INTEGRALS ALLIED TO AIRY’S INTEGRALS

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1. Introductory. Airy’s integrals

\( \int_0^\infty \cos (\lambda^3 \pm z \lambda) \, d\lambda \)

can be expressed ([1], [2]) in terms of Bessel functions. In this paper integrals of the types

\( \int_0^\infty \cos (\lambda^n \pm z \lambda) \lambda^{k-1} \, d\lambda \)

are discussed. Various subsidiary formulae are given in § 2, some integrals of the type

\( \int_0^\infty \exp (-\lambda^n) f(\lambda) \lambda^{k-1} \, d\lambda \)

are evaluated in § 3, and from these the integrals of the Airy type are derived in § 4.

2. Formulae required in the proof. The first of these is the Gamma function formula

\( \Gamma(z) \Gamma\left(z + \frac{1}{m}\right) \ldots \Gamma\left(z + \frac{m-1}{m}\right) = (2\pi)^{\frac{m}{2}} \Gamma(mz). \) .................(1)

The second is Ragab’s formula ([2], p. 406, ex. 27, [3])

\[ K_{\mu}(z) K_{\nu}(z) = \frac{1}{4z} \sum \frac{\Gamma \left( \frac{\mu + \nu}{2}, \frac{1 + \mu - \nu}{2}, \frac{1 - \mu + \nu}{2}, \frac{1 - \mu - \nu}{2} \right)}{\frac{1}{2} \pi i} e^{iz^2}. \] ..............(2)

The following two formulae are also required.

If \( m \) is a positive integer and if \( R(k) > 0 \), ([2], p. 406, ex. 30),

\( \int_0^\infty e^{-\lambda^k} \frac{1}{m^{\frac{1}{m}}} \sum p_{q+1} = \frac{k + \nu}{m} \lambda^\nu \) \ldots (3)

where \( \alpha_{p+1} = \frac{k + \nu}{m}, \nu = 0, 1, 2, \ldots, m-1. \)

If \( m \) is a positive integer and if \( |\mathrm{amp} \, z| < \pi \), ([2], p. 407, ex. 32),

\( \frac{1}{2\pi i} \int_\gamma e^{iz^2} \frac{1}{m^{\frac{1}{m}}} \sum p_{q+1} = \frac{k + \nu}{m} \lambda^\nu \) \ldots (4)

where the contour starts at \( -\infty \) on the real axis, passes round the origin in the positive direction, and returns to \( -\infty \), and \( \beta_{p+1} = \frac{k + \nu}{m}, \nu = 0, 1, 2, \ldots, m-1. \)

3. Some infinite integrals. Consider the integral

\( \int_0^\infty \exp (-\lambda^n + z \lambda^l) \lambda^{k-1} \, d\lambda, \)

where \( n \) and \( l \) are positive integers such that \( l < n \), and \( R(k) > 0 \). On expanding \( \exp (z \lambda^l) \) in powers of \( z \) and putting \( \lambda = \mu z^{1/m} \), this becomes

\( \frac{1}{n} \sum_{r=0}^\infty \frac{z^r}{r!} \Gamma \left( \frac{k + r}{n} \right), \)

and therefore
\[
\int_0^\infty \exp \left( -\lambda^n + z\lambda^l \right) \lambda^{k-1} d\lambda
\]

\[
= \frac{1}{n} \sum_{t=0}^{n-1} \Gamma \left( \frac{k+t}{n} \right) \frac{z^t}{t!} \left\{ \begin{array}{c}
\frac{k+t}{n}, \frac{k+t+n}{n}, \ldots, \frac{k+t+(l-1)n}{n} : l^{-t} \left( \frac{z}{n} \right)^n, \\
\frac{t+1}{n}, \frac{t+2}{n}, \ldots, \frac{t+n}{n}
\end{array} \right\}, \quad \cdots (5)
\]

the asterisk indicating that the parameter \( n/n \) is omitted.

Now assume that \( n \) is odd, replace \( z \) by \(-1/z\), and apply (1) with \((k+t)/(nl)\) for \( z \) and \( l \) for \( m \) and also with \((t+1)/n\) for \( z \) and \( n \) for \( m \); then the equation can be written

\[
\int_0^\infty \exp \left( -\lambda^n \right) E \left( \frac{z}{\lambda^n} \right) \lambda^{k-1} d\lambda
\]

\[
= n^{3/n} \left( \frac{2\pi}{l+1} \right)^{n-1} \sum_{t=0}^{n-1} \left( -m^{-1/nz} \right) \left( \frac{2\pi}{l+1} \right)^{n-1} \left\{ \begin{array}{c}
\frac{k+t}{n}, \frac{k+t+n}{n}, \ldots, \frac{k+t+(l-1)n}{n} : l^{-t} \left( \frac{z}{n} \right)^n, \\
\frac{t+1}{n}, \frac{t+2}{n}, \ldots, \frac{t+n}{n}
\end{array} \right\}, \quad \cdots (6)
\]

