22

Anomalies

In the Standard Model, the fermion fields of the leptons and quarks interact through the mediation of vector bosons. As we remarked in Chapter 10, the renormalisability of the Model requires the vector boson fields to be introduced through the mechanisms of local gauge symmetry. Renormalisation requires the insertion of counter terms in the Lagrangian (Chapter 8). It is important that the counter terms maintain the local gauge symmetries, along with their corresponding conserved currents. As a consequence, one of the global current conservation laws of the Standard Model, that we have obtained by treating the fields as classical fields, has to be modified when the classical fields are quantised. This is an example of an *anomaly.* We shall see that baryon number and lepton number are not strictly conserved quantities in quantum field theory.

22.1 The Adler–Bell–Jackiw anomaly

Bell and Jackiw and, independently, Adler were the first to find an anomaly in a field theory (see Treiman *et al.*, 1985). They were concerned with the axial vector current associated with the chiral symmetries introduced in Section 16.7. To appreciate the nature of this anomaly, consider the model Lagrangian density

$$\mathcal{L} = \bar{\psi}[\gamma^\mu(\mathrm{i}\partial_\mu - qA_\mu) - m]\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{22.1}$$

This has the local gauge symmetry of electromagnetism; it is invariant under the transformation

$$\begin{aligned} \psi(x) &\to \psi'(x) = \mathrm{e}^{-\mathrm{i}q\chi(x)}\psi(x), \\ A_\mu(x) &\to A'_\mu(x) = A_\mu(x) + \partial_\mu\chi(x). \end{aligned} \tag{22.2}$$

If $m = 0$, $\mathcal{L}$ also has a global chiral symmetry: it is then invariant under the transformation

$$\psi(x) \rightarrow \psi'(x) = \mathrm{e}^{\mathrm{i}\alpha\gamma^5}\psi(x), \tag{22.3}$$

as may easily be verified using the properties of the γ matrices (Section 5.5).

Applying the transformation (22.3) to the Lagrangian density (22.1), with α taken to be infinitesimal and space and time dependent, gives an infinitesimal change $\delta\mathcal{L}$ in $\mathcal{L}$ which (after an integration by parts in the action) may be taken to be

$$\delta\mathcal{L} = \alpha(x)[\partial_\mu j_A^\mu - 2\mathrm{i}m\bar{\psi}\gamma^5\psi],$$

where

$$j_A^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi \tag{22.4}$$

is the *axial current.* (See Problem 5.6.)

It follows from Hamilton's principle that, for fields that obey the field equations,

$$\partial_\mu j_A^\mu = 2\mathrm{i}m\bar{\psi}\gamma^5\psi. \tag{22.5}$$

If $m = 0$, the axial current is conserved:

$$\partial_\mu j_A^\mu = 0 \quad \text{if} \quad m = 0. \tag{22.6}$$

The results (22.5) and (22.6) have been obtained treating the fields as classical fields. In quantum field theory the fields become quantum operators, and the currents can be calculated in perturbation theory. It is found that in order to keep the electric charge conserved and maintain electromagnetism as a local gauge symmetry, perturbation theory requires

$$\partial_\mu j_A^\mu = 2\mathrm{i}m\bar{\psi}\gamma^5\psi - \frac{e^2}{2\pi^2}\varepsilon^{\mu\nu\lambda\rho}\partial_\mu A_\nu \partial_\lambda A_\rho. \tag{22.7}$$

With $m = 0$ the axial current is not conserved, but instead

$$\partial_\mu j_A^\mu = -\frac{e^2}{2\pi^2}\varepsilon^{\mu\nu\lambda\rho}\partial_\mu A_\nu \partial_\lambda A_\rho. \tag{22.8}$$

This is the Adler–Bell–Jackiw axial anomaly. It is found to be the only anomalous term in $\partial_\mu j_A^\mu$. Using Problem 4.3, we can write (22.8) in the explicitly gauge invariant form

$$\partial_\mu j_A^\mu = -\frac{e^2}{\pi^2}\mathbf{E}\cdot\mathbf{B}. \tag{22.9}$$

