[[y^3]]$, and $u_1, u_2 \in k[[x^3, xy, y^3, z]]^*$. By replacing x with $x + ty^2$ and y with $y + t'x^2$, we eliminate a_{410} and a_{140}. Then, by replacing F with $(1 + x^3 b_{60} + x^4 y b_{71} + y^3 c_{06} + xy^4 c_{17})F$, we eliminate all b_{hi} and c_{hi}. Finally, we replace x, y, z, F with suitable multiples and achieve $u_1 = u_2 = 1$. (For example, we let $F = u_2^{-3} F'$, $x = u_2^{-1} x'$, $y = u_2^{-1} y'$, and $z = (u_1 u_2^3)^{-1/2} z'$.)

If it does not have $x^2 y^2$ but has $x^4 y$ or xy^4, then it is E_8^1. By replacing F with a unit multiple, we may assume that it has $x^4 y$ and does not have xy^4. We can eliminate $x^h y^i z$, and we have

$$F = z^2 u_1 + x^3 + y^3 + x^4 y u_2 + \sum_{(h,i,j) \in S_1} a_{hij} x^h y^i z^j + \sum_{(h,i) \in S_2} b_{hi} x^h y^i + \sum_{(h,i) \in S_3} c_{hi} x^h y^i,$$
$$S_1 = \{(6,0,0), (3,3,0), (0,6,0)\}, \qquad S_2 = \{(9,0)\}, \qquad S_3 = \{(0,9), (1,7), (2,5), (3,6)\},$$

with a_{hij}, b_{hi}, c_{hi}, and u_1, u_2 as in the previous case. By replacing y with $y + tx^2$, we eliminate a_{600}. By replacing x with $x + t'y^2$ and F with $(1 + t'' y^3)F$, we eliminate a_{330} and a_{060}. By replacing F with $(1 + x^2 y^2 c_{25} + xy^4 c_{17} + y^6 c_{09})F$, then with $(1 + x^6 b_{90} + x^3 y^3 c_{36})F$, we eliminate all b_{hi} and c_{hi}. We replace x, y, z, F with suitable multiples and achieve $u_1 = u_2 = 1$.

Hereafter, assume $p = 2$ and that F_2 is a square. If $\sqrt{F_2}$ is homogeneous, then we may assume $F_2 = x^2$, we may assume F_3 has $y^2 z$ or z^3, and then we have $(a,b,c) = (1,1,0)$, and hence $F \in k[[x^2, xy, y^2, z]]$. If $\sqrt{F_2}$ is not homogeneous, then we may assume $F_2 = x^2 + z^2$ and

https://doi.org/10.1017/nmj.2022.20 Published online by Cambridge University Press

$a=1$ and $c=0$, and then again we have $(a,b,c)=(1,1,0)$, and hence $F \in k[[x^2,xy,y^2,z]]$, and F_3 has xyz, y^2z, z^3, or x^2z.

Assume (F_2 is x^2 or x^2+z^2 and) F_3 contains y^2z. Furthermore, since $F \notin (x,y)^2$, we see that F has z^l ($l \geq 2$). We have $F \equiv x^2+y^2z+z^l (\mathrm{mod}\ (x^4,x^3y,x^2y^2,xy^3,y^4,x^2z,xyz,y^2z^2,z^{l+1}))$. Write $F=F_0(x^2,y^2,z)+xyF_1(x^2,y^2,z)$. Then there exist unique $f,g \in k[[z]]$ such that $F_0 \in (x^2-f(z)^2, y^2-g(z)^2)$, and they satisfy $l=\min\{2\operatorname{ord}_z(f), 2\operatorname{ord}_z(g)+1\}$. If l is even, then, by replacing y with $y-xg/f$, we may assume $g=0$. If l is odd, then, by replacing x with $x-yf/g$, we may assume $f=0$. We eliminate a_{hij} with $h \geq 2$, $(h,i,j) \neq (2,0,0)$, by replacing F with $(1+a_{hij}x^{h-2}y^iz^j)F$, and a_{hij} with $i \geq 2$, $(h,i,j) \neq (0,2,0),(0,2,1)$, by replacing z with $z+a_{hij}x^hy^{i-2}z^j$. We obtain $F=x^2+y^2z+z^lu(z)+xye(z)$, where $e(z) \in k[[z]]$ and $u(z) \in k[[z]]^*$. We have $e(z) \neq 0$, since if $e(z)=0$, then $F=F_0 \in ((x-f(z))^2,(y-g(z))^2)$, which is absurd. Write $e(z)=z^kv(z)$, $k \geq 1$, and $v(z) \in k[[z]]^*$. It is $D_{2k+l}^{\lfloor l/2 \rfloor}$. If l is even, then, since $F_0 \in (x^2-f(z)^2, y^2-g(z)^2)$ and $g=0$, we have $z^lu(z)=f(z)^2$ and hence $u(z)$ is a square, and then by replacing x with $u(z)^{1/2}x$ and by replacing F with a unit multiple, we obtain $F=x^2+y^2zu'(z)+z^l+xye(z)$ for some $u'(z) \in k[[z]]^*$. Similarly, if l is odd, then, since $f=0$, we have $z^lu(z)=zg(z)^2$ and hence $u(z)$ is a square, and then (by replacing y) we obtain $F=x^2u'(z)+y^2z+z^l+xye(z)$. By replacing x,y,z,F with unit multiples, we can achieve $u'=v=1$.

Assume $F_2=x^2$ and F_3 has z^3 but no y^2z. To define an RDP, F must have y^4 and must have xyz or xy^3. If F has xyz, then it is E_7^3. We have

$$
\begin{gathered}
F=x^2+y^4+z^3u_1+xyzu_2+\sum_{(h,i,j)\in S_1} a_{hij}x^hy^iz^j \\
+\sum_{(h,i)\in S_2} b_{hi}x^hy^i+\sum_{(h,i)\in S_3} c_{hi}x^hy^i+\sum_{(h,i)\in S_4} d_{hi}x^hy^i, \\
S_1=\{(3,1,0),(1,3,0),(1,5,0),(0,2,2),(2,0,1),(2,0,2),(0,4,1),(0,4,2)\}, \\
S_2=\{(4,0),(5,1)\}, \qquad S_3=\{(0,6),(1,7)\}, \qquad S_4=\{(2,2)\},
\end{gathered}
$$

where $a_{hij} \in k$, $b_{hi}=\sum_{j=0}^2 b_{hij}z^j$ with $b_{hij} \in k[[x^2]]$, $c_{hi}=\sum_{j=0}^2 c_{hij}z^j$ with $c_{hij} \in k[[y^2]]$, $d_{hi} \in k[[x^2,xy,y^2]]$, and $u_1,u_2 \in k[[x^2,xy,y^2,z]]^*$. We may assume $u_1,u_2 \equiv 1 (\mathrm{mod}\ \mathfrak{m})$. We replace z with $z+a_{310}x^2+a_{130}y^2+a_{150}y^4$, x with $x+tyz$ (which eliminates a_{022}), y with $y+a_{201}x+a_{202}xz$, and x with $x+a_{041}y^3+a_{042}y^3z$, and thus eliminate all a_{hij} ($(h,i,j) \in S_1$). We replace F with $(1+x^2b_{40}+y^2c_{06})F$, F with $(1+x^3yb_{51}+xy^3c_{17})F$, and z with $z+d_{22}xy$, and thus eliminate all b_{hi}, c_{hi}, and d_{hi}. We replace x,y,z,F with suitable multiples and achieve $u_1=u_2=1$.

Next, if F does not have xyz but has xy^3, then it is E_8^3. We have

$$
\begin{gathered}
F=x^2+y^4+z^3u_1+xy^3u_2+\sum_{(h,i,j)\in S_1} a_{hij}x^hy^iz^j+\sum_{(h,i)\in S_2} b_{hi}x^hy^i+\sum_{(h,i)\in S_3} c_{hi}x^hy^i, \\
S_1=\{(2,0,1),(2,0,2),(0,4,1),(0,2,2),(1,1,2),(0,6,0),(0,4,2),(0,6,1),(0,6,2), \\
(2,2,0),(2,2,1),(2,2,2)\}, \qquad S_2=\{(4,0),(3,1),(4,2))\}, \qquad S_3=\{(0,8)\},
\end{gathered}
$$

with a_{hij}, b_{hi}, c_{hi}, and u_1,u_2 as in the previous case. We may assume $u_1,u_2 \equiv 1 (\mathrm{mod}\ \mathfrak{m})$. We replace F with $(1+a_{201}z+a_{202}z^2)F$, x with $x+tyz$ and z with $z+t'y^2$ (which eliminates a_{041} and a_{022}), z with $z+a_{112}xy$, x with $x+t''y^3$ (which eliminates a_{060}), x with $x+a_{042}yz^2+a_{061}y^3z$, x with $x+a_{062}y^3z^2$, and y with $y+(a_{220}+a_{221}z+a_{222}z^2)x$, and thus

https://doi.org/10.1017/nmj.2022.20 Published online by Cambridge University Press

eliminate all a_{hij} $((h,i,j)\in S_1)$. We replace F with $(1+x^2b_{40}+xyb_{31}+x^2y^2b_{42}+y^4c_{08})F$, and thus eliminate all b_{hi} and c_{hi}. We replace x,y,z,F with suitable multiples and achieve $u_1=u_2=1$.

Assume $F_2=x^2+z^2$ and F_3 does not have y^2z and has xyz. F moreover needs xy^i, y^iz, or y^l. Replacing z with $z+xy^{i-1}$ (resp. x with $x+y^{i-1}z$), we may assume that there are no xy^i (resp. y^iz) of low degree. Thus, we have

$$F\equiv x^2+z^2+xyz+y^{2n-2}$$
$$(\mathrm{mod}\ (x^4,x^3y,x^2y^2,xy^m(m>n),y^{2n},x^2z,xyz^2,y^2z^2,y^mz(m>n),z^3)),$$

$n\geq 3$, and this is $D_{2n}^{n-1}[*]$. We eliminate x^hy^i $(h,i\geq 1,\ (h,i)\neq(1,1))$ by replacing z with $z+a_{hi0}x^{h-1}y^{i-1}$, and x^hz^j and y^iz^j similarly. We eliminate x^h, y^i, z^j $(h\geq 3,\ i\geq 2n-1,\ j\geq 3)$ by replacing F with a unit multiple. We obtain $F=x^2+z^2+xyzu+y^{2n-2}$ for some $u\in k[[x^2,xy,y^2,z]]^*$, and we can achieve $u=1$.

Assume $F_2=x^2+z^2$ and F_3 does not have y^2z nor xyz. By replacing F with $(1+a_{201}x^2z)^{-1}F$, we may assume that F does not have x^2z. Then F has z^3 and F moreover needs xy^3, and then it is E_7^2. We have

$$F=x^2+z^2+z^3u_1+xy^3u_2+\sum_{(h,i,j)\in S_1}a_{hij}x^hy^iz^j+\sum_{(h,i)\in S_2}b_{hi}x^hy^i+\sum_{(h,i)\in S_3}c_{hi}x^hy^i,$$
$$S_1=\{(0,4,0),(0,4,1),(0,6,0),(0,2,2),(3,1,0),(2,2,0),(1,1,2),(4,0,0),(2,0,2)\},$$
$$S_2=\{(4,0),(3,1),(2,2)\},\qquad S_3=\{(0,4)\},$$

with a_{hij}, b_{hi}, c_{hi}, and u_1,u_2 as in the case of E_7^3, and moreover $b_{hi}\in(x^2,z)$ and $c_{hi}\in(y^4,y^2z,z^2)$. We may assume $u_1,u_2\equiv 1(\mathrm{mod}\ \mathfrak{m})$. We replace z with $z+a_{040}^{1/2}y^2$, x with $x+a_{041}yz$, x with $x+ty^3$ (which eliminates a_{060}), F with $(1+a_{022}y^2)F$, y with $y+t'x$ and z with $z+t''xy$ and F with $(1+t'''xy)F$ (which eliminates a_{310}, a_{220}, and a_{112}), z with $z+t''''x^2$ and F with $(1+t'''''x^2)F$ (which eliminates a_{400} and a_{202}), and thus eliminate all a_{hij} $((h,i,j)\in S_1)$. We replace x with $x+yc_{04}$ and F with $(1+b_{40}x^2+b_{31}xy+b_{22}y^2)F$ to eliminate all b_{hi} and c_{hi}. We replace x,y,z,F with suitable multiples and achieve $u_1=u_2=1$. ∎

4.3 μ_n-actions on RDPs

In this section, we classify μ_n-actions on RDPs under each of the following assumptions.

- w is not fixed by μ_n (Proposition 4.12).
- $n=p^e$, and the subgroup scheme μ_p fixes w and is symplectic (Proposition 4.13).
- w is fixed by μ_n and the action is symplectic (Proposition 4.14).

In Propositions 4.13 and 4.14, we use the convention that a smooth point is of type A_0.

