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THE SPACE OF TOTALLY BOUNDED ANALYTIC FUNCTIONS

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0. Introduction

This paper is a continuation of our project on "inverse interpolation", begun in [**6**]. In brief, the task of inverse interpolation is to deduce some property of a function f from some given property of the set L of its Lagrange interpolants. In the present work, the property of L is that it be a uniformly bounded set of functions when restricted to the domain of f. In particular (see Section 3), when the domain is a disc, we deduce sharp bounds on the successive derivatives of f. As a result, f must extend to be an analytic function (of restricted growth) in the concentric disc of thrice the original radius.

The class of all these "totally bounded" functions forms a Banach space $TB\mathbb{D}$ in a natural way, and has a companion space, the space $TC\mathbb{D}$ of totally convergent functions. To belong to $TC\mathbb{D}$, we require not only that the interpolants p be uniformly bounded, but that the remainders $f-p$ go uniformly to 0 on $\mathbb{D}$ as the degree of interpolation goes to ∞. We study $TB\mathbb{D}$ in some detail. Our study is to some extent parallel with the theory of the Bloch spaces B_0 and B studied extensively in [**1**] and [**10**], and in Section 4 we compare our spaces with theirs more closely.

1. Definitions and basic properties of $TB\mathbb{D}$ and $TC\mathbb{D}$

Definition 1.1. Given a complex-valued function f on a set $S\subseteq\mathbb{C}$, a polynomial p, say of degree n, is called a Lagrange interpolant to f on S if there exist $n+1$ distinct points $z_0, z_1, \dots, z_n \in S$ such that $f(z_j)=p(z_j)$ for $j=0,1,\dots,n$. Of course $f-p$ is permitted to have more than $n+1$ zeros on S. The z_j are called the nodes (or knots) of the interpolant. In case several of the z_j coincide, the usual conventions are made about interpolation of derivatives at the points of coincidence. We denote by $L(f)$ the set of all Lagrange interpolants of f. (If all the knots coincide, say at z_0, then the interpolant will be called a *Taylor* interpolant as well—it will be a partial sum

$$S_n(z:z_0)=\sum_{k=0}^{n}\frac{f^{(k)}(z_0)}{k!}(z-z_0)^k$$

of the Taylor series for f around z_0.)

We remark that if p, as above, is a Lagrange interpolant of f, then it must be given by the Lagrange interpolation formula

$$p(z)=\sum_{j=0}^{n} f(z_j)l_j(z) \tag{1.0}$$

where

$$l_j(z)=\frac{(z-z_0)(z-z_1)\cdots(z-z_{j-1})(z-z_{j+1})\cdots(z-z_n)}{(z_j-z_0)(z_j-z_1)\cdots(z_j-z_{j-1})(z_j-z_{j+1})\cdots(z_j-z_n)}, \tag{1.0$'$}$$

$$j=0,1,\ldots,n.$$

We write

$$R_n[f:z_0,z_1,\ldots,z_n;z]=f(z)-p(z).$$

Definition 1.2. We say that a function $f:S\to\mathbb{C}$ is totally bounded on S if there is a finite number M such that $|p(z)|\leq M$ for every $z\in S$ and every $p\in L(f)$. We write

$$\|f\|_{TBS}=\sup\{\|p\|_{\infty,S}:p\in L(f)\}, \tag{1.1}$$

where

$$\|g\|_{\infty,S}=\sup\{|g(z)|:z\in S\}.$$

We denote the class of all such functions by TBS.

Remark. $f\in TBS$ if and only if (a) f is bounded on S and (b) $\{\|f-p\|_{\infty,S}, p\in L(f)\}$ is bounded. This gives rise to a convenient equivalent norm, that differs from the above norm by at most a factor of 2.

Definition 1.3. We say that a function $f:S\to\mathbb{C}$ is totally convergent if

(a) f is totally bounded on S and
(b) $\limsup\limits_{\deg p\to\infty}\{\|f-p\|_{\infty,S}:p\in L(f)\}=0.$

We write TCS for the class of totally convergent functions on S.

Note. We are using the unorthodox shorthand of, say, $TC\mathbb{D}$ for $TC(\mathbb{D})$. This helps particularly when these expressions occur as subscripts.

Proposition 1.1. *If S is an infinite set, then TBS and TCS are infinite-dimensional Banach spaces and TCS is a closed subspace of TBS.*

The proof is along the usual lines, and we omit it.