It is interesting to note that from (22.8) we can construct a current

$$j_{\text{total}}^\mu = j_A^\mu + \frac{e^2}{4\pi^2}\varepsilon^{\mu\nu\lambda\rho}A_\nu F_{\lambda\rho}, \tag{22.10}$$

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which evidently is conserved:

$$\partial_\mu j^\mu_{\text{total}} = 0. \tag{22.11}$$

j^μ_{total} is gauge dependent (it contains A_ν) and hence lacks immediate physical significance. Nevertheless it follows from (22.11) that the charge

$$Q(t) = \int j^{\text{o}}_{\text{total}}\, \mathrm{d}^3 x \tag{22.12}$$

is constant in time. $Q(t)$ is a gauge invariant quantity.

22.2 Cancellation of anomalies in electroweak currents

In the Standard Model, there are anomalies that have an origin and structure similar to the axial anomaly described in Section 22.1. In particular in the electroweak sector the gauge bosons couple to currents that have both vector and axial vector components, as, for example, in (12.15) where

$$j_e^\mu = e_{\text{L}}^\dagger \bar{\sigma}^\mu \nu_{\text{L}} = \bar{e}\gamma^\mu (1/2)(1-\gamma^5)\nu_{\text{e}}. \tag{22.13}$$

It is the mix of vector and axial vector that gives rise to anomalies that threaten the renormalisability of the electroweak sector. Detailed calculations show that, in a theory that has only leptons and no quarks, anomalies do spoil the conservation laws of the currents that couple to the bosons. Conversely, in a theory with only quarks and no leptons there are again anomalies. Remarkably, in a theory which includes both leptons and quarks the anomalies cancel exactly, provided that the number of lepton families is equal to the number of quark families, and then the electroweak gauge currents are strictly conserved (t'Hooft, 1976). Thus equality in the number of lepton families and quark families is of fundamental importance to the renormalisability of the Standard Model.

There are no serious anomalies associated with the gluon fields of the strong interaction.

22.3 Lepton and baryon anomalies

We now turn to the currents that, classically, arise from global symmetries and conserve the number of leptons and the number of quarks. We will first consider the situation if neutrinos are shown to be Dirac fermions. For Dirac neutrinos there is a conserved lepton current given by (22.25)

$$\begin{aligned} J^\mu_{\text{lepton}}(x) = \sum_{\alpha=\text{e},\mu,\text{r}} \big[&\alpha_{\text{L}}^\dagger(x)\,\tilde{\sigma}^\mu \alpha_{\text{L}}(x) + \alpha_{\text{R}}^\dagger(x)\,\sigma^\mu \alpha_{\text{R}}(x) + \nu_{\alpha\text{L}}^\dagger(x)\,\tilde{\sigma}^\mu \nu_{\alpha\text{L}}(x) \\ &+ \nu_{\alpha\text{R}}^\dagger(x)\,\sigma^\mu \nu_{\alpha\text{R}}(x)\big]. \end{aligned} \tag{22.14}$$

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and classically

$$\partial_\mu(J^\mu_{\text{lepton}}) = 0. \tag{22.15}$$

On quantisation, this current is not conserved. The divergence equation has to be modified in a way reminiscent of (22.8) and becomes

$$\partial_\mu\left(J^\mu_{\text{lepton}}\right) = \frac{3}{64\pi^2}\varepsilon^{\mu\nu\lambda\rho}\left[\frac{1}{2}g_2^2\mathrm{Tr}\left(\mathrm{W}_{\mu\nu}\mathrm{W}_{\lambda\rho}\right) - g_1^2\mathrm{B}_{\mu\nu}\mathrm{B}_{\lambda\rho}\right]. \tag{22.16}$$

The fields $\mathbf{W}_{\mu\nu}$, $\mathbf{B}_{\mu\nu}$, and the coupling constants g_1 and g_2, were introduced in Chapter 11.