LEMMA 4.11. *Let X be a k-scheme equipped with a μ_{p^2}-action. Let $\pi_1\colon X\to X_1=X/\mu_p$ be the quotient morphism by the action of the subgroup scheme $\mu_p\subset\mu_{p^2}$. If $w\in X$ is non-fixed by the action of μ_p, then $\pi_1(w)\in X_1$ is non-fixed by the action of μ_{p^2}/μ_p.*

Proof. Let $\mathcal{O}_{X,w}=B=\bigoplus_{i\in\mathbb{Z}/p^2\mathbb{Z}}B_i$ be the corresponding decomposition. Since w is non-fixed by μ_p, there exists $y\in\mathfrak{m}_w\subset B$ with $1+y\in\bigoplus_{i\equiv 1(\mathrm{mod}\ p)}B_i$ (Lemma 2.9). Then, since $y^p\in\mathfrak{m}_{\pi_1(w)}\subset\mathcal{O}_{X_1,\pi_1(w)}$ satisfies $1+y^p=(1+y)^p\in B_p$, we conclude by Proposition 2.8 that $\pi_1(w)$ is non-fixed by μ_{p^2}/μ_p. ∎

https://doi.org/10.1017/nmj.2022.20 Published online by Cambridge University Press

Table 5. Non-fixed μ_n-actions on RDPs (listed in order of appearance in the proof).

p	n	p^e	p^f	$\mathrm{wt}(x,y,1+z)$	Equation	RDP
Any	$p^e r$	p^e	p^f	$1,-1,p^f r$	$xy+z^{p^{e-f}m}$	$A_{p^{e-f}m-1}$
$\neq 2$	$2p^e$	p^e	1	$0,p^e,2$	$x^2+y^2+z^{p^e m}$	$A_{p^e m-1}$
5	30	5	1	$15,10,6$	$x^2+y^3+z^5$	E_8^0
3	12	3	1	$3,6,4$	$x^2+y^3+yz^3$	E_7^0
3	12	3	1	$6k,3,4$	$x^2+z^3+y^4$	E_6^0
3	6	3	1	$3,3k,2$	$x^2+z^3+y^4$	E_6^0
3	30	3	1	$15,6,10$	$x^2+z^3+y^5$	E_8^0
3	18	9	3	$3,2,6$	$x^2+y^3+z^3(1+z)$	E_6^0
2	$4m-2$	2	1	$2m,2,2m-1$	$x^2+yz^2+xy^m$	D_{2m+1}^0
2	$4(m-1)$	(*)	2^{e-1}	$1,2,2(m-1)$	$x^2+yz^2+y^m(1+z)$	D_{2m}^0
2	6	2	1	$0,2,3$	$x^2+xz^2+y^3$	E_6^0
2	24	8	2	$3,2,6$	$x^2+y^3+z^4(1+z)$	E_8^0
2	12	4	2	$3,2,6$	$x^2+y^3+z^4(1+z)$	E_8^0
2	$4m-2$	2	1	$2m-2,2,2m-1$	$z^2+x^2y+xy^m$	D_{2m}^0
2	$4m$	(*)	2^{e-1}	$1,-2,2m$	$z^2+x^2y+y^m(1+z)$	D_{2m+1}^0
2	6	2	1	$2,2k,3$	$z^2+x^3+y^3$	D_4^0
2	18	2	1	$6,4,9$	$z^2+x^3+xy^3$	E_7^0
2	30	2	1	$10,6,15$	$z^2+x^3+y^5$	E_8^0
2	12	4	2	$3,2,6$	$x^2+y^3+z^2(1+z)$	D_4^0
2	4	4	2	$1,0,2$	$x^2(1+y)(1+z)+z^2+y^{2m+1}$	D_{4m}^0
2	8	8	4	$2,1,4$	$x^2+z^2(1+z)+xy^2$	D_5^0
2	12	4	2	$3,1,6$	$x^2+z^2(1+z)+xy^3$	E_7^2
2	20	4	2	$5,2,10$	$x^2+z^2(1+z)+y^5$	E_8^0
2	2^e $(e\geq 2)$	2^e	2	$0,-1,2$	$x^2+y^2(1+z)+xz^{2^{e-1}m}$	$D_{2^e m}^0$

(*): In the cases where $n=4(m-1)$ or $n=4m$, p^e is the highest power of $p=2$ dividing n.

Suppose X is a scheme equipped with a μ_n-action, $n=p^e r$ with $p\nmid r$, and $w\in X$ is a closed point fixed by $\mu_r\subset\mu_n$. Let f be the maximal integer with $0\leq f\leq e$ such that the subgroup scheme $\mu_{p^f}\subset\mu_{p^e}\subset\mu_n$ fixes w. We say that $\mu_{p^f r}$ is the stabilizer of w and denote it by $\mathrm{Stab}(w)$.

Proposition 4.12. *Let $\mathcal{O}_{X,w}$, together with a μ_n-action, be as in the beginning of §4, and assume w is an RDP. Write $n=p^e r$ with $p\nmid r$, and $\mathrm{Stab}(w)=\mu_{p^f r}$. Suppose $\mathrm{Stab}(w)\subsetneq\mu_n$ (hence $f<e$, in particular $e\geq 1$). Then there exist $x,y,z\in\mathfrak{m}$ generating $\mathfrak{m}$, with $x,y,1+z$ homogeneous such that, up to replacing r with a multiple and up to $\mathrm{Aut}(\mu_n)=(\mathbb{Z}/n\mathbb{Z})^*$, the weights and the type of singularity are as in Table 5.*

In this case, we do not pursue the exact equation, and the equations in Table 5 are merely examples.

Proof. We first show that there exist elements $x,y,z\in\mathfrak{m}$ generating $\mathfrak{m}$ and satisfying $\mathrm{wt}(x,y,1+z)=(a,b,c)$ with $c=n/p^{e-f}=p^f r$. Since the stabilizer of the μ_n-action is $\mu_{p^f r}\subsetneq\mu_n$, there exist $i\in\mathbb{Z}/n\mathbb{Z}$ and a homogeneous element t of weight i such that $t\notin\mathfrak{m}$ and $\gcd\{i,n\}=p^f r$. We may assume $t\equiv 1(\mathrm{mod}\ \mathfrak{m})$, and then $z':=-1+t\in\mathfrak{m}$ satisfies $\mathrm{wt}(1+z')=i$. Take an integer q such that $qi=p^f r$ (in $\mathbb{Z}/n\mathbb{Z}$), and then $1+z:=(1+z')^q$ satisfies $z\in\mathfrak{m}$ and $\mathrm{wt}(1+z)=p^f r$. Now, take $x'^{(1)},x'^{(2)},x'^{(3)}$ generating $\mathfrak{m}$. We may assume each $x'^{(h)}$ is homogeneous with respect to $\mu_{p^f r}$, that is, there exists $i^{(1)},i^{(2)},i^{(3)}\in\mathbb{Z}/n\mathbb{Z}$ such that

https://doi.org/10.1017/nmj.2022.20 Published online by Cambridge University Press

$x'^{(h)} = \sum_{j=0}^{p^{e-f}-1} x'^{(h,j)}$ with $\mathrm{wt}(x'^{(h,j)}) = i^{(h)} + jp^f r$. Let $x^{(h)} := \sum_{j=0}^{p^{e-f}-1}(1+z)^{-j}x'^{(h,j)}$. Then $\mathrm{wt}(x^{(h)}) = i^{(h)}$ and, since $x^{(h)} \equiv x'^{(h)} (\mathrm{mod}\ (z))$, the elements $x^{(1)}, x^{(2)}, x^{(3)}, z$ generate $\mathfrak{m}$. We can omit one of these four elements, which cannot be z since $\mathfrak{m}$ is not generated by homogeneous elements.

In this proof, by a *monomial*, we mean a polynomial of the form $x^i y^j z^{p^{e-f}l}(1+z)^m$ with $0 \le m < p^{e-f}$. Any polynomial (resp. formal power series) is uniquely expressed as a finite (resp. possibly infinite) sum of monomials with k-coefficients, and we say that a polynomial or a formal power series *has* a monomial if its coefficient is nonzero. Expressions such as $F = x^2 + y^3(1+z) + \cdots$ will indicate that F has these monomials. However, when we say of degree m part F_m of F, this is understood with respect to the usual monomials $x^i y^j z^l$.

Assume the degree 2 part F_2 of F is either irreducible or the product of two distinct homogeneous linear factors. Then we may assume $F = xy(1+z)^i + \cdots$, and F must have $z^{m'}(1+z)^j$, we may assume $i = j = 0$ by replacing y and F, and it is $A_{m'-1}$ with $m' = p^{e-f}m$.

Assume F_2 is the product of two distinct nonhomogeneous linear factors. Then we have $p \ne 2$ and we may assume $F = x^2(1+z)^i + y^2(1+z)^j + z^{m'}(1+z)^k + \cdots$ and $b \equiv a + r/2$ (mod r). We may assume $k = 0$ by replacing F with $(1+z)^{-k}F$ and $i = j = 0$ by replacing x and y with $x(1+z)^{i(p^e+1)/2}$ and $y(1+z)^{j(p^e+1)/2}$. We have $2a = 2b = 0$ and then we have $f = 0$ and $r = 2$. (Otherwise, a, b, c cannot generate $\mathbb{Z}/p^e r\mathbb{Z}$.) It is $A_{m'-1}$ with $m' = p^e m$, and we may assume $(a, b) = (0, n/2)$.

Assume $p \ge 5$ and F_2 is a square. We may assume $F_2 = x^2$, $F_3 = y^3$, $F = x^2 + y^3 + \cdots$, and then F must have $z^5(1+z)^i$ and we have $p = 5$, it is E_8^0, and we may assume $i = 0$. By $2a = 3b = 0$, we have $n \mid 30$ and we may assume $a = 15$ and $b = 10$.

Assume $p = 3$ and F_2 is a square. We may assume $F = x^2 + \cdots$. We may assume F_3 mod (x) is either y^3, z^3, or $y^3 + z^3$. If $F = x^2 + y^3 + \cdots$, then F must have yz^3 and it is E_7^0. If $F = x^2 + z^3 + \cdots$, then F must have y^4 or y^5 and it is E_6^0 or E_8^0. If $F = x^2 + y^3 + z^3(1+z)^i + \cdots$, then we may assume $i = 1$ (if $i = 0$, then by replacing z with $y(1+z)^l + z$, we reduce this case to the previous case) and then it is E_6^0.

Assume $p = 2$ and F_2 is a square. We may assume F_2 is x^2, z^2, $x^2 + z^2$, $x^2 + y^2$, or $x^2 + y^2 + z^2$.

Assume $F_2 = x^2$. We may assume $F = x^2 + \cdots$. Then F must have $yz^2(1+z)^i$ or $y^3(1+z)^j$. If F has $yz^2(1+z)^i$, then we may assume $i = 0$ and F must have $xy^m(1+z)^i$ $(m \ge 2)$ or $y^m(1+z)$ $(m \ge 3)$. In the former case, we may assume $i = 0$ (by replacing x with $x(1+z)^i$) and we have D_{2m+1}^0 with $(a, b, c) = (2m, 2, 2m-1)$, $n \mid 2(2m-1)$. In the latter case, we have D_{2m}^0 with $(a, b, c) = (1, 2, 2(m-1))$, $n \mid 4(m-1)$. Now, assume that F does not have $yz^2(1+z)^i$ and has $y^3(1+z)^j$. We may assume $j = 0$. Then F must have either $xz^2(1+z)^i$ or $z^4(1+z)^i$. If F has $xz^2(1+z)^i$, then we may assume $i = 0$ and then $F = x^2 + y^3 + xz^2 + \cdots$ is E_6^0, and we have $e = 1$, $f = 0$, $a = 0$, and $c = n/2$. If F does not have $xz^2(1+z)^i$ and has $z^4(1+z)^i$, then we may assume $i = 1$ and then $F = x^2 + y^3 + z^4(1+z) + \cdots$ is E_8^0, and we have $e - f \le 2$, $f = 1$, $(a, b, c) \equiv (1, 2/3, 2) (\mathrm{mod}\ 2^e)$.

Assume $F_2 = z^2$. We have $e - f = 1$. Suppose F has $x^2y(1+z)^i$. We may assume $i = 0$. We may assume F does not have monomials z^2M $(M \ne 1)$ nor x^2M $(M \ne 1, y)$ of low degree (by replacing F or y with $(1+M)^{-1}F$ or $y + M$, respectively). F must have xy^m or $y^m(1+z)$ and then it is D_{2m}^0 or D_{2m+1}^0. Now, suppose F does not have $x^2y(1+z)^i$ nor $xy^2(1+z)^i$ and has $x^3(1+z)^i$. We may assume $i = 0$. Then F must have $y^3(1+z)^j$ (D_4^0), $xy^3(1+z)^j$ (E_7^0), or $y^5(1+z)^j$ (E_8^0).

https://doi.org/10.1017/nmj.2022.20 Published online by Cambridge University Press

Assume $F_2 = x^2 + z^2$. We may assume $F = x^2 + z^2(1+z)^i + \cdots$. If $i = 0$, then by replacing z with $z + x$ or $z + x(1+z)$, we reduce this case to the previous case. Assume $i = 1$. F cannot have $x^3(1+z)^i$ nor $xz^2(1+z)^i$. (If F has $x^3(1+z)^i$, then we have $2a = c = 3a + ic$ and this implies $(2i+1)c = 0$, contradicting $c = n/p^{e-f}$. Other cases are similar.) We may assume that F does not have $yz^2(1+z)^i$. If F has $y^3(1+z)^i$, then F does not have $xy^2(1+z)^j$ and it is D_4^0. If F does not have $y^3(1+z)^i$ and has $x^2y(1+z)^i$, then F cannot have $xy^2(1+z)^j$, and F must have $y^{2m+1}(1+z)^j$, and it is D_{4m}^0. If F does not have $y^3(1+z)^i$ nor $x^2y(1+z)^i$ and has $xy^2(1+z)^i$, then it is D_5^0. If F does not have $y^3(1+z)^i$ nor $xy^2(1+z)^i$ nor $x^2y(1+z)^i$, then F must have $xy^3(1+z)^i$ or $y^5(1+z)^i$, and (we may assume $i = 0$ and) it is E_7^2 or E_8^0. (This E_7^2 is the only example of D_n^r or E_n^r with $r > 0$ in this proposition.)

Assume $F_2 = x^2 + y^2$. Write $F = x^2 + y^2(1+z)^j + \cdots$. If $j = 0$, then by replacing x with $x + y(1+z)^k$, we reduce to the $F_2 = x^2$ case. Suppose $j = 1$. We may assume F does not have monomials x^2M $(M \neq (1+z)^i)$ nor y^2M $(M \neq (1+z)^i)$ of low degree (by replacing F or z with $(1+M)^{-1}F$ or $z + M$, respectively). F must have $xz^{2m}(1+z)^k$ or $yz^{2m}(1+z)^k$, by symmetry, we may assume F has $xz^{2m}(1+z)^k$, we may assume $k = 0$, and then it is D_{4m}^0.