Proposition 1.2. *If S is an open set in $\mathbb{C}$, then every function f in TBS is analytic on S.*

Proof. Let $z_0, z_1, z_2, \ldots$ be a countable dense subset of S, and let p_N be the Lagrange interpolant to $f \in TBS$, with nodes at $z_0, z_1, \ldots, z_N$. The family $\{p_N\}$ is uniformly bounded on S, by hypothesis, and so there exists a subsequence $\{p_{N_k}\}$ that converges uniformly on compact subsets of S to an analytic function F on S. Clearly, $f(z_i) = F(z_i)$ for all $i = 0, 1. 2, \ldots$. But, by looking at *linear* interpolants of f, we see that f must be continuous on S. Hence $f \equiv F$, and the proposition is proved.

For most of the rest of this paper, we will take S to be $\mathbb{D} = \{z \in \mathbb{C}, |z| < 1\}$, the open unit disc. An immediate question is whether, say, $TC\mathbb{D}$ contains any functions f that are *not* polynomials. But as I. D. Berg has pointed out to us, no infinite-dimensional Banach space is the *union* of an ascending sequence of finite-dimensional subspaces, as the Baire Category Theorem shows. Hence, there must exist non-polynomial functions in $TC\mathbb{D}$. The next two results exhibit some explicitly, and the main result of Section 3 will show us how to construct many more. In particular, it follows from that result that if $f \in TB\mathbb{D}$, then f must have an extension that is analytic in $3\mathbb{D} = \{z \in \mathbb{C} : |z| < 3\}$.

Theorem 1.1. *Let w be a complex number, and consider the function $1/(w-z)$. This function belongs to $TC\mathbb{D}$ exactly when $|w| > 3$, and belongs to $TB\mathbb{D}$ exactly when $|w| \geqq 3$. (Hence $TC\mathbb{D} \subsetneqq TB\mathbb{D}$.)*

Remark. Although Theorem 1.1 follows easily from the later result, Theorem 1.3, we believe it is instructive to prove it directly by examining the Lagrange interpolants to $(w-z)^{-1}$.

Proof. Let $p(z)$ be the Lagrange interpolant of degree $\leqq n$ to $f(z) = (w-z)^{-1}$, with nodes $z_0, z_1, \ldots, z_n \in \mathbb{D}$. Thus $R(z) = f(z) - p(z)$ vanishes at $z_0, z_1, \ldots, z_n$. Let $q(z) = (w-z)R(z)$. This is a polynomial of degree $\leqq n+1$ that vanishes at $z_0, z_1, \ldots, z_n$, and hence

$$q(z) = A(z-z_0)(z-z_1)\cdots(z-z_n),$$

where A is a constant. Since $q(w) = 1$, we see that

$$q(z) = \frac{(z-z_0)(z-z_1)\cdots(z-z_n)}{(w-z_0)(w-z_1)\cdots(w-z_n)}$$

and hence

$$R(z) = \frac{1}{w-z}\frac{(z-z_0)(z-z_1)\cdots(z-z_n)}{(w-z_0)(w-z_1)\cdots(w-z_n)}.$$

Suppose that $|w| \geqq 3$. Then for $z \in \mathbb{D}$,

$$|R(z)| \leqq \frac{2^n}{(|w|-1)^{n+1}}.$$

So it is clear that if $|w| \geqq 3$ then $(w-z)^{-1} \in TB\mathbb{D}$, and that if $|w|>3$ then $(w-z)^{-1} \in TC\mathbb{D}$. Now consider arbitrary $w \in \mathbb{C}$. If $|w| \leqq 1$, then $(w-z)^{-1}$ is not even a bounded analytic function on $\mathbb{D}$, so we need only consider $|w|>1$. Using rotational symmetry, we may take w real and positive; $1<w \leqq 3$. The idea now is to consider the Lagrange interpolant at nodes $z_0, z_1, \ldots, z_n$ very close to 1, evaluated at a point z very close to -1. To shorten the argument, we will pass to the limit, and take the Taylor interpolant around 1 evaluated at -1. We have

$$\frac{1}{w-z}=\frac{1}{w-1} \frac{1}{1-\dfrac{z-1}{w-1}}=\frac{1}{w-1}\left[1+\left(\frac{z-1}{w-1}\right)+\left(\frac{z-1}{w-1}\right)^2+\cdots\right].$$

In this expression, put $z=-1$. It can now easily be seen that if $1<w<3$ then the nth remainder R_n is unbounded, whereas if $w=3$, then R_n oscillates boundedly, and Theorem 1.1 is proved.

We remark on the phenomenon exhibited here that the "worst" interpolants are the Taylor interpolants around a suitable boundary point evaluated at the diametrically opposite boundary point. This is a recurrent theme in this paper.