The total quark number is also classically conserved but the same anomalous term as in (22.15) arises when the quark fields are quantised for each colour. Summing over the three colours we have

$$\partial_\mu J^\mu_{\text{quark}} = 3\partial_\mu J^\mu_{\text{lepton}}. \tag{22.17}$$

Since baryon number is one third of the quark number, this can also be written

$$\partial_\mu J^\mu_{\text{baryon}} = \partial_\mu J^\mu_{\text{lepton}}, \tag{22.18}$$

where $J^\mu_{\text{lepton}} = J^\mu_{\text{e}} + J^\mu_{\text{muon}} + J^\mu_{\text{tau}}$.

Thus if neutrinos are Dirac particles, anomalies reduce the two classically conserved currents of the Standard Model to one that can be taken as $J^\mu_{\text{baryon}} - J^\mu_{\text{lepton}}$. The independent current $J^\mu_{\text{baryon}} + J^\mu_{\text{lepton}}$ is not conserved.

Let us now consider the lepton number current. This is not conserved but, as we found with the chiral anomaly, there is nevertheless an associated current that is conserved, and we may write

$$\partial_\mu\left(J^\mu_{\text{lepton}} - J^\mu_{\text{T}}\right) = 0, \tag{22.19}$$

where

$$J^\mu_{\text{T}} = \frac{3}{32\pi^2}\varepsilon^{\mu\nu\lambda\rho}\left[\frac{1}{2}g_2^2\mathrm{Tr}\left(\mathrm{W}_\nu\mathrm{W}_{\lambda\rho} - (ig_2/3)\,\mathrm{W}_\nu\mathrm{W}_\lambda\mathrm{W}_\rho\right) - g_1^2B_\nu B_{\lambda\rho}\right]. \tag{22.20}$$

J^μ_{T} is called the *topological current*, and

$$N_{\text{T}} = \int J^0_{\text{T}}\,\mathrm{d}^3\mathbf{x} \tag{22.21}$$

is the *topological number*.

The lepton number is defined to be

$$N_{\text{lepton}} = \int J^0_{\text{lepton}}\mathrm{d}^3\mathbf{x}, \tag{22.22}$$

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and it follows from (22.19) that $N_{\text{lepton}} - N_{\text{T}}$ is constant in time. If N_{T} changes by ΔN_{T}, then N_{lepton} changes by ΔN_{lepton}, and $\Delta N_{\text{lepton}} = \Delta N_T$.

22.4 Gauge transformations and the topological number

Is the topological number a gauge invariant? For simplicity we shall restrict our discussion to fields that are gauge transforms of the vacuum field configuration. Then from (11.4b) and (11.6)

$$B_\mu = (2/g_1)\,\partial_\mu\theta, \tag{22.23}$$
$$\mathbf{W}_\mu = (2i/g_2)\left(\partial_\mu\mathbf{U}\right)\mathbf{U}^\dagger. \tag{22.24}$$

The field strengths $B_{\mu\nu}$ and $\mathbf{W}_{\mu\nu}$ are of course zero everywhere. Also we shall only consider gauge transformations in a local region of space, so that $\theta \to 0$ and $\mathbf{U} \to \mathbf{I}$ as $r \to \infty$. The topological number for this vacuum configuration is

$$N_{\text{T}} = -\frac{1}{8\pi^2}\int \varepsilon^{0ijk}\text{Tr}\left\{(\partial_i \text{U})\,\text{U}^\dagger\left(\partial_j\text{U}\right)\text{U}^\dagger\,(\partial_k\text{U})\,\text{U}^\dagger\right\}\text{d}^3\mathbf{x}, \tag{22.25}$$

using (22.24) in (22.20).