Assume $F_2 = x^2 + y^2 + z^2$. Write $F = x^2(1+z)^i + y^2(1+z)^j + z^2(1+z)^k + \cdots$, $i,j,k \in \{0,1\}$. If $i = j$, then we reduce this case to $F = x^2 + z^2$ case by replacing x with $x + y(1+z)^l$. If $i \neq j$, then either $i = k$ or $j = k$ and then we reduce this case to $F = x^2 + z^2$ case by replacing z with $z + x(1+z)^l$ or $z + y(1+z)^l$. ∎

Proposition 4.13. *Let $\mathcal{O}_{X,w}$, together with a μ_n-action, be as in the beginning of §4. Suppose $n = p^e$ with $p > 0$ and $e \geq 2$. Let $\mathrm{Stab}(w) = \mu_{p^f}$ $(0 \leq f \leq e)$. Suppose $f > 0$ and that the subgroup scheme $\mu_p \subset \mathrm{Stab}(w)$ acts symplectically. Then one of the following is true.*

- *w is $A_{p^{e-f}m-1}$ for some integer $m \geq 1$.*
- *w is E_7^2 and $(p^f, p^e) = (2,4)$.*
- *w is D_{2n+1}^{n-1} $(n \geq 2)$ or E_8^3, and $(p^f, p^e) = (4,4)$.*

Proof. Let $\mathcal{O}_{x,w} = B = \bigoplus_{i \in \mathbb{Z}/p^2\mathbb{Z}} B_i$ be the corresponding decomposition.

Assume w is a smooth point. Since μ_p acts symplectically, the maximal ideal $\mathfrak{m}$ is generated by two elements $x \in \bigoplus_{i \equiv a (\mathrm{mod}\ p)} B_i$ and $y \in \bigoplus_{i \equiv b (\mathrm{mod}\ p)} B_i$ for some $a, b \in \mathbb{Z}/p^e\mathbb{Z}$ with $a, b \not\equiv 0$ and $a + b \equiv 0 (\mathrm{mod}\ p)$. Since $a, b \not\equiv 0 (\mathrm{mod}\ p)$, we may assume moreover $x \in B_a$ and $y \in B_b$. Then w is fixed by the whole group scheme μ_{p^e} and hence $e = f$. This case is done (with $m = 1$: recall the convention that a smooth point is of type A_0).

Hereafter, we assume w is an RDP. Let $\varepsilon = 0$ if $e = f$ and $\varepsilon = 1$ if $e > f$. By arguing as in the beginning of the proof of Proposition 4.12 and by using Proposition 4.9, $\mathfrak{m}$ is generated by three elements x, y, z with $x \in B_a$, $y \in B_b$, and $\varepsilon + z \in B_c$, and we may assume $a \equiv -b \not\equiv 0$ (mod p), and if $e > f$ then we may moreover assume $c = p^f$.

If $e > f$, then, since μ_p acts symplectically, it follows (from the classification given in Proposition 4.12) that either the RDP is $A_{m'-1}$ and then we may assume $F = xy + z^{m'} + \cdots$ and hence $p^{e-f} \mid m'$, or the RDP is E_7^2 and $(p^f, p^e) = (2,4)$.

Hereafter, assume $e = f$. If the RDP is A_{m-1}, then there is nothing to prove. The remaining possibilities are given in Table 4 (Proposition 4.9) and in particular we have $p \leq 3$.

https://doi.org/10.1017/nmj.2022.20 Published online by Cambridge University Press

Table 6. Symplectic μ_n-actions on RDPs.

p	n	X		X/μ_n	Multiplicity
Any	Any	A_{m-1}	$(m \geq 1)$	A_{mn-1}	m
$\neq 2$	2	A_{m-1}	$(m \geq 4$ even$)$	$D_{m/2+2}$	2
$\neq 2$	4	A_{m-1}	$(m \geq 3$ odd$)$	D_{m+2}	1
$\neq 2$	2	D_{m+2}		D_{2m+2}	$m+1$
$\neq 2,3$	3	D_4		E_6	2
2	3	D_4^r	$(r=0,1)$	E_6^r	2
$\neq 2,3$	2	E_6		E_7	3
3	2	E_6^r	$(r=0,1)$	E_7^r	3
3	3	E_6^1		E_6^1	2
3	3	E_8^1		E_6^0	2
2	2	D_{2m}^r	$(1 \leq r \leq m-1)$	$D_{4r}^{(3r-m)^+}$	$2r$
2	2	$D_{2m}^{m-1}[*]$		$D_{m+2}^{\lfloor m/2 \rfloor}$	2
2	2	D_{2m+1}^r	$(1 \leq r \leq m-1)$	$D_{4r+2}^{(3r-m+1)^+}$	$2r+1$
2	2	E_7^2		D_5^0	2
2	2	E_7^3		E_7^3	3
2	2	E_8^3		E_7^2	3
2	4	D_{2m+1}^{m-1}		D_{2m+1}^{m-1}	1
2	4	E_8^3		D_5^0	1

Assume $p=3$ and the RDP is E_6^1 or E_8^1 (as in Proposition 4.9). We may assume that $F = z^2 + x^3 + y^3 + \cdots$ with $\mathrm{wt}(x,y,z) \equiv (1,-1,0) (\mathrm{mod}\ 3)$. Then F cannot be homogeneous since $\mathrm{wt}(x^3) \not\equiv \mathrm{wt}(y^3) (\mathrm{mod}\ 3^2)$.

Assume $p=2$ and the RDP is D_n or E_n. By the classification in Proposition 4.9, we have $(a,b,c) \equiv (1,1,0) (\mathrm{mod}\ 2)$, and $F_2 \notin kz^2$. If $(a,b,c) \equiv (1,\pm 1,0) (\mathrm{mod}\ 4)$, then $F \in (x,y)^2$ and F cannot define an RDP. If $(a,b,c) \equiv (1,1,2) (\mathrm{mod}\ 4)$ (then we may assume $F_2 = x^2$), or if $(a,b,c) \equiv (1,-1,2) (\mathrm{mod}\ 4)$ and $F_2 = x^2$, then $F \in (x^2, xyz, xy^3, z^3, z^2y^2, zy^4, y^6)$ and hence F cannot define an RDP. Hence, we may assume $(a,b,c) \equiv (1,-1,2) (\mathrm{mod}\ 4)$ and $F_2 = x^2 + y^2$, and hence $p^e = 4$. If F has xyz, then F must have z^{2n-1} for some $n \geq 2$, and then it is D_{2n+1}^{n-1}. If F does not have xyz, then F must have z^3 and F must also have x^3y or xy^3, and then it is E_8^3. ∎

Proposition 4.14. *Let $p \geq 0$. Let $\mathcal{O}_{X,w}$, together with a μ_n-action $(n>1)$, be as in the beginning of §4. Suppose the action fixes w and is symplectic. Then p, n, the type of singularity at w, and the quotient singularity are as in Table 6.*

Proof. If $p \nmid n$, then this is Remark 3.5 (Table 2). If $n = p$, then this is Proposition 4.9 (Table 4). If $n = p^e$ with $e \geq 2$, then by Proposition 4.13, the possibilities are D_{2n+1}^{n-1}, E_8^3, and A_{m-1} (with quotient A_{mn-1}). In the other cases, we conclude by comparing the tables of the tame case and the $n=p$ case. For example, E_6^r with $(p,n) = (3,6)$ is impossible since the μ_2-quotient E_7^r of E_6^r does not admit a symplectic μ_3-action. ∎

To a point w fixed by a μ_n-action, we define its multiplicity $m(w)$ inductively by:

- if w is a smooth point, then $m(w) = 1$, and
- if w is an RDP, then $m(w) = \sum_{w' \in \mathrm{Fix}(\mu_n \curvearrowright \mathrm{Bl}_w X)} m(w')$.

https://doi.org/10.1017/nmj.2022.20 Published online by Cambridge University Press

The multiplicity for each case is displayed in Table 6. We observe that $m(\pi_r(w)) = rm(w)$ for any divisor $r < n$ of n, where $\pi_r : X \to X/\mu_r$ is the quotient map and $m(\pi_r(w))$ is the multiplicity with respect to the $\mu_{n/r}$-action on X/μ_r.

4.4 μ_n-actions on smooth points

Let $B = \mathcal{O}_{X,w}$, together with a μ_n-action, be as in the beginning of §4, and assume it is smooth. Assume the μ_n-action fixes w and is symplectic at w. As shown in Lemma 4.8(1), there exists $j \in (\mathbb{Z}/n\mathbb{Z})^*$, unique up to sign, such that the maximal ideal of B is generated by two homogeneous elements of respective weights j and $-j$. (We say that the weights of the μ_n-action on the tangent space are j and $-j$.)

Now, let $\tilde{Y}$ be the minimal resolution of $Y = X/\mu_n$ at $\pi(w)$, and let $\pi' \colon X' = X \times_Y \tilde{Y} \to \tilde{Y}$. Let e_k ($k = 1, \dots, n-1$) be the exceptional curves of $\tilde{Y}$, ordered in a way that $e_k \cap e_{k'} \neq \emptyset$ if and only if $|k - k'| \leq 1$. The μ_n-action induces a decomposition $\pi'_* \mathcal{O}_{X'} = \bigoplus_{i \in \mathbb{Z}/n\mathbb{Z}} (\pi'_* \mathcal{O}_{X'})_i$. Let $I_i := \operatorname{Im}(((\pi'_* \mathcal{O}_{X'})_i)^{\otimes n} \to \mathcal{O}_{\tilde{Y}})$ for $i = 0, \dots, n-1$. Then I_i are described as follows. (Clearly, $I_0 = \mathcal{O}_{X'}$.)

LEMMA 4.15. *$(\pi'_* \mathcal{O}_{X'})_i$ and I_i are invertible sheaves. After possibly reversing the ordering of the exceptional curves, we have an equality*

$$I_i = \mathcal{O}_{\tilde{Y}}\Big(-\sum_k f_n\big((j^{-1} i \bmod n), k\big) e_k\Big),$$

for each $i = 1, \dots, n-1$.

Here, $j^{-1} i \bmod n$ denotes the unique integer $h \in \{0, 1, \dots, n-1\}$ satisfying $hj \equiv i (\bmod n)$, and the function $f_n : \{1, 2, \dots, n-1\}^2 \to \mathbb{Z}$ is defined as

$$f_n(h,k) = \begin{cases} hk, & (k \leq n-h), \\ (n-h)(n-k), & (k \geq n-h). \end{cases}$$

Proof. Straightforward. ∎

§5. Tame quotients of K3 surfaces and abelian surfaces

The following fact should be known to experts. (For example, if X is a K3 surface in characteristic 0 and G is symplectic and commutative, then this is a result of Nikulin.) We give a proof since we could not find a complete reference (covering all characteristics).

THEOREM 5.1. *Let X be either an abelian surface or an RDP K3 surface in characteristic $p \geq 0$, and G a finite group of order not divisible by p acting on X.*

If X is an RDP K3 surface and G is symplectic, then the quotient X/G is an RDP K3 surface.

If X is an abelian surface and G is symplectic, then X/G is either an abelian surface or an RDP K3 surface.

If X is an RDP K3 surface and G is non-symplectic, then X/G is either an RDP Enriques surface or a rational surface.

If X is an abelian surface and G is non-symplectic, then X/G is either an RDP Enriques surface, a (quasi-)hyperelliptic surface, a surface birational to a ruled surface, or a rational surface.

https://doi.org/10.1017/nmj.2022.20 Published online by Cambridge University Press

Proof. By Proposition 3.4, we may assume X is smooth. Let $\pi : X \to X/G = Y$ be the quotient morphism, and let $\tilde{Y} \to Y$ be the minimal resolution.

We have $b_1(\tilde{Y}) = b_1(Y) \le b_1(X)$, where $b_i(X) = \dim H^i_{\text{ét}}(X, \mathbb{Q}_l)$ is the ith l-adic Betti number for an auxiliary prime $l \neq \operatorname{char} k$. Hence, if X is a K3 surface (hence $b_1(X) = 0$), then $\tilde{Y}$ cannot be abelian, (quasi-)hyperelliptic, nor nonrational ruled.

First, suppose G acts non-symplectically. If some nontrivial $g \in G$ satisfy $\dim \operatorname{Fix}(g) = 1$, then by the usual ramification formula, K_Y has negative coefficients at the corresponding divisors of Y, and hence $\kappa(\tilde{Y}) = -\infty$. If Y has a non-RDP singularity, then $K_{\tilde{Y}}$ has negative coefficients at the corresponding exceptional curves, and hence $\kappa(\tilde{Y}) = -\infty$. In either case, $\tilde{Y}$ is either ruled or rational. Suppose neither is the case. Then K_Y is an RDP surface with numerically trivial K_Y. Since we have

$$H^0(Y^{\text{sm}}, \Omega^2_Y) = H^0(\pi^{-1}(Y^{\text{sm}}), \Omega^2_X)^G = H^0(X, \Omega^2_X)^G = 0,$$

K_Y is not trivial. Then, by the classification of such surfaces (see Table 1), $\tilde{Y}$ is either an Enriques surface or a (quasi-)hyperelliptic surface. This settles the non-symplectic case.

Now, suppose G acts symplectically. By Proposition 3.4, Y is an RDP surface with K_Y trivial. By the classification of surfaces with trivial canonical divisor, Y is an RDP K3 surface, an abelian surface, a non-classical RDP Enriques surface ($p = 2$), or a (quasi-)hyperelliptic surface ($p = 2, 3$). (Note that abelian and (quasi-)hyperelliptic surface admit no smooth rational curves.)

Since Y has only RDP singularities, we have $h^1(\tilde{Y}, \mathcal{O}_{\tilde{Y}}) = h^1(Y, \mathcal{O}_Y)$. Since $p \nmid |G|$, $(\mathcal{O}_X)^G \subset \mathcal{O}_X$ is a direct summand and hence we have $h^1(Y, \mathcal{O}_Y) = h^1(Y, (\pi_* \mathcal{O}_X)^G) \le h^1(Y, \pi_* \mathcal{O}_X) = h^1(X, \mathcal{O}_X)$. Therefore, if X is a K3 surface, then Y cannot be a nonclassical RDP Enriques surface.

It remains to show that if X is an abelian surface, then Y cannot be a (quasi-)hyperelliptic surface. (If $p \neq 2, 3$, then this is clear since (quasi-)hyperelliptic surfaces always have nontrivial canonical divisor.) If X is an abelian surface and Y is a (quasi-)hyperelliptic surface, then since a (quasi-)hyperelliptic surface admits no smooth rational curves, no element of $G \setminus \{1\}$ has fixed points. It suffices to show that any fixed-point-free symplectic automorphism g of an abelian surface X of finite order not divisible by p is a translation, since the quotient of an abelian variety by a finite group of translations is an abelian variety.