Proposition 1.3. *If $f \in H^1 3\mathbb{D}$ (and hence if $f \in H^\infty 3\mathbb{D}$) then $f \in TB\mathbb{D}$, and* $\|f\|_{TBD} \leqq 2\|f\|_{H^1 3D}$.

Proof. Here, of course, $3\mathbb{D}=\{z \in \mathbb{C}:|z|<3\}$, and $H^1 3\mathbb{D}$ is those functions f analytic in $3\mathbb{D}$ such that

$$\|f\|_{H^1 3D} \doteq \sup_{r<3} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(r e^{i\theta})| \, d\theta<\infty.$$

The proposition is an easy consequence of the integral form for the remainder (see [2])

$$R_k[f: z_0, z_1, \ldots, z_k; z]=\frac{1}{2\pi i} \int_\gamma \left[\prod_{i=0}^{k} \frac{z-z_i}{t-z_i}\right] f(t) \frac{dt}{t-z} \tag{1.2}$$

where γ is any simple closed contour that surrounds all the z_i and z and lies in the region of analyticity of f. We need only take γ to be the circle $z=r e^{i\theta}$, $-\pi<\theta \leqq \pi$, and let $r \rightarrow 3$, taking the obvious estimates.

Remark. It is clear that $H^1 3\mathbb{D} \neq TB\mathbb{D}$, as is seen from $f(z)=(3-z)^{-1}$.

Theorem 1.2. *$TB\mathbb{D}$ is not separable.*

Proof. By considering Taylor interpolants as in the proof of Theorem 1.1, one can show that for $|\lambda|=|\lambda'|=1$, $\lambda \neq \lambda'$, $|\lambda-\lambda'|<1/10$, say

$$\left\|\frac{1}{3\lambda-z}-\frac{1}{3\lambda'-z}\right\|_{TBD} \geqq \frac{1}{10}. \tag{1.3}$$

Thus $TB\mathbb{D}$ has uncountably many elements at a fixed mutual positive distance (1/10), and is hence not separable. The details are as follows. We write

$$\frac{1}{3\lambda-z}-\frac{1}{3\lambda'-z}=\frac{1}{2\lambda-(z-\lambda)}-\frac{1}{(3\lambda'-\lambda)-(z-\lambda)}$$

$$=\frac{1}{2\lambda}\frac{1}{1-\dfrac{z-\lambda}{2\lambda}}-\frac{1}{3\lambda'-\lambda}\frac{1}{1-\dfrac{z-\lambda}{3\lambda'-\lambda}}$$

$$=\frac{1}{2\lambda}\left[1+\frac{z-\lambda}{2\lambda}+\left(\frac{z-\lambda}{2\lambda}\right)^2+\cdots\right]-\frac{1}{3\lambda'-\lambda}\left[1+\left(\frac{z-\lambda}{3\lambda'-\lambda}\right)+\left(\frac{z-\lambda}{3\lambda'-\lambda}\right)^2+\cdots\right].$$

Truncating these series at the $(n-1)$st power, we get

$$p_n(z)=\frac{1-\left(\dfrac{z-\lambda}{2\lambda}\right)^n}{3\lambda-z}-\frac{1-\left(\dfrac{z-\lambda}{3\lambda'-\lambda}\right)^n}{3\lambda'-z}.$$

Choosing now $z=-\lambda$, we have

$$p_n(-\lambda)=\frac{1-(-1)^n}{4\lambda}-\frac{1-(-1)^n\left(\dfrac{2\lambda}{3\lambda'-\lambda}\right)^n}{3\lambda'+\lambda}.$$

Now, choosing n even, say $n=2m$, we get, for m large,

$$|p_{2m}(-\lambda)|\sim\left|\frac{1}{3\lambda'+\lambda}\right|$$

and the result follows.

Definition 1.4. For f holomorphic in $\mathbb{D}$, define

$$|||f|||=\sup\left\{\frac{|f^{(n)}(z)|}{n!}\,2^n : n=0,1,2,\ldots;\, z\in\mathbb{D}\right\}.$$

Theorem 1.3. *For all such f,*

$$\tfrac{1}{2}|||f|||\leqq\|f\|_{TBD}\leqq 2|||f|||.$$

Corollary 1. *If $f\in TB\mathbb{D}$, then $f\in H3\mathbb{D}$, i.e. f has a holomorphic extension to $3\mathbb{D}=\{z\in\mathbb{C}:|z|<3\}$.*