It can be shown that N_{T} is an integer multiple of 3, 0, ±3, ±6, ... We can illustrate this by considering unitary transformations of the form

$$\mathbf{U}(x) = \cos f(r)\mathbf{I} + \text{i}\sin f(r)(\hat{\mathbf{r}}\cdot\tau), \tag{22.26}$$

taking $\alpha = f(r)\hat{\mathbf{r}}$ in (B.9). Here $f(r)$ is a function with the property that $f(r) \to 0$ as $r \to \infty$, so that $\mathbf{U} \to \mathbf{I}$ as $r \to \infty$. If $\mathbf{U}(\mathbf{x})$ is to be defined at $r = 0$, then $\sin f(r)$ must vanish there (since $\hat{\mathbf{r}}$ is not defined at $r = 0$). Thus we require $f(0) = n\pi$ where n is an integer. Subject only to the boundary conditions at $r = 0$ and $r \to \infty$, $f(r)$ can be any continuous and differentiable function.

If $n = 0$, $f(r)$ can be deformed continuously to give $f(r) = 0$, $\mathbf{U} = \mathbf{I}$, for all r; transformations like this are called 'small' unitary transformations. If $n \neq 0$ there is no way in which $f(r)$ can be deformed continuously to give $\mathbf{U} = \mathbf{I}$ for all r; these are 'large' unitary transformations. Direct computation of (22.25) with $\mathbf{U}$ of the form (22.26) gives

$$N_{\text{T}} = \frac{6}{\pi}\int_0^{n\pi}\sin^2 f\,\text{d}f = 3n. \tag{22.27}$$

It appears that in a theory with no fermions there would be many inequivalent representations of the vacuum state, characterised by a topological number N_{T}. Neglecting the fermions, and treating the $SU(2) \times U(1)$ gauge fields and the Higgs field classically, it is found that to change N_{T} continuously by one unit involves field distortions that require energy. Estimates suggest the energy barrier in field configurations is of height a few times $(4\pi/g_2^2)\ M_\mathrm{w} \sim 100\ M_\mathrm{w}$. Treating the fields as quantum fields, t'Hooft (1976) found that quantum tunnelling can take place through the barrier, but the probability per unit volume in space-time of a change in N_T is very small because of a very small tunnelling factor $\exp(-16\pi^2/g_2^2) \approx 10^{-173}$.

22.5 The instability of matter, and matter genesis

Including the fermions in the Standard Model, if the Higgs and gauge fields pass over the energy barrier separating different topological sectors, the fermion fields must also evolve. Suppose, for example, that $\Delta N_\mathrm{lepton} = -3$ and, from (22.18), $\Delta N_\mathrm{baryon} = -3$. These conditions are satisfied by, for example, the decay ${}^3_2\mathrm{He} \to \mathrm{e}^+ + \mu^+ + \bar{\nu}_\tau$.

With suppression factors like 10^{-173}, it is unlikely that any helium nucleus in our galaxy has ever decayed in this way since helium nuclei were formed.

It is nevertheless an intriguing possibility that the matter content of the Universe could have been generated by an anomaly mechanism. In the Big Bang model of cosmology, at the very early stage in its evolution the Universe was intensely hot, at a temperature high compared even with the barrier height separating the different topological sectors. Thermal fluctuations over the barrier would produce matter or antimatter depending on the sign of ΔN_T. In the beginning the net baryon and lepton numbers might both have taken the symmetrical value zero. To generate the observed preponderance of matter over antimatter requires *CP* violation, and this is an attribute of the Standard Model.

The modifications are straightforward if neutrinos are Majorana fermions. For example, with the Majorana Lagrange density of (21.11), (22.19) becomes

$$\partial_\mu \left(J^\mu_\mathrm{lepton} - J^\mu_\mathrm{T} \right) = m_{\alpha\beta} \left(\nu_\alpha^\mathrm{T} \sigma^2 \nu_\beta + \nu_\beta^+ \sigma^2 \nu_\alpha^* \right) \tag{22.28}$$

as can be shown by making an infinitesimal, space time dependent, phase change on all the lepton fields (see the method of section (22.1)). If neutrinos are Majorana particles then, with the anomalies, no global conservation laws remain.