Suppose g is such an automorphism. Endow X with a group variety structure (i.e., choose an origin) and write $g(x) = h(x) + a$ with h an automorphism of the group variety (i.e., h fixes the origin) and a a point. Then h is symplectic (since g and the translation by a are symplectic) and of finite order dividing $\operatorname{ord}(g)$, since $x = g^{\operatorname{ord}(g)}(x) = h^{\operatorname{ord}(g)}(x) + (h^{\operatorname{ord}(g)-1}(a) + \cdots + a)$. If $h = \mathrm{id}$, then g is a translation. Suppose $h \neq \mathrm{id}$. Then h acts on the tangent space of each fixed point via $\mathrm{SL}_2(k)$ (since h is symplectic and of finite order not divisible by p), and hence $\operatorname{Fix}(h)$ is isolated. Hence, $h - \mathrm{id}$ has finite kernel and hence is surjective. Let x be a point with $h(x) - x = -a$. Then $g(x) = x$, contradiction. ∎

§6. μ_p-quotients of RDP K3 surfaces and abelian surfaces

The following theorems are the μ_p-analogue of Theorem 5.1.

THEOREM 6.1. *The quotient of an RDP K3 surface by a symplectic μ_p-action is again an RDP K3 surface.*

https://doi.org/10.1017/nmj.2022.20 Published online by Cambridge University Press

THEOREM 6.2. *The quotient X/μ_p of an RDP K3 surface by a non-symplectic action of μ_p is either a rational surface (possibly with non-RDP singularities) or an RDP Enriques surface. The quotient is an RDP Enriques surface if and only if the action is fixed-point-free, and this can happen only if $p=2$.*

THEOREM 6.3. *A μ_{p^e}-action on an abelian variety is always symplectic, in the sense that the one-dimensional space of top differential forms is of weight 0, and the quotient is again an abelian surface.*

Proof of Theorem 6.3. It suffices to consider μ_p-actions. A μ_p-action on A corresponds [GP, Th. VII.7.2(ii)] to an element $v \in H^0(A,T) \cong T_0 A$ satisfying $v^p = v$, and hence the action can be identified with the translation action by a subgroup scheme of A. Then the quotient is again an abelian variety, and hence this action is symplectic by Proposition 2.10. ∎

Proof of Theorem 6.1. By Theorem 4.6(1,2), $Y = X/\mu_p$ is an RDP surface. Then, by Proposition 2.10, K_Y is trivial, and by the classification of surfaces, Y is either an RDP K3 surface, an abelian surface, a nonclassical RDP Enriques surface, or a (quasi-)hyperelliptic surface.

Since $\pi : X \to Y$ is purely inseparable, we have $\dim H^i_{\text{ét}}(X,\mathbb{Q}_l) = \dim H^i_{\text{ét}}(Y,\mathbb{Q}_l)$, in particular $b_1(Y) = b_1(X) = 0$ (where $b_i = \dim H^i_{\text{ét}}$ is the ith Betti number). Since Y is an RDP surface and since $\mathcal{O}_Y = (\pi_* \mathcal{O}_X)_0$ is a direct summand of $\pi_* \mathcal{O}_X$, we have $h^1(\tilde{Y},\mathcal{O}_{\tilde{Y}}) = h^1(Y,\mathcal{O}_Y) \le h^1(Y,\pi_*\mathcal{O}_X) = h^1(X,\mathcal{O}_X) = 0$. Hence, Y is an RDP K3 surface. ∎

Proof of Theorem 6.2. Let X be an RDP K3 surface equipped with a nontrivial non-symplectic μ_p-action. Let $Y = X/\mu_p$. We have $b_1(Y) = b_1(X) = 0$, and hence as in the tame case (§5), the minimal resolution $\tilde{Y}$ of Y cannot be abelian, (quasi-)hyperelliptic, nor non-rational ruled.

If D has non-isolated fixed points, then by the Rudakov–Shafarevich formula $K_X \sim (p-1)(D) + \pi^* K_Y$ [RS, Cor. 1 to Prop. 3], or by Proposition 2.10, $\kappa(\tilde{Y}) = -\infty$. If D has an isolated fixed point $w \in X$, then by Theorem 4.6(3), $\pi(w) \in Y$ is a non-RDP singularity, and then hence $\kappa(\tilde{Y}) = -\infty$. In either case, Y is a rational surface.

Now, assume D is fixed-point-free. Then, by Theorem 4.6(1), Y is an RDP surface and, by the Rudakov–Shafarevich formula, K_Y is torsion. Moreover, it follows from Proposition 2.10 that the space $H^0(Y^{\mathrm{sm}},(\Omega^2)^{\otimes n})$ is 0 if $0<n<p$ and is generated by a nonvanishing multicanonical form if $n=p$. Thus, $K_{\tilde{Y}}$ is nonzero and p-torsion. By the classification of surfaces, it follows that Y is an RDP Enriques surface. Then, since $2K_Y \sim 0$ and $H^0(Y^{\mathrm{sm}},\mathcal{O}(nK_Y)) = 0$ for $0<n<p$, we have $p=2$. ∎

There is also the following relation with the height of K3 surfaces. The height is an invariant of a K3 surface in positive characteristic which is either ∞ or an integer in $\{1,\dots,10\}$. See §8 for more details.

COROLLARY 6.4. *Let X be an RDP K3 surface in characteristic p equipped with a nontrivial μ_p-action. If X is of finite height, then the action is symplectic and the quotient is an RDP K3 surface.*

Proof. We assume that the action is non-symplectic and show that then X is not of finite height. By Theorem 6.2, the quotient $Y = X/\mu_p$ is a rational surface or an RDP

Enriques surface. Hence, X admits a purely inseparable finite morphism $Y^{(1/p)} \to X$ from a rational surface or an RDP Enriques surface. Hence, $H^2_{\text{ét}}(X, \mathbb{Q}_l)$, which is isomorphic to $H^2(Y^{(1/p)}, \mathbb{Q}_l)$, is generated by algebraic cycles, which is impossible if X is of finite height by Lemma 8.3(2). ∎

REMARK 6.5. In a subsequent paper [Mat4, Th. 1.3], we prove that also the converse holds, and we moreover determine the height in terms of the singularities of X and Y.

We call a proper birational morphism $X' \to X$ between RDP surfaces to be a *partial resolution* if it is dominated by the minimal resolution $\tilde{X}$ of X. The following proposition often enables us to reduce assertions on μ_p-actions to simpler cases.

PROPOSITION 6.6. *Let X be an RDP surface equipped with a μ_p-action.*

1. *Among partial resolutions of X to which the μ_p-action extends, there exists a unique maximal one, which we call the maximal partial resolution of X.*
2. *A partial resolution $X' \to X$ is maximal if and only if it satisfies the property* $\operatorname{Sing}(X') \cap \pi^{-1}(\operatorname{Sing}(X'/\mu_p)) = \emptyset$.
3. *The action on X is fixed-point-free if and only if the action on the maximal partial resolution is fixed-point-free.*
4. *Suppose X is an RDP K3 surface. The action on X is symplectic if and only if the action on the maximal partial resolution is symplectic.*

We say that an RDP surface X equipped with a μ_p-action is *maximal* if it is the maximal partial resolution of itself.

Proof. Note that isomorphism classes of partial resolutions are in one-to-one correspondence to subsets of the set of exceptional curves of $\tilde{X} \to X$.

(1) Let X' be a partial resolution. Let $\tilde{D}$ and D' be the rational derivations induced by D on $\tilde{X}$ and X'. Clearly, D' is regular and thus corresponds to a μ_p-action if and only if the coefficient of (D') for each exceptional curve of $X' \to X$ is nonnegative. Therefore, the contraction of all exceptional curves on $\tilde{X}$ with negative coefficients in $(\tilde{D})$ is the maximal partial resolution.

(2) First, suppose X' does not satisfy the property. There is an RDP $w \in X'$ such that $\pi(w)$ is not smooth. If w is a fixed RDP, then let $X'_1 = \operatorname{Bl}_w X'$. If w is a non-fixed RDP, then let $X'_1 = X' \times_{Y'} Y'_1$, where Y' is the μ_p-quotient of X' and $Y'_1 \to Y'$ is the minimal resolution at $\pi(w)$. Then the μ_p-action extends to X'_1 (by Proposition 2.8 in the former case, clear in the latter case) and X'_1 is a partial resolution of X' (clear in the former case, by Theorem 4.6(1) in the latter case). Thus, X' is not maximal.

Conversely, suppose X' satisfies the property. Let $w \in \operatorname{Sing}(X')$. To show that X' is maximal, it suffices to show that every exceptional curve of $\tilde{X}$ above w appears in $(\tilde{D})$ with negative coefficient, since then it must be contracted in the maximal partial resolution. This follows from the Rudakov–Shafarevich formula $K_{\tilde{X}} \sim (p-1)(\tilde{D}) + \pi^* K_{\tilde{Y}}$ [RS, Cor. 1 to Prop. 3], where $\tilde{Y} = \tilde{X}/\mu_p$: for each exceptional curve, its coefficient in $K_{\tilde{X}}$ (resp. $\pi^* K_{\tilde{Y}}$) is 0 (resp. positive) since $\tilde{X}$ is the minimal resolution of the RDP w (resp. since $\pi(w)$ is a smooth point). Alternatively, one can use the explicit computation of $(\tilde{D})$ given in Lemma 6.7. (w is non-fixed since $\pi(w)$ is a smooth point [Theorem 4.6(2,3)].)

We also observe that the maximal partial resolution of X can be constructed by repeatedly applying the procedure of constructing X'_1 from X'. Indeed, the total index of RDPs strictly decreases through the procedure.

https://doi.org/10.1017/nmj.2022.20 Published online by Cambridge University Press

(3) It suffices to show that the procedure above preserves the (non-)fixed-point-freeness. If w is a non-fixed RDP, then there is no fixed point above w by the functoriality of the fixed-point scheme. If w is a fixed RDP, then there is a fixed point above w by Corollary 2.12.

(4) The spaces of 2-forms in question are naturally isomorphic. ∎

LEMMA 6.7. *Let X be an RDP surface, and let D be a derivation of multiplicative type. Let $w \in X$ be a non-fixed RDP of index n, and suppose the image of w is a smooth point. Let $\tilde{X}$ be the minimal resolution at w, and let $\tilde{D}$ be the induced rational derivation on $\tilde{X}$. Let $(\tilde{D})$ and $\langle \tilde{D} \rangle$ be the divisorial and isolated fixed locus of $\tilde{D}$ above w. Then every exceptional curve of $\tilde{X}$ above w appears in $(\tilde{D})$ with negative coefficient, and we have $\deg\langle \tilde{D} \rangle = \frac{p-2}{p-1}n$ and $(\tilde{D})^2 = -\frac{2}{p-1}n$.*

Proof. We compute $(\tilde{D})$ and $\langle \tilde{D} \rangle$ by a straightforward calculation using the classification of non-fixed RDPs given in Proposition 4.7. See [Mat3, Lem. 3.11] for the result. Then the assertions follow. ∎

PROPOSITION 6.8. *Suppose each of X and Y is either an RDP K3 surface or an RDP Enriques surface. Let $\pi \colon X \to Y$ be a μ_p-quotient morphism. Suppose X is maximal with respect to the μ_p-action. Then the total index N_1 of non-fixed RDPs and the number N_2 of fixed points on X are as follows:*

$$(N_1, N_2) = \begin{cases} (24\frac{p-1}{p+1}, 24\frac{1}{p+1}), & \text{if } (\tilde{X}, \tilde{Y}) \text{ is } (\mathrm{K3}, \mathrm{K3}), \\ (12\frac{2p-1}{p+1}, 12\frac{p-2}{p^2-1}) = (12, 0), & \text{if } (\tilde{X}, \tilde{Y}) \text{ is } (\mathrm{K3}, \mathrm{Enr}), \\ (12\frac{p-2}{p+1}, 12\frac{2p-1}{p^2-1}) = (0, 12), & \text{if } (\tilde{X}, \tilde{Y}) \text{ is } (\mathrm{Enr}, \mathrm{K3}), \\ (12\frac{p-1}{p+1}, 12\frac{1}{p+1}), & \text{if } (\tilde{X}, \tilde{Y}) \text{ is } (\mathrm{Enr}, \mathrm{Enr}). \end{cases}$$

In the cases where $(\tilde{X}, \tilde{Y})$ is $(\mathrm{K3}, \mathrm{Enr})$ or $(\mathrm{Enr}, \mathrm{K3})$, only $p = 2$ is possible.

Proof. Let D be the corresponding derivation of multiplicative type. Since $\tilde{Y}$ is of Kodaira dimension 0, $\mathrm{Fix}(D)$ consists only of isolated points (possibly none).

Define the rational derivation $\tilde{D}$ on $\tilde{X}$ as in Lemma 6.7. Since the fixed locus of $\tilde{D}$ consists of those above non-fixed RDPs on X and the 0-cycle of fixed points on X, by Lemma 6.7, we have

$$(\tilde{D})^2 = -\frac{2}{p-1}N_1, \quad \deg\langle \tilde{D} \rangle = N_2 + \frac{p-2}{p-1}N_1,$$

and by the Katsura–Takeda formula (Proposition 2.13), we have

$$\deg c_2(\tilde{X}) = (N_2 + \frac{p-2}{p-1}N_1) + 0 + \frac{2}{p-1}N_1 = N_2 + \frac{p}{p-1}N_1.$$

On the other hand, we have $\dim H^2_{\text{ét}}(X, \mathbb{Q}_l) = \dim H^2_{\text{ét}}(\tilde{X}, \mathbb{Q}_l) - N_1 = b_2(\tilde{X}) - N_1$. Since Y is an RDP surface whose RDPs are precisely the images (which are A_{p-1}) of the fixed smooth points of X, we have $\dim H^2_{\text{ét}}(Y, \mathbb{Q}_l) = \dim H^2_{\text{ét}}(\tilde{Y}, \mathbb{Q}_l) - (p-1)N_2 = b_2(\tilde{Y}) - (p-1)N_2$. Since $X \to Y$ is purely inseparable, we have $\dim H^2_{\text{ét}}(X, \mathbb{Q}_l) = \dim H^2_{\text{ét}}(Y, \mathbb{Q}_l)$. Therefore, we have $(p-1)N_2 - N_1 = b_2(\tilde{Y}) - b_2(\tilde{X})$.