On generalising, using (3) and (4), it is found that, if \( n \) and \( l \) are positive integers such that \( n \) is odd and \( l<n \), and if \( R(k)>0 \),

\[
\exp \left( -\lambda^n \right) E \left( \frac{z}{\lambda^n} \right) \lambda^{k-1} d\lambda
\]

\[
= n^{2\pi l - \pi - \pi - 2\pi / 2} \left( \frac{2\pi}{l+1} \right)^{n-1} \left( \frac{2\pi}{l+1} \right)^{n-1} \left\{ \begin{array}{c}
\frac{k+t}{n}, \frac{k+t+n}{n}, \ldots, \frac{k+t+(l-1)n}{n} : l^{-t} \left( \frac{z}{n} \right)^n, \\
\frac{t+1}{n}, \frac{t+2}{n}, \ldots, \frac{t+n}{n}
\end{array} \right\}, \quad \cdots (7)
\]

Note. If \( n \) is even the argument of the \( E \)-function should be multiplied by \( e^{z/n} \).

For example, on applying formula (2) it is found that

\[
\int_0^\infty \exp \left( -\lambda^n \right) K_{\mu} \left( z\lambda^n \right) K_{\nu} \left( z\lambda^n \right) \lambda^{k-1} d\lambda
\]

\[
= (2\sqrt{2})^{-1} n^{-3/2} (2l+1)(n) (2\pi)^{1-n} \sum_{t=0}^{n-1} \left( \begin{array}{c}
\frac{k+t+2l}{2n}, \frac{k+t+2l+n}{2n}, \ldots, \frac{1+\mu+\nu+2t}{2n}, \\
\frac{t+1}{n}, \frac{t+2}{n}, \ldots, \frac{1+2t+2n-2}{2n}
\end{array} \right) \left( \frac{e^{inz}}{(2l)^2} \right)^n, \quad \cdots (8)
\]

where \( n \) and \( l \) are positive integers such that \( n \) is odd and \( n>2l \), and \( R(k)>-l \).

4. Integrals of the Airy type. In formula (5) swing the line of integration through a positive angle \( \pi/(2n) \), so that \( \lambda \) becomes \( \eta_{\pi/2n} \), and let \( z = \pm \xi_{(n-1)\pi/(2n)} \), where \( \xi \) is real and positive; then
INTEGRALS ALLIED TO AIRY’S INTEGRALS

\[ \int_0^\infty \exp \left( -i\eta^n \pm ix \eta^l \right) \eta^{k-1} \, d\eta = \frac{1}{n} \sum_{t=0}^{n-1} \Gamma \left( \frac{k+tl}{n} \right) \left( \frac{\pm x}{t!} \right) \]

\[ \times e^{it(n-1)\pi/2n} \left\{ \begin{array}{c} \frac{k+tl}{n} \ , \ \frac{k+tl+n}{n} \ , \ \ldots \ , \ \frac{k+tl+(l-1)n}{n} \ ; \ e^{i(n-1)t\pi/2l} \left( \frac{\pm x}{n} \right)^n \end{array} \right\} \]  

\( n > R(k) > 0 \) and \( n > l \).

On putting \( n = 3, l = 1, k = 1 \), and equating real parts, Airy’s integrals are obtained.

The same method may be applied for other values of \( n \) and \( l \).

For instance, if \( n = 5, l = 3, \) and if \( k \) is real and such that \( 0 < k < 5, x \) real and positive,

\[ \int_0^\infty \cos \left( \eta^5 \pm x \eta^3 \right) \eta^{k-1} \, d\eta = \frac{1}{5} \sum_{t=0}^{4} \Gamma \left( \frac{k+3t}{5} \right) \cos \left( \frac{k-2t}{10} \pi \right) \left( \frac{\pm x}{t!} \right) \]

\[ \times F \left\{ \begin{array}{c} \frac{k+3t}{15} \ , \ \frac{k+3t+5}{15} \ , \ \frac{k+3t+10}{15} \ ; \ 
\frac{t+1}{5} \ , \ \frac{t+2}{5} \ , \ \ldots \ , \ \frac{t+5}{5} \end{array} \right\} \right\} \]  

\( \ldots \ldots \ ) (10)

Note. From these formulae numerous others can be deduced. For instance, if the cosines with arguments \( \pm x \) on the left of (10) are added, integrals of the type

\[ \int_0^\infty \cos \eta^5 \cos (x \eta^3) \eta^{k-1} \, d\eta \]

are obtained. The second cosine and the functions on the right can then be expressed as \( E \)-functions and generalised.

REFERENCES