Corollary 2. *If f is totally bounded in $\mathbb{D}$ and if g is a function on $\mathbb{D}$ with $L(f)=L(g)$* (as unstructured *sets* of polynomials), *then $f=g$.*

Proof. By Corollary 1 to Theorem 1.3, f actually extends to be analytic in $3\mathbb{D}$. Since g is also totally bounded, the same is true for g. Now let $p_n(z)$ be the nth Taylor interpolant of f, knotted at the origin. Thus, $p_n(z)$ converges uniformly to f on compact subsets of $3\mathbb{D}$, and in particular, uniformly on $2\mathbb{D}$. Now $p_n(z)$ interpolates to g at points $z_{n,0}, z_{n,1}, \ldots, z_{n,n}$ in $\mathbb{D}$ (with due regard to multiplicity). The $z_{n,j}$ either have infinitely many cluster points in $\bar{\mathbb{D}}=\{z\in\mathbb{C}:|z|\leqq 1\}$, or a cluster point of infinite order there. Since f and g are analytic in $3\mathbb{D}$, it follows from the identity theorem that $f=g$.

Remark. It can now be easily shown that $f\in TC\mathbb{D}$ iff $\|f^{(n)}\|_{\infty,\mathbb{D}}2^n/n!\to 0$ as $n\to\infty$. Thus, if $0<\alpha<1$, then $(3-z)^{-\alpha}\in TC\mathbb{D}$. So there are many $f\in TC\mathbb{D}$ for which $f^2\notin TC\mathbb{D}$, and therefore $TC\mathbb{D}$ is not an algebra. Similarly, $TB\mathbb{D}$ is not an algebra, since $(3-z)^{-2}\notin TB\mathbb{D}$. A similar proof shows that TBG is not an algebra for any region G. For by rotation, translation, and dilation, we may suppose that $(-1,1)\in G$ and $\operatorname{diam} G=2$, and consider $(3-z)^{-1}$ and $(3-z)^{-2}$ as above. (See the proof of Theorem 1.1.)

Remark. It follows easily from the method of proof of Theorem 1.3 that $f\in TB\mathbb{D}$ iff f is Taylor totally bounded ($f\in TTB\mathbb{D}$), which means that there is a uniform bound on the partial sums of the Taylor series expansion of f around *any* point $z_0\in\mathbb{D}$. We do not know whether this remains true if we replace $\mathbb{D}$ by a generic region G.

Proof of Theorem 1.3. Given $z_0\in\mathbb{D}$, look at the nth partial sum $S_n(z:z_0)$ in the Taylor expansion

$$f(z)=\sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n.$$

Since $S_n(z:z_0)$ is a limiting case of a Lagrange interpolant of f, we must have $|S_n(z:z_0)|\leqq\|f\|_{TB\mathbb{D}}$. Thus

$$\left|\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n\right|=|S_n(z:z_0)-S_{n-1}(z:z_0)|\leqq 2\|f\|_{TB\mathbb{D}},$$

and the first asserted inequality follows on choosing $z=-z_0$ and applying the maximum modulus theorem. In the other direction, now take $f\in TB\mathbb{D}$, and choose $z_0, z_1, \ldots, z_n; z\in\mathbb{D}$, where we choose the z_i distinct. We claim that

$$n!\,|f[z_0, z_1, \ldots, z_n]|\leqq\sup\{|f^{(n)}(\xi)|:\xi\in\mathbb{D}\},\tag{1.4}$$

where $f[z_0, z_1, \ldots, z_n]$ is the nth order divided difference of f (see[2]). There is no loss of generality in supposing that f vanishes at $z_1, z_2, \ldots, z_n$, since we can otherwise subtract a polynomial interpolant of degree $\leqq n-1$ that agrees with f at $z_1, z_2, \ldots, z_n$.

This affects neither the nth order divided difference nor the nth derivative. Thus,

$$f[z_0, z_1, \ldots, z_n] = \frac{f(z_0)}{(z_0 - z_1)\cdots(z_0 - z_n)}.$$

Now for fixed $z_0, z_1, \ldots, z_n \in \mathbb{D}$, consider the following extremal problem:

(1.5) maximize $|f(z_0)|$ overall holomorphic functions f on $\mathbb{D}$ such that $\|f^{(n)}\|_{\infty,\mathbb{D}} \leqq 1$ and $f(z_1) = f(z_2) = \cdots = f(z_n) = 0$.