Combining the two equalities, we determine (N_1, N_2) in terms of p. In two cases, only $p = 2$ is possible since $12\frac{p-2}{p^2-1}$ or $12\frac{2p-1}{p^2-1}$ must be an integer. ∎

https://doi.org/10.1017/nmj.2022.20 Published online by Cambridge University Press

§7. Possible orders of symplectic μ_n-actions on RDP K3 surfaces

The following theorem is again parallel to the case of automorphisms of finite tame order, but the proof (for μ_p-actions) is quite different.

THEOREM 7.1. *Let X be an RDP K3 surface in characteristic p equipped with a symplectic μ_n-action ($n > 1$, divisible by p or not). Then:*

1. $n \le 8$.
2. *The number of fixed points, counted with the multiplicities defined after Proposition 4.14, depends only on n and is as in Table 7.*
3. *Assume all fixed points are smooth points. Then the decomposition of $\bigoplus_{w \in \mathrm{Fix}(\mu_n)} T_w^* X$ with respect to the μ_n-action is concentrated on the subset $(\mathbb{Z}/n\mathbb{Z})^* \subset \mathbb{Z}/n\mathbb{Z}$, and for every $i \in (\mathbb{Z}/n\mathbb{Z})^*$, the ith summand has dimension as in Table 7.*

If $p \nmid n$ (resp. if $n = p$), then assertion (3) means that every primitive nth root of 1 (resp. every element of $\mathbb{F}_p^*$) appears as an eigenvalue of a fixed generator of $\mu_n \cong \mathbb{Z}/n\mathbb{Z}$ (resp. of the corresponding derivation) on the space $\bigoplus_{w \in \mathrm{Fix}(\mu_n)} T_w^* X$ with equal multiplicity.

For each p and each $n \le 8$, there indeed exists an RDP K3 surface equipped with a symplectic μ_n-action in characteristic p. See §9.1 for explicit examples.

Proof of Theorem 7.1 for the case $p \nmid n$. We may assume X is smooth.

Assertions (1) and (2) are proved by Nikulin [Ni, §5] ($p = 0$) and Dolgachev–Keum [DK2, Th. 3.3] ($p > 0$). (Both proofs overlooked the case $n = 14$, but their arguments for the nonexistence of the case $n = 15$ apply to case $n = 14$.)

(3) (If $p = 0$, then this follows from the argument in [Mu, Prop. 1.2]. We give another proof, applicable to all $p \ge 0$.)

Let $w \in X$, and let $\mu_r = \mathrm{Stab}(w) \subset \mu_n$ be its stabilizer group. Suppose the decomposition of $T_w^* X$ $(= \mathfrak{m}_w / \mathfrak{m}_w^2)$ with respect to the μ_r-action is concentrated on two (not necessarily distinct) weights $j_1, j_2 \in \mathbb{Z}/r\mathbb{Z}$. Then since the action on $\Omega^2_{X,w} \cong \bigwedge^2 T_w^* X$ is trivial, we have $j_1 + j_2 = 0$. We have $j_1, j_2 \in (\mathbb{Z}/r\mathbb{Z})^*$, since if $\gcd(j_1, r) = \gcd(j_2, r) = r' > 1$, then $\mu_{r'}$ acts trivially, contradicting the assumption that the action is faithful. This already proves the assertion if $n = 2, 3, 4, 6$, since up to sign there is only one element in $(\mathbb{Z}/n\mathbb{Z})^*$.

For each divisor $r \ne 1$ of n and for each $0 \le j \le \lfloor r/2 \rfloor$, let $S_{r,j}$ be the set of the points $w \in X$ with $\mathrm{Stab}(w) = \mu_r$ and with μ_r acting on $T_w^* X$ by weights j and $-j$. Then $S_{r,j}$ is a finite set and it is empty if $j \notin (\mathbb{Z}/r\mathbb{Z})^*$. Let $\tilde{S}_{r,j} = S_{r,j}/\mu_n$ be the set of μ_n-orbits of points of $S_{r,j}$. Let $N_{r,j} = |S_{r,j}|$ and $\tilde{N}_{r,j} = |\tilde{S}_{r,j}| = N_{r,j}/(n/r)$.

Let $\rho : \tilde{Y} \to Y = X/\mu_n$ be the minimal resolution (then $\tilde{Y}$ is a smooth K3 surface), and let $\pi' \colon X' = X \times_Y \tilde{Y} \to \tilde{Y}$. Let $\pi'_* \mathcal{O}_{X'} = \bigoplus_{i \in \mathbb{Z}/n\mathbb{Z}} (\pi'_* \mathcal{O}_{X'})_i$ be the decomposition induced by the μ_n-action. Then, by Lemma 4.15, $(\pi'_* \mathcal{O}_{X'})_i$ are invertible sheaves, and they are described as follows. For each $i \in \mathbb{Z}/n\mathbb{Z}$, let C_i be the corresponding class of Cartier divisors. For each orbit $w \in \tilde{S}_{r,j}$, its image $\pi(w)$ is an RDP of type A_{r-1}, and let $e_{w,k}$ $(k = 1, \ldots, r-1)$ be the

Table 7. Symplectic μ_n-actions on K3 surfaces.

n	2	3	4	5	6	7	8
Number of fixed points	8	6	4	4	2	3	2
Dimension of each summand	16	6	4	2	2	1	1

https://doi.org/10.1017/nmj.2022.20 Published online by Cambridge University Press

exceptional curves in $\tilde{Y}$ above $\pi(w)$, ordered in a way that $e_{w,k} \cap e_{w,k'} \neq \emptyset$ if and only if $|k-k'| \leq 1$. Then, after possibly reversing the ordering, we have a linear equivalence

$$-rC_i \sim \sum_{r|n, r\neq 1} \sum_{j=0}^{\lfloor r/2 \rfloor} \sum_{w \in \tilde{S}_{r,j}} \sum_{k=1}^{r-1} f_r((j^{-1}i \bmod r), k) e_{w,k}$$

(see Lemma 4.15 for the definition of the function $f_r : \{1, \dots, r-1\}^2 \to \mathbb{Z}$) for each $i \neq 0$.

Let m be any integer with $1 \leq m \leq r-1$. Using the equality

$$\left(\sum_{k=1}^{r-1} \frac{f_r(m,k)}{r} e_{w,k} \right) \cdot e_{w,k'} = \begin{cases} -1, & \text{if } k' = r-m, \\ 0, & \text{otherwise}, \end{cases}$$

we compute that

$$\left(\sum_{k=1}^{r-1} \frac{f_r(m,k)}{r} e_{w,k} \right)^2 = -\frac{m(r-m)}{r}.$$

Hence, we have

$$-C_i^2 = \sum_{r|n, r\neq 1} \sum_{j=0}^{\lfloor r/2 \rfloor} \tilde{N}_{r,j} \cdot \frac{(j^{-1}i \bmod r)(r - j^{-1}i \bmod r)}{r}$$

and this must belong to $2\mathbb{Z}$.

Assume $n=5$. Then we have $N_{5,j} = \tilde{N}_{5,j}$, $N_{5,1} + N_{5,2} = 4$, and

$$-C_1^2 = \frac{4}{5} N_{5,1} + \frac{6}{5} N_{5,2} \in 2\mathbb{Z}.$$

Hence, $(N_{5,1}, N_{5,2}) = (2,2)$.

Assume $n=7$. Then we have $N_{7,j} = \tilde{N}_{7,j}$, $N_{7,1} + N_{7,2} + N_{7,3} = 3$, and

$$-C_1^2 = \frac{6}{7} N_{7,1} + \frac{12}{7} N_{7,2} + \frac{10}{7} N_{7,3} \in 2\mathbb{Z}.$$

Hence, $(N_{7,1}, N_{7,2}, N_{7,3}) = (1,1,1)$.

Assume $n=8$. By assertion (2) for the cases $n = 2,4,8$, we have $\tilde{N}_{2,1} = 1$, $\tilde{N}_{4,1} = 1$, $\tilde{N}_{8,1} + \tilde{N}_{8,3} = 2$, and

$$-C_1^2 = \frac{1}{2} \tilde{N}_{2,1} + \frac{3}{4} \tilde{N}_{4,1} + \frac{7}{8} \tilde{N}_{8,1} + \frac{15}{8} \tilde{N}_{8,3} \in 2\mathbb{Z}.$$

Hence, $(N_{8,1}, N_{8,3}) = (\tilde{N}_{8,1}, \tilde{N}_{8,3}) = (1,1)$. ∎

REMARK 7.2. By above, we have $C_i^2 = -4$ for any $2 \leq n \leq 8$ and any $1 \leq i \leq n-1$. As we see below, this holds also if p divides n. This implies $\chi(\tilde{Y}, (\pi'_* \mathcal{O}_{X'})_i) = 0$ for $i \neq 0$. Then we obtain $\chi(Y, (\pi_* \mathcal{O}_X)_i) = 0$ for $i \neq 0$, since $\rho_*((\pi'_* \mathcal{O}_{X'})_i) = (\pi_* \mathcal{O}_{X'})_i$ and $R^q \rho_*((\pi'_* \mathcal{O}_{X'})_i) = 0$ for $q > 0$. This is finer than the equality

$$\sum_{i \neq 0} \chi(Y, (\pi_* \mathcal{O}_X)_i) = \chi(X, \mathcal{O}_X) - \chi(Y, \mathcal{O}_Y) = 2 - 2 = 0.$$

Proof of Theorem 7.1 for the case $n = p$. We may assume that X is maximal. (For assertion (2), the multiplicity is by definition compatible with blowups at fixed points.) This means that all fixed points are smooth points, and that the singularities of the quotient

https://doi.org/10.1017/nmj.2022.20 Published online by Cambridge University Press

surface Y are all A_{p-1} and are precisely the images of the fixed points. Let D be the corresponding derivation.

As in the previous case, let $\tilde{Y}$ be the minimal resolution of Y (hence a smooth K3 surface) and let $\pi' \colon X' = X \times_Y \tilde{Y} \to \tilde{Y}$. The sheaf $\pi'_* \mathcal{O}_{X'}$ admits a decomposition to invertible sheaves $(\pi'_* \mathcal{O}_{X'})_i$. For each $i \in \mathbb{Z}/p\mathbb{Z}$, let C_i be the corresponding class of Cartier divisor. As in the previous case, we have

$$-C_i^2 = \sum_{j=0}^{\lfloor p/2 \rfloor} N_{p,j} \cdot \frac{(j^{-1}i \bmod p)(p - j^{-1}i \bmod p)}{p}.$$

By Proposition 6.8, we have $N := \sum_j N_{p,j} = 24/(p+1)$. Hence, $p \in \{2,3,5,7,11,23\}$. If $p = 23$, then $N = 1$ and the exceptional curves generate a negative definite sublattice of rank $p-1 = 22$ of the indefinite lattice $\operatorname{Pic}(\tilde{Y})$ of rank ≤ 22, contradiction. If $p = 11$, then $N = 2$ and then C_i^2 (for any $i \in (\mathbb{Z}/p\mathbb{Z})^*$) cannot be an integer since the sum of two nonzero squares in $\mathbb{F}_{11}$ cannot be zero. Hence, we have $p \in \{2,3,5,7\}$, and we can determine the multiplicities of the weights as in the $p \nmid n$ case. ∎

Corollary 7.3. *Let X be an RDP K3 surface equipped with a symplectic μ_q-action with $q = 5,7$. Here, both $p = q$ and $p \neq q$ are allowed.*

- *If $q = 7$, then any fixed point is a smooth point.*
- *If $q = 5$, then any fixed point is a smooth point or an RDP of type A_1.*

Proof. Let $w \in X$ be a fixed point. By Proposition 4.14, w is of type A_{m-1} for some $m \geq 1$. Let $\pm i \in (\mathbb{Z}/q\mathbb{Z})^*$ be the nonzero weights of $\mathfrak{m}_w/\mathfrak{m}_w^2$ with respect to the μ_q-action. Let $\tilde{X}$ be the minimal resolution of X at w (to which the μ_q-action extends). One can calculate the local equation to show that all (smooth) fixed point of $\tilde{X}$ above w has weights $\pm i$. Since there are m such points, it follows from Theorem 7.1(3) that $m \leq 1$ if $q = 7$ and $m \leq 2$ if $q = 5$. ∎

Proof of Theorem 7.1 for the case $n = p^e$ $(e \geq 2)$. For each $0 \leq j \leq e$, let $\pi_j \colon X \to X_j = X/\mu_{p^j}$ be the quotient morphism by the subgroup scheme $\mu_{p^j} \subset \mu_{p^e}$, and for each $0 \leq j \leq e-1$, let D_j be the derivation on X_j corresponding to the action of $\mu_{p^{j+1}}/\mu_{p^j}$.

Let $w \in X$ be a μ_p-fixed point. Let $\mu_{p^f} = \operatorname{Stab}(w)$ (then $1 \leq f \leq e$). Then, by Remark 4.4, μ_{p^f} acts symplectically at w. By Proposition 4.13, either w is of type A_{m-1} for some $m \geq 1$ with $p^{e-f} \mid m$, or w is D_{2m+1}^{m-1} or E_7^2 or E_8^3 and $p^e = 4$. (Again, we use the convention that a smooth point is of type A_0.) Then since each D_j $(j < f)$ is symplectic at $\pi_j(w)$, we observe that $\pi_f(w) \in X_f$ is of type $A_{p^e m-1}$ or D_5^0 or D_{2m+1}^{m-1}. (Since X has a μ_p-fixed point and since X_f is an RDP K3 surface, this already implies $p^e - 1 < 22$.)

By Lemma 4.11, any preimage of any fixed point of D_j is again fixed. In other words, the fixed points of D_j on X_j are precisely the images of the $\mu_{p^{j+1}}$-fixed points on X.

For each $1 \leq f \leq e$, let $\tilde{S}_f \subset X$ be the points with stabilizer equal to μ_{p^f}. For each $w \in \tilde{S}_f$, let $m(w)$ be its multiplicity of w defined after Proposition 4.14 with respect to the μ_{p^f}-action. Then, by Propositions 4.13 and 4.14, we have $p^{e-f} \mid m(w)$. Let $M_f = \sum_{w \in \tilde{S}_f} m(w)$ for each $1 \leq f \leq e$, then by above $p^{e-f} \mid M_f$. Using the equality mentioned after Proposition 4.14 and assertion (2) for D_{e-1} and D_{e-2} on X_{e-1} and X_{e-2}, we obtain $p^{e-1} M_e = p^{e-2}(M_e + M_{e-1}) = 24/(p+1)$, and hence $M_e = 24/(p^{e-1}(p+1))$ and $M_{e-1}/p = 24(p-1)/(p^e(p+1))$. Since M_{e-1}/p is an integer, p^e divides 24. Therefore, $p^e = 2^2, 2^3$. Moreover, we obtain $M_f = p^{e-f} \cdot 24(p-1)/(p^e(p+1))$ $(1 \leq f \leq e-1)$ by applying assertion (2) to D_j $(0 \leq j \leq e-1)$.

https://doi.org/10.1017/nmj.2022.20 Published online by Cambridge University Press

Assertion (3) is trivial if $n=4$. Suppose $n=8$. For each $1\le f\le 3$ and $0\le j\le 2^f/2$, let $\tilde{S}_{2^f,j}$ be the set of the points with stabilizer μ_{2^f} and with primitive weights $\pm j\in(\mathbb{Z}/2^f\mathbb{Z})^*$, and let $\tilde{N}_{2^f,j}=(2^{e-f})^{-1}\sum_{w\in\tilde{S}_{2^f,j}}m(w)$. We have $\sum_j\tilde{N}_{2^f,j}=(2^{e-f})^{-1}M_f$ for each $1\le\ f\ \le\ e$. Then we again have $\tilde{N}_{2,1}=1$, $\tilde{N}_{4,1}=1$, $\tilde{N}_{8,1}+\tilde{N}_{8,3}=2$, and

$$-C_1^2=\frac{1}{2}\tilde{N}_{2,1}+\frac{3}{4}\tilde{N}_{4,1}+\frac{7}{8}\tilde{N}_{8,1}+\frac{15}{8}\tilde{N}_{8,3}\in 2\mathbb{Z}.$$

Hence, $(\tilde{N}_{8,1},\tilde{N}_{8,3})=(1,1)$. ∎

Proof of Theorem 7.1 for the remaining cases. First, we show that if $n=pq$ where q is a prime $\neq p$, then $n=6$. We have $\mu_n=\mu_p\times\mu_q\cong\mu_p\times\mathbb{Z}/q\mathbb{Z}$. We may assume that X is maximal with respect to the μ_p-action. Let $\pi_q\colon X\to X_q=X/\mu_q$ and $\pi_p\colon X\to X_p=X/\mu_p$. Note that $w\in X$ is fixed by the μ_p-action if and only if $\pi_q(w)\in X_q$ is fixed by the μ_p-action. Let a_1 and a_q be the number of μ_q-orbits of length 1 and q of μ_p-fixed points of X (which are all smooth by assumption). Then the μ_p-fixed points of X_q consists of a_1 points of type A_{q-1} and a_q smooth points. Applying assertion (2) to the μ_p-actions on X and X_q we have $a_1+qa_q=qa_1+a_q=24/(p+1)$. Therefore, $a_1=a_q=24/(p+1)(q+1)$ and hence $(a_1,\{p,q\})=(2,\{2,3\}),(1,\{2,7\}),(1,\{3,5\})$.

The cases $(a_1,\{p,q\})=(1,\{2,7\}),(1,\{3,5\})$ are impossible since letting $w\in X$ be the unique μ_{pq}-fixed point (which is a smooth point), if $pq=14$ then $\pi_2(w)\in X_2$ is a μ_7-fixed RDP of type A_1, and if $pq=15$ then $\pi_3(w)\in X_3$ is a μ_5-fixed RDP of type A_2, both contradicting Corollary 7.3.

Now, we consider general n. It remains to show that the cases $(p,n)=(2,12),(3,12)$ are impossible.

Assume $(p,n)=(3,12)$. As above, we may assume X is maximal with respect to the μ_3-action. There are exactly six μ_3-fixed points, all smooth. By the above argument for $(p,n)=(3,6)$, exactly two of them are μ_2-fixed, and among the images of these two points in X/μ_2, exactly one is (μ_4/μ_2)-fixed. This is impossible since non-(μ_4/μ_2)-fixed points in X/μ_2 come by pairs.

Now, assume $(p,n)=(2,12)$. As in the proof of the $n=p^e$ case (applied to the μ_4-action), let $\tilde{S}_1$ be the set of μ_2-fixed non-μ_4-fixed points, and then we have $M_1=\sum_{w\in\tilde{S}_1}m(w)=4$ and $2\mid m(w)$. Hence, $|\tilde{S}_1|$ is 1 or 2. Since the μ_3-action on X preserves this 1- or 2-point set $\tilde{S}_1$, it acts on $\tilde{S}_1$ trivially, and hence fixes at least four μ_2-fixed points (counted with multiplicity $m(w)$), contradicting the observation $a_1=2$ for μ_6-actions.

Assertion (3) for $n=6$ is trivial. ∎

§8. Possible orders of $\boldsymbol{\mu_n}$-actions on RDP K3 surfaces

Let $S_{\mathrm{cyc}}(p)$ (resp. $S_\mu(p)$) be the set of positive integers n for which there exists an RDP K3 surface equipped with an automorphism of order n (resp. a μ_n-action) in characteristic p. We clearly have $S_{\mathrm{cyc}}(0)=S_\mu(0)$ and $S_{\mathrm{cyc}}(p)^{p'}=S_\mu(p)^{p'}$, where $(-)^{p'}$ denotes the subset of prime-to-p elements.

Remark 8.1. Keum [Ke, Main Theorem] proved the following results on $S_{\mathrm{cyc}}(p)$. (This set is denoted by Ord_p in his paper.) The sets $S_{\mathrm{cyc}}(p)$ for $p\neq 2,3$ are given by

$$\begin{aligned}S_{\mathrm{cyc}}(0)&=\{n:\phi(n)\le 20\}\\&=\{1,\dots,22,24,25,26,27,28,30,32,33,34,36,38,40,42,44,48,50,54,60,66\},\end{aligned}$$

https://doi.org/10.1017/nmj.2022.20 Published online by Cambridge University Press

and

$$S_{\mathrm{cyc}}(p) = \begin{cases} S_{\mathrm{cyc}}(0), & \text{if } p = 7 \text{ or } p \geq 23, \\ S_{\mathrm{cyc}}(0) \setminus \{p, 2p\}, & \text{if } p = 13, 17, 19, \\ S_{\mathrm{cyc}}(0) \setminus \{44\}, & \text{if } p = 11, \\ S_{\mathrm{cyc}}(0) \setminus \{25, 50, 60\}, & \text{if } p = 5. \end{cases}$$

Moreover, $S_{\mathrm{cyc}}(p)^{p'} = S_{\mathrm{cyc}}(0)^{p'}$ for all $p \geq 2$. (The sets $S_{\mathrm{cyc}}(2)$ and $S_{\mathrm{cyc}}(3)$ are not determined.)

In this section, we determine the set $S_\mu(p)$ for all p.

THEOREM 8.2. *We have*

$$S_\mu(p) = \begin{cases} S_{\mathrm{cyc}}(0), & \textit{if } p \neq 2, 3, 5, 11, \\ S_{\mathrm{cyc}}(0) \setminus \{33, 66\}, & \textit{if } p = 11, \\ S_{\mathrm{cyc}}(0) \setminus \{25, 40, 50\}, & \textit{if } p = 5, \\ S_{\mathrm{cyc}}(0) \setminus \{27, 33, 48, 54, 66\}, & \textit{if } p = 3, \\ S_{\mathrm{cyc}}(0) \setminus \{34, 40, 44, 48, 50, 54, 66\}, & \textit{if } p = 2. \end{cases}$$

In particular, there exists an RDP K3 surface equipped with a nontrivial μ_p-action in characteristic p if and only if $p \leq 19$.

We need some preparations. The *height* h of a K3 surface X in characteristic $p > 0$, whose definition we do not recall here, is either ∞ or an integer in $\{1, \ldots, 10\}$, and X is called *supersingular* or *of finite height*, respectively. If $h < \infty$, then the inequality $\rho \leq 22 - 2h$ holds (Lemma 8.3(2)), where $\rho = \operatorname{rank} \operatorname{Pic}(X)$ is the Picard number. This implies that if $\rho \geq 21$, then X is supersingular.

In fact, the Tate conjecture for K3 surfaces, now a theorem, states conversely that if X is supersingular, then $\rho = 22$ (see Lemma 8.3(4) for references). In this case, the $\mathbb{Z}_p$-lattice $H^2_{\mathrm{crys}}(X/W(k))^{F=p}$ is isomorphic to $\operatorname{Pic}(X) \otimes \mathbb{Z}_p$ [O2, Cor. 1.6], and the discriminant group of $H^2_{\mathrm{crys}}(X/W(k))^{F=p}$ (isomorphic to the discriminant group of $\operatorname{Pic}(X)$) is of the form $(\mathbb{Z}/p\mathbb{Z})^{2\sigma_0}$ for an integer $\sigma_0 \in \{1, \ldots, 10\}$. This σ_0 is called the *Artin invariant* of X. Here, the *discriminant group* of a nondegenerate lattice (resp. nondegenerate $\mathbb{Z}_p$-lattice) L is defined to be the finite group L^*/L, where $L^* = \operatorname{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$ (resp. $L^* = \operatorname{Hom}_{\mathbb{Z}_p}(L, \mathbb{Z}_p)$) is the dual of L.

We define the *crystalline transcendental lattice* $T(X) = T_{\mathrm{crys}}(X) \subset H^2_{\mathrm{crys}}(X/W(k))$ to be the orthogonal complement of the image of $\operatorname{Pic}(X) \otimes W(k)$, where $W(k)$ is the ring of Witt vectors over k. We have $\rho + \operatorname{rank} T(X) = 22$.

We collect some facts.

LEMMA 8.3. *Let X be a K3 surface in characteristic $p > 0$.*

1. $\operatorname{Aut}(X)$ *acts on $H^2_{\mathrm{crys}}(X/W(k))$ and $H^2_{\text{ét}}(X, \mathbb{Z}_l)$ (for any prime $l \neq p$) faithfully, and the characteristic polynomial of any element is independent of the cohomology (and l) and has coefficients in $\mathbb{Q}$.*
2. *If X is of finite height h, then $\rho \leq 22 - 2h$.*
3. *Let $g \in \operatorname{Aut}(X)$ and suppose it acts on $H^0(X, \Omega^2)$ by a primitive Nth root of 1. If X is of finite height and $p \geq 3$, then the characteristic polynomial of g^* on $T_{\mathrm{crys}}(X)$ is*

the product of cyclotomic polynomials $\Phi_{Np^{e_i}}$ *with nonnegative integers* e_i*. In particular,* $\phi(N) \mid \operatorname{rank} T_{\mathrm{crys}}(X)$*, in particular* $\operatorname{rank} T_{\mathrm{crys}}(X) \geq \phi(N)$*.*

4. *If X is supersingular, then $\rho = 22$.*
5. *Let $g \in \operatorname{Aut}(X)$ and define N as in (3). If X is supersingular of Artin invariant σ_0, then $N \mid (p^{\sigma_0}+1)$.*

An immediate consequence of (5) is that, letting $g \in \operatorname{Aut}(X)$ and N be as in (3), if there exists no integer σ_0 with $N \mid (p^{\sigma_0}+1)$, then X is not supersingular. This applies to, for example, $(p,N) = (3,8), (5,4)$.

Proof.

(1) [Ke, Th. 1.4].

(2) [I, Prop. II.5.12].

(3) See [Mat2, Lem. 2.4(3)], which deduces the assertion from [J, Th. 3.2].

(4) This assertion, the Tate conjecture for supersingular K3 surfaces, is proved by Madapusi Pera [Mad, Th. 1] for characteristic ≥ 3 and by Kim and Madapusi Pera [KMP, Th. A.1] for characteristic 2.

The assertion under the assumption that X admits an elliptic fibration, which is true, for example, if $\rho \geq 5$ (which is always the case when we use this assertion in this paper), was proved much earlier by Artin [A1, Th. 1.7].

(5) This is proved by Nygaard [Ny2, Th. 2.1] under the assumption $p \neq 2$. The argument is in fact valid for $p = 2$ (see [Mat2, Rem. 2.2]). ∎

Proof of Theorem 8.2. Let $S'_\mu(p)$ be the set on the right-hand side of the statement. If n is a positive integer not divisible by p, then μ_n-action is equivalent to the action a cyclic group of order n and, as noted in Remark 8.1, Keum [Ke, Main Theorem] proved that n is the order of some automorphism of a K3 surface in characteristic p if and only if $n \in S'_\mu(p)$ (equivalently $n \in S_\mu(0)$).

If $n \in S'_\mu(p)$ and $p \mid n$, then the examples given in Example 9.6 show that $n \in S_\mu(p)$.

Now, take $n \in S_\mu(p)$ with $p \mid n$, and we will show $n \in S'_\mu(p)$. Write $n = p^e r$ with $p \nmid r$. Since a smooth K3 surface never admits a μ_p-action, an example X must have an RDP w. Since μ_{p^e}-fixed RDPs can be blown up, we may assume w is not μ_{p^e}-fixed. Note that such RDPs are classified in Proposition 4.12. We show in each case that n belongs to $S'_\mu(p)$.

Suppose w is D_m or E_m. Let $\mu_{p^f s} = \operatorname{Stab}(w) \subset \mu_{p^e r}$ (with $p \nmid s$). Then the pair $(w, p^e s)$ appears in Table 5 and we have $(r/s)m < 22$. Then we observe that $n \in S'_\mu(p)$ except in the following cases: $(p,n,s,(r/s)w) = (2,54,9,3E_7^0), (2,40,5,D_{21}^0), (2,34,17,D_{18}^0), (2,34,17,D_{19}^0)$. These exceptional cases do not occur, since it follows that μ_s acts trivially on the classes of every $(21,21,18,19)$ exceptional curves, which gives a too large invariant subspace of $H^2_{\text{ét}}(\tilde{X},\mathbb{Q}_l)$ for an order s automorphism (which must act on $H^2_{\text{ét}}$ faithfully with a characteristic polynomial with coefficients in $\mathbb{Q}$ by Lemma 8.3(1)).