By **[5]**, the extremal function is (see also **[9]**)

$$\tilde{f}(z) = \frac{1}{n!}(z - z_1)(z - z_2)\cdots(z - z_n).$$

Hence, if $f(z_1) = \cdots = f(z_n) = 0$ and $\|f^{(n)}\|_{\infty,\mathbb{D}} \leqq 1$, then

$$|f[z_0, z_1, \ldots, z_n]| \leqq \frac{1}{n!} \frac{|(z_0 - z_1)\cdots(z_0 - z_n)|}{|(z_0 - z_1)\cdots(z_0 - z_n)|} = \frac{1}{n!},$$

and hence, for general f,

$$|f[z_0, z_1, \ldots, z_n]| \leqq \frac{1}{n!} \sup\{|f^{(n)}(z)| : z \in \mathbb{D}\}.$$

Since (1.4) holds for *any* set of $n+1$ points in $\mathbb{D}$, replacing n by $n+1$ we have

$$|f[z, z_0, \ldots, z_n]| \leqq \frac{1}{(n+1)!} \|f^{(n+1)}\|_{\infty,\mathbb{D}}.$$

Now the second estimate of Theorem 1.3 is easily proved, since

$$\begin{aligned} |R_n[f : z_0, z_1, \ldots, z_n; z]| &= |(z - z_0)(z - z_1)\cdots(z - z_n)|\,|f[z, z_0, \ldots, z_n]| \\ &\leqq \frac{2^{n+1}}{(n+1)!} \|f^{(n+1)}\|_{\infty,\mathbb{D}}, \end{aligned}$$

from which the desired inequality $\|f\|_{TBD} \leqq 2|||f|||$ immediately follows.

Conjecture. *For any region G and any f holomorphic in G, define*

$$|||f||| = \sup\left\{\frac{|f^{(n)}(z)|}{n!} e_G(z)^n : n = 0, 1, 2, \ldots; z \in G\right\},$$

where $e_G(z) = \sup\{|z - w| : w \in \partial G\}$. Then

$$\tfrac{1}{2}|||f||| \leqq \|f\|_{TBG} \leqq 2|||f|||.$$

The first inequality is easy to prove, as above, and we have an approach to a proof of the second inequality, via an integral formula for the remainders R_k in terms of $f^{(k+1)}$. The proof we gave for $\mathbb{D}$ may not work for general regions G, since the associated extremal problem corresponding to (1.5) does not seem to have an easy solution if, say, G is not convex.

Theorem 1.4. *$TC\mathbb{D}$ is the closure, in the $TB\mathbb{D}$ norm, of the polynomials (and hence is separable).*

Proof. Denote by P the set of all polynomials, and by $\bar{P}$ its closure in $TB\mathbb{D}$. Clearly, $\bar{P} \subseteq TC\mathbb{D}$. Now we need a lemma.

Lemma 1.1. *Suppose $f_n, f \in TC\mathbb{D}$. Then $f_n \to f$ in the $TB\mathbb{D}$ norm $\Leftrightarrow$ both*

(i) *$f_n(z) \to f(z)$ uniformly in $(1+\varepsilon)\mathbb{D}$ for some $\varepsilon \in 0$, and*

(ii) *$R_k[f_n: z_0, z_1, \ldots, z_k; z] \to 0$ as $k \to \infty$ uniformly for $n = 0, 1, 2, \ldots$ and $z_0, z_1, \ldots, z_k; z \in \mathbb{D}$.*

Remark. It follows from Theorem 1.3 that (i) may be replaced by

(i′) *$f_n(z) \to f(z)$ uniformly in $(1+\varepsilon)\mathbb{D}$ for each ε with $0 < \varepsilon < 2$.*

Proof. The $\Rightarrow$ implication is an easy consequence of Theorem 1.3 and the formula (1.2) for $R_k[f_n - f]$. For the reverse implication, suppose that (i) and (ii) hold for a particular $\varepsilon > 0$. Look at the estimate

$$|R_k[f - f_n: z_0, z_1, \ldots, z_k; z]| \leqq |R_k[f: z_0, z_1, \ldots, z_k; z]| + |R_k[f_n: z_0, z_1, \ldots, z_k; z]|. \qquad (*)$$

Now given $\delta > 0$, choose $k(\varepsilon)$ so that $k > k(\varepsilon)$ implies that $|R_k[f_m: z_0, z_1, \ldots, z_k; z]| < \delta$ for all m and all $z_0, z_1, \ldots, z_k; z \in \mathbb{D}$, and also $|R_k[f: z_0, z_1, \ldots, z_k; z]| < \delta$. We get $|R_k[f - f_n: z_0, z_1, \ldots, z_k; z]| < 2\delta$ for $k > k(\varepsilon)$, independently