Suppose w is $A_{m'-1}$. Let $\mu_{p^f s} = \operatorname{Stab}(w) \subset \mu_{p^e r}$ (with $p \nmid s$). We have $0 \leq f < e$. It follows from Proposition 4.12 that $p^{e-f} \mid m'$, so write $m' = p^{e-f} m$ with $m \geq 1$, and that $\mu_{p^f t}$ acts on w symplectically where either $s = t$ or $(s,t) = (2,1)$. Then $X/\mu_{p^f t}$ has $(r/s)A_{p^f t m' - 1}$. We have $22 > (r/s)(p^f t m' - 1) = (r/s)(p^e t m - 1) \geq r(t/s)m(p^e - 1)$. If $s = t$, then this implies $(p^e - 1)r \leq (p^e - 1)rm < 22$, and if $(s,t) = (2,1)$, then this implies ($p \neq 2$ and) $2 \mid r$ and $(p^e - 1)r \leq (p^e - 1)rm < 44$. We observe that this condition implies either $n \in S'_\mu(p)$ or $(p,n) = (5,40), (3,48), (2,34)$. It remains to show that each of the latter three cases is impossible.

If $(p,n)=(3,48)$, then $(m,s,t)=(1,2,1)$. Since μ_s does not act symplectically, the generator of μ_{16} acts on $H^0(\tilde{X},\Omega^2)$ by a primitive 16th root of unity. If $\tilde{X}$ is of finite height, then by Lemma 8.3(3), we have $\operatorname{rank} T(\tilde{X}) \geq \phi(16)=8$ and we have $\rho(\tilde{X})>8\cdot 2=16$ (from $8A_2$), contradicting $\rho+\operatorname{rank} T(\tilde{X})=22$. By the remark after Lemma 8.3, $\tilde{X}$ cannot be supersingular.

If $(p,n)=(2,34)$, then $(m,s,t)=(1,1,1)$. Let e_i ($i\in\mathbb{Z}/17\mathbb{Z}$) be the exceptional curves above the μ_{17}-orbit of w, numbered in a way that a generator $g\in\mu_{17}$ acts by $g(e_i)=e_{i+1}$. Let $L\subset\operatorname{Pic}(\tilde{X})$ be the sublattice generated by e_i's, and $L'=\operatorname{Pic}(\tilde{X})\cap\mathbb{Q}L$ its primitive closure.

First, suppose $\tilde{X}$ is of finite height. Then, by Corollary 6.4 the μ_2-action is symplectic. We may assume that X is maximal, in which case the number of RDPs on X (which are all of type A_1) is 8 by Proposition 6.8, which is not compatible with the μ_{17}-action.

Next, suppose $\tilde{X}$ is supersingular. By Lemma 8.3(5), the only possible Artin invariant is $\sigma_0=4$. By Lemma 8.4 (applied to $L_1=L'$, $L_2=L'^{\perp}$, $M=\operatorname{Pic}(\tilde{X})$, $\bar{M}=\operatorname{Pic}(\tilde{X})^*$, and $\bar{L}_i=L_i^*$), we have

$$\begin{aligned}\operatorname{rank}(\operatorname{disc}(L')) &\leq \dim_{\mathbb{F}_2}(\operatorname{disc}(\operatorname{Pic}(\tilde{X})))+\operatorname{rank}(\operatorname{disc}(L'^{\perp}))\\ &\leq \dim_{\mathbb{F}_2}(\operatorname{disc}(\operatorname{Pic}(\tilde{X})))+\operatorname{rank}(L'^{\perp})\\ &= 2\sigma_0+(22-\operatorname{rank} L')=13.\end{aligned}$$

Since $\operatorname{disc}(L)\cong(\mathbb{Z}/2\mathbb{Z})^{\oplus 17}$ has rank 17, we obtain $L\subsetneq L'$. Let $V\subset 2^{\mathbb{Z}/17\mathbb{Z}}$ be the set of subsets $S\subset\mathbb{Z}/17\mathbb{Z}$ such that $(1/2)\sum_{i\in S}e_i\in L'$. Then V is naturally a g-stable $\mathbb{F}_2$-vector space, and is nonzero, and we can identify it with a nonzero $\mathbb{F}_2[x]$-submodule V of $\mathbb{F}_2[x]/(x^{17}-1)$. Clearly, $V=Q(x)\cdot\mathbb{F}_2[x]/(x^{17}-1)$ for some $Q(x)\in\mathbb{F}_2[x]$ dividing $x^{17}-1$. Using the factorization $x^{17}-1=(x-1)F_{17,1}(x)F_{17,2}(x)$ in $\mathbb{F}_2[x]$, where

$$\begin{aligned}F_{17,1}(x)&=x^8+x^7+x^6+x^4+x^2+x+1 \quad \text{and}\\ F_{17,2}(x)&=x^8+x^5+x^4+x^3+1\end{aligned}$$

are irreducible, it follows that V contains at least one of

$$\begin{aligned}(x^{17}-1)/(x-1)&=x^{16}+\cdots+1,\\ (x^{17}-1)/F_{17,1}(x)&=x^9+x^8+x^6+x^3+x+1, \quad \text{or}\\ (x^{17}-1)/F_{17,2}(x)&=x^9+x^6+x^5+x^4+x^3+1.\end{aligned}$$

Hence, there exists a set $S\in V$ with $\#S=17$ or $\#S=6$. But then, $((1/2)\sum_{i\in S}e_i)^2=(1/2)^2\cdot\#S\cdot(-2)\notin 2\mathbb{Z}$, contradiction.

If $(p,n)=(5,40)$, then $(m,s,t)=(1,2,1)$ and there are four non-μ_5-fixed A_4 on which g acts transitively, where g is a fixed generator of $\mu_8\subset\mu_{40}$, and g^4 is non-symplectic. Moreover, $\operatorname{Fix}(g^4)$ is one-dimensional, passing through the four points of type A_4.

By the remark after Lemma 8.3, $\tilde{X}$ cannot be supersingular. This implies that the μ_5-action is symplectic (Corollary 6.4) and hence the quotient $Y=X/\mu_5$ is an RDP K3 surface.

We will show in a subsequent paper [Mat3, Prop. 2.15(4)] that the μ_p-action induces a canonical nonzero element $v\in H^0(Y^{\mathrm{sm}},\Omega^2)\otimes\operatorname{Der}(Y)$, which we can write $v=\omega_Y\otimes D_Y$ (uniquely up to k^*), such that D_Y is p-closed and satisfies $Y^{D_Y}=X^{(p)}$. It is characterized by $D(f)^pD_Y(h)\omega_Y=d(f^p)\wedge dh$ for local sections f of $\mathcal{O}_X$ and h of $\mathcal{O}_Y$. Since the μ_5-action in our case is g-invariant, it follows that $v=\omega_Y\otimes D_Y$ is g-invariant.

https://doi.org/10.1017/nmj.2022.20 Published online by Cambridge University Press

Fix a decomposition $v = \omega_Y \otimes D_Y$. We have $D_Y^p = \phi D_Y$ for some meromorphic function ϕ. Since both Y and the quotient $Y^{D_Y} \cong X^{(p)}$ are RDP K3 surfaces, it follows from the Rudakov–Shafarevich formula [RS, Cor. 1 to Prop. 3] that D_Y has only isolated fixed points, and this implies that ϕ is holomorphic, hence constant. We have $\phi \neq 0$, since if $\phi = 0$, then, as we will prove in a subsequent paper [Mat3, Lem. 3.6 or Th. 4.6], the α_p-action corresponding to D_Y must have quotient singularities different from A_{p-1}, but X has only RDPs of type A_{p-1}. Since $\mathrm{Fix}(D_Y)$ is isolated and $Y^{D_Y} = Y^{g^*(D_Y)}$, we have $g^*(D_Y) = \lambda D_Y$ with $\lambda \in H^0(Y, \mathcal{O})^* = k^*$, and since $D_Y^p = \phi D_Y$ with $\phi \in k^*$, we have $\lambda^{p-1} = 1$, and hence $(g^4)^*(D_Y) = D_Y$.

On the other hand, since $\mathrm{Fix}(g^4 \curvearrowright Y)$ is homeomorphic to $\mathrm{Fix}(g^4 \curvearrowright X)$ and hence is one-dimensional, we have $(g^4)^*(\omega_Y) = -\omega_Y$. This contradicts the g-invariance of $v = \omega_Y \otimes D_Y$. ∎

LEMMA 8.4. *Let $L_1 \subset \bar{L}_1$, $L_2 \subset \bar{L}_2$, and $L_1 \oplus L_2 \subset M \subset \bar{M} \subset \bar{L}_1 \oplus \bar{L}_2$ be sequences of abelian groups, where the bars have no specific meaning. Assume that the projection $\bar{M} \to \bar{L}_1$ are surjective and that $M \cap (\bar{L}_1 \oplus 0) = L_1 \oplus 0$. Then we have* $\mathrm{rank}(\bar{L}_1/L_1) \leq \mathrm{rank}(\bar{M}/M) + \mathrm{rank}(\bar{L}_2/L_2)$*, where the rank of an abelian group is the minimum number of generators.*

Proof. We may assume $L_i = 0$. The assumption then implies that $M \to \bar{L}_2$ is injective. Since the rank behaves subadditively with respect to subgroups, quotients, and extensions, we obtain $\mathrm{rank}(\bar{L}_1) \leq \mathrm{rank}(\bar{M}) \leq \mathrm{rank}(M) + \mathrm{rank}(\bar{M}/M) \leq \mathrm{rank}(\bar{L}_2) + \mathrm{rank}(\bar{M}/M)$. ∎

§9. Examples

For a projective variety with projective coordinates (x_i), we use the notation $\mathrm{wt}(x_i) = (n_i)$ to mean that $\mathrm{wt}(x_j/x_i) = (n_j - n_i)$ on the affine piece $(x_i \neq 0)$ for each i. Note that $\mathrm{wt}(x_i) = (n_i)$ is equivalent to $\mathrm{wt}(x_i) = (a + n_i)$. We use a similar notation for subvarieties of $\mathbb{P}(3,1,1,1)$.

9.1 Symplectic actions

EXAMPLE 9.1 (Symplectic $\mu_4 \times \mu_4$-action). The quartic surface $X = (w^4 + x^4 + y^4 + z^4 + wxyz = 0)$ in characteristic $p = 2$ is an RDP K3 surface. It has six RDPs, all of type A_3, at the points where two of w, x, y, z are 0 and the others are 1. This surface admits a symplectic action of $G = H_1 \times H_2$, where $H_1 = \mu_4$ and $H_2 = \mu_4$ act by $\mathrm{wt}(w,x,y,z) = (0,0,1,-1)$ and $\mathrm{wt}(w,x,y,z) = (0,1,0,-1)$, respectively.

With respect to the action of the subgroup scheme $\mu_2 \subset H_1$, the two RDPs at $(0,0,1,1)$ and $(1,1,0,0)$ are fixed and the other four RDPs are non-fixed and non-maximal. The quotient surface by this μ_2-action is $(W^2 + X^2 + Y^2 + Z^2 + AB = WX - A^2 = YZ - B^2 = 0)$ in $\mathbb{P}^5$, where $W = w^2, \dots, Z = z^2$ and $A = wx, B = yz$, with two RDPs of type A_7 at $(Y = Z = B = W + X = W + A = 0), (W = X = A = Y + Z = Y + B = 0)$ and four of type A_1 at $(WX = A = YZ = B = W + X + Y + Z = 0)$.

The quotient morphism by the subgroup scheme $\mu_2 \times \mu_2$ (resp. the full group G) is the relative Frobenius morphism $X \to X^{(2)}$ (resp. $X \to X^{(4)}$).

EXAMPLE 9.2 (Symplectic $\mu_3 \times \mu_3$-action). The surface $X = (v^3 + w^3 + x^3 + y^3 + z^3 + vwx = v^2 - yz = 0) \subset \mathbb{P}^4$ in characteristic $p = 3$ is an RDP K3 surface, and has $2A_5$ at $(1,0,0,1,1), (0,1,-1,0,0)$ and $4A_2$ at $(0,1,0,-1,0), (0,1,0,0,-1), (0,0,1,-1,0), (0,0,1,0,-1)$. This surface admits a symplectic action of $G = H_1 \times H_2$, where $H_1 = \mu_3$ and $H_2 = \mu_3$ act by $\mathrm{wt}(v,w,x,y,z) = (0,1,-1,0,0)$ and $\mathrm{wt}(v,w,x,y,z) = (0,0,0,1,-1)$, respectively. Let D_1, D_2

https://doi.org/10.1017/nmj.2022.20 Published online by Cambridge University Press

Table 8. Examples of symplectic μ_n-actions on RDP K3 surfaces.

n	p	Monomials	wt(w,x,y,z)
5	5	$w^3x, x^3z, z^3y, y^3w, w^2z^2, wxyz, x^2y^2$	1,2,3,4
6	2,3	$w^4, wy^3, wxyz, x^3z, z^4, w^2z^2, x^2y^2$	0,1,2,3
7	7	$w^4, x^3z, z^3y, y^3x, wxyz$	0,1,2,4
8	2	$w^4, x^4, y^3z, yz^3, wxyz$	0,2,1,5

be the corresponding derivations. The fixed points of D_1 (resp. D_2) is the first (resp. second) A_5 point. The fixed points of D_1+D_2 (resp. D_1-D_2), which corresponds to the diagonal (resp. anti-diagonal) subgroup of G, are the first and the fourth (resp. the second and the third) A_2 points. The quotient morphism by G is the relative Frobenius morphism $X \to X^{(3)}$.

Example 9.3 (Symplectic μ_n-action ($n=5,6,7,8$)). For each $n=5,6,7,8$, let F be a linear combination of the monomials listed in Table 8, in characteristic p, and then $X=(F=0)\subset \mathbb{P}^3$ admits a μ_n-action with the indicated weights. If F is a generic such polynomial, then X is an RDP K3 surface and the μ_n-action is symplectic. The fixed locus is $X\cap\{(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)\}$.

For example, for $n=5,6,7,8$, respectively, the polynomials with coefficients $(1,1,1,1,0,0,0)$, $(1,1,1,1,1,0,0)$, $(1,1,1,1,1)$, $(1,1,1,1,1)$ satisfy the condition.

9.2 Non-symplectic actions

Example 9.4 (Non-symplectic μ_2-action with Enriques quotient in characteristic 2). Following [BM1, §3], let L_1,L_2,L_3 be three linear polynomials in 12 variables, and let $X\subset\mathbb{P}^5$ be the intersection of three quadrics F_1,F_2,F_3 defined by $F_h=L_h(x_k^2, x_ix_j, y_k^2, y_ix_j+x_iy_j+y_iy_j)_{1\le k\le 3, 1\le i<j\le 3}\in k[x_1,x_2,x_3,y_1,y_2,y_3]$. Then, for generic L_h, X is an RDP K3 surface (with 12 RDPs of type A_1), μ_2 acts on ($\mathbb{P}^5$ and) X by $\mathrm{wt}(x_i, y_i+x_i)=(0,1)$ without any fixed point on X, and the quotient X/μ_2 is an Enriques surface.

Example 9.5 (Non-symplectic μ_2-action with rational quotient in characteristic 2). The quartic surface $w^2(xy+z^2)+x^4+y^4+z^4+yz(y^2+z^2)=0$ is an RDP K3 surface, and the μ_2-action with $\mathrm{wt}(w,x,y,z)=(0,1,1,1)$ is non-symplectic. The fixed locus consists of the curve $(w=0)$ and the RDP $(x=y=z=0)$ of type A_1. The image of this RDP in the quotient surface is a non-RDP singularity.

In the following example, for two polynomials $A(t),B(t)$ with $\deg A\le 8$ and $\deg B\le 12$, *the elliptic (or quasi-elliptic) surface defined by the equation* $y^2=x^3+A(t)x+B(t)$ is an abbreviation for the projective surface that is the union of four affine surfaces

$$\begin{aligned}
&\operatorname{Spec} k[x,y,t]/(-y^2+x^3+A(t)x+B(t)),\\
&\operatorname{Spec} k[x',y',t^{-1}]/(-y'^2+x'^3+t^{-8}A(t)x'+t^{-12}B(t)),\\
&\operatorname{Spec} k[z,w,t]/(-z+w^3+A(t)wz^2+B(t)z^3),\\
&\operatorname{Spec} k[z',w',t^{-1}]/(-z'+w'^3+t^{-8}A(t)w'z'^2+t^{-12}B(t)z'^3),
\end{aligned}$$

glued by the relations $x'=t^{-4}x$, $y'=t^{-6}y$, $z=y^{-1}$, $w=xy^{-1}$, $z'=y'^{-1}=t^6y^{-1}$, and $w'=x'y'^{-1}=t^2xy^{-1}$. For generic A and B, this is an RDP K3 surface.

Table 9. Examples of non-symplectic μ_n-actions on RDP K3 surfaces.

p	n	Equation	$\mathrm{wt}(x,y,t)$	RDPs	References
19	38	$y^2=x^3+t^7x+t$	$2,3,6$	A_{18}	[Ko]
17	34	$y^2=x^3+t^7x+t^2$	$4,23,6$	A_{16}	[Ko]
13	26	$y^2=x^3+t^5x+t$	$2,3,6$	A_{12}	[Ko]
11	44	$y^2=x^3+x+t^{11}$	$22,11,2$	$2A_{10}$	[Ko]
7	42	$y^2=x^3+t^7x+1$	$14,21,4$	$3A_6$	?
7	28	$y^2=x^3+x+t^7$	$14,7,2$	$2A_6$	[Ko]
5	60	$y^2=x^3+t(t^{10}-1)$	$2,3,6$	$2E_8^0$	[Ke]
3	60	$y^2=x^3+t(t^{10}-1)$	$2,3,6$	$10A_2$	[Ke]
3	42	$y^2=x^3+t(t^7-1)$	$2,3,6$	$7A_2$	[B]*
3	36	$y^2=x^3-t(t^6-1)$	$2,3,6$	$2E_6^0$	[Ko]*
3	24	$y^2=x^3+t^2(t^8-1)$	$2,3,3$	$8A_2$	?
2	60	$y^2=x^3+t(t^{10}-1)$	$2,3,6$	$5D_4^0$	[Ke]
2	38	$y^2=x^3+t^7x+t$	$2,3,6$	$19A_1$	[Ko]
2	36	$y^2=x^3-t(t^6-1)$	$2,3,6$	$3D_4^0$	[Ko]*
2	32	$y^2=x^3+t^2x+t^{11}$	$18,11,2$	D_{20}^0	[O1]
2	26	$y^2=x^3+t^5x+t$	$2,3,6$	$13A_1$	[Ko]
2	24	$y^2=x^3+t^5(t^4+1)$	$2,3,6$	E_8^0	[B]
p	n	Equation	$\mathrm{wt}(w,x,y,z)$	RDPs	References
7	42	$w^2=x^5y+y^5z+z^5x$	$-1,0,-2,8$	$3A_6$	?
3	42	$w^2=x^5y+y^5z+z^5x$	$-1,0,-2,8$	$7A_2$	?
2	42	$w^2=x^5y+y^5z+z^5x$	$-1,0,-2,8$	$21A_1$	?
2	22	$w^2=x^5y+y^5z+xy^2z^3$	$-1,0,-2,8$	$11A_1$	?
p	n	Equation	$\mathrm{wt}(w,x,y,z)$	RDPs	References
2	28	$w^4+x^3y+y^3z+z^3x=0$	$0,1,-3,9$	$7A_3$	?

EXAMPLE 9.6 (Non-symplectic μ_n-actions). Table 9 proves the existence part of Theorem 8.2 for n divisible by p. The first group consists of elliptic (or quasi-elliptic) RDP K3 surfaces, the second of double sextics, and the third of quartics. Only the non-μ_n-fixed RDPs are listed, except in the example for $(p,n)=(2,32)$, the D_{20}^0 point is fixed and after blowing-up this point, we find a non-fixed D_{18}^0 point.

The examples are characteristic p reductions of the examples (of an automorphism of order n) in characteristic 0 obtained, respectively, by Brandhorst [B, Th. 5.9], Keum [Ke, Exam. 3.2], Kondo [Ko, §§3 and 7], and Oguiso [O1, Prop. 2], except that for the ones marked "?" we could not find a reference. An asterisk means that we made a coordinate change $t\mapsto t^{-1}$.

9.3 μ_p-actions on abelian surfaces

As shown in Theorem 6.3, the nontrivial μ_p-actions of abelian surfaces A, up to automorphisms of μ_p, are precisely the translations by subgroup schemes of $A[p]$ isomorphic to μ_p.

REMARK 9.7. In the case of finite order automorphisms on abelian surfaces, there are examples with non-abelian quotients. Kummer surfaces in characteristic $\neq 2$ are the minimal resolution of the RDP K3 quotient (with $16A_1$) by the symplectic involution $x\mapsto -x$ on abelian surfaces (for characteristic 2, see Remark 3.6). Furthermore, certain non-symplectic (or sometimes symplectic) actions give (quasi-)hyperelliptic quotients. It seems that there are no μ_p-analogue of these actions.

Remark 9.8. If we consider *rational* vector fields (i.e., possibly with poles) of multiplicative type, there are other kinds of examples. See [KT, Exam. 6.2] for a rational vector field of multiplicative type on an abelian surface (in characteristic 2) with a general type quotient.

Acknowledgments. I thank Hiroyuki Ito, Kazuhiro Ito, Shigeyuki Kondo, Christian Liedtke, Gebhard Martin, Hisanori Ohashi, Shun Ohkubo, and Tomoaki Shirato for helpful comments and discussions. I also thank the anonymous referee for a number of corrections and suggestions.

References

[A1] M. Artin, *Supersingular $K3$ surfaces*, Ann. Sci. Éc. Norm. Supér. (4) **7** (1974), 543–567 (1975).

[A2] M. Artin, "Coverings of the rational double points in characteristic p" in *Complex Analysis and Algebraic Geometry*, Iwanami Shoten, Tokyo, 1977, 11–22.

[BM1] E. Bombieri and D. Mumford, *Enriques' classification of surfaces in char. p. III*, Invent. Math. **35** (1976), 197–232.

[BM2] E. Bombieri and D. Mumford, "Enriques' classification of surfaces in char. p. II" in *Complex Analysis and Algebraic Geometry*, Iwanami Shoten, Tokyo, 1977, 23–42.

[B] S. Brandhorst, *The classification of purely non-symplectic automorphisms of high order on K3 surfaces*, J. Algebra **533** (2019), 229–265.

[DK1] I. V. Dolgachev and J. Keum, *Wild p-cyclic actions on $K3$-surfaces*, J. Algebraic Geom. **10** (2001), 101–131.

[DK2] I. V. Dolgachev and J. Keum, *Finite groups of symplectic automorphisms of $K3$ surfaces in positive characteristic*, Ann. of Math. (2) **169** (2009), 269–313.

[EH+] T. Ekedahl, J. M. E. Hyland, and N. I. Shepherd-Barron, *Moduli and periods of simply connected Enriques surfaces*, preprint, arXiv:1210.0342

[GP] P. Gille and P. Polo, eds., *Schémas en Groupes (SGA 3). Tome I. Propriétés générales Des schémas en Groupes*, Doc. Math (Paris) [Math. Doc. (Paris)] **7**, Soc. Math. France, Montrouge, 2011 (in French). Séminaire de Géométrie Algébrique du Bois Marie 1962–64. [Algebraic Geometry Seminar of Bois Marie 1962–64]; A seminar directed by M. Demazure and A. Grothendieck with the collaboration of M. Artin, J.-E. Bertin, P. Gabriel, M. Raynaud and J-P. Serre; Revised and annotated edition of the 1970 French original.

[I] L. Illusie, *Complexe de de Rham-Witt et cohomologie cristalline*, Ann. Sci. Éc. Norm. Supér. (4) **12** (1979), 501–661 (in French).

[J] J. Jang, *A lifting of an automorphism of a K3 surface over odd characteristic*, Int. Math. Res. Not. IMRN 2016, 1787–1804.

[Ka] T. Katsura, "On Kummer surfaces in characteristic 2" in *Proceedings of the International Symposium on Algebraic Geometry (Kyoto Univ., Kyoto, 1977)*, Kinokuniya Book Store, Tokyo, 1978, 525–542.

[KT] T. Katsura and Y. Takeda, *Quotients of abelian and hyperelliptic surfaces by rational vector fields*, J. Algebra **124** (1989), 472–492.

[Ke] J. H. Keum, *Orders of automorphisms of K3 surfaces*, Adv. Math. **303** (2016), 39–87.

[KMP] W. Kim and K. Madapusi Pera, *2-adic integral canonical models*, Forum Math. Sigma **4** (2016), e28, 34 pp.

[Ko] S. Kondō, *Automorphisms of algebraic $K3$ surfaces which act trivially on Picard groups*, J. Math. Soc. Japan **44** (1992), 75–98.

[Mad] K. Madapusi Pera, *The Tate conjecture for K3 surfaces in odd characteristic*, Invent. Math. **201** (2015), 625–668.

[Mat1] Y. Matsumoto, *Extendability of automorphisms of K3 surfaces*, to appear in Math. Res. Lett., preprint, arXiv:1611.02092v2

[Mat2] Y. Matsumoto, *Degeneration of K3 surfaces with non-symplectic automorphisms*, to appear in Rend. Sem. Mat. Univ. Padova, preprint, arXiv:1612.07569v4

[Mat3] Y. Matsumoto, *μ_p- and α_p-actions on K3 surfaces in characteristic p*, to appear in J. Algebraic Geom., preprint, https://www.ams.org/journals/jag/0000-000-00/S1056-3911-2022-00804-X/, DOI: http://doi.org/10.1090/jag/804

[Mat4] Y. Matsumoto, *Inseparable maps on W_n-valued Ext groups of non-taut rational double point singularities and the height of K3 surfaces*, to appear in J. Commut. Algebra, preprint, arXiv:1907.04686v3

[Mu] S. Mukai, *Finite groups of automorphisms of $K3$ surfaces and the Mathieu group*, Invent. Math. **94** (1988), 183–221.

[Ni] V. V. Nikulin, *Finite automorphism groups of Kähler $K3$ surfaces*, Trudy Moskov. Mat. Obshch. **38** (1979), 75–137 (in Russian). English translation: Trans. Moscow Math. Soc. **1980**, 71–135.

[Ny1] N. O. Nygaard, *A p-adic proof of the nonexistence of vector fields on $K3$ surfaces*, Ann. of Math. (2) **110** (1979), 515–528.

[Ny2] N. O. Nygaard, *Higher de Rham–Witt complexes of supersingular $K3$ surfaces*, Compos. Math. **42** (1980/81), 245–271.

[O1] K. Oguiso, *A remark on the global indices of $\mathbb{Q}$-Calabi–Yau 3-folds*, Math. Proc. Camb. Philos. Soc. **114** (1993), 427–429.

[O2] A. Ogus, "*Supersingular K3 crystals*" in *Journées de Géométrie Algébrique de Rennes (Rennes, 1978)*, Astérisque **64**, Soc. Math. France, Montrouge, 1979, 3–86.

[RS] A. N. Rudakov and I. R. Shafarevich, *Inseparable morphisms of algebraic surfaces*, Izv. Akad. Nauk SSSR Ser. Mat. **40** (1976), 1269–1307, 1439 (in Russian). English translation: Math. USSR-Izv. **10** (1976), 1205–1237.

[S] C. S. Seshadri, "*L'opération de Cartier. Applications*" in *Séminaire C. Chevalley, 3ième année: 1958/59. Variétés de Picard*, Éc. Norm. Supér., Paris, 1960, 1–26 (in French).

Yuya Matsumoto
Department of Mathematics
Faculty of Science and Technology
Tokyo University of Science
2641 Yamazaki, Noda, Chiba 278-8510
Japan
matsumoto.yuya.m@gmail.com
matsumoto_yuya@ma.noda.tus.ac.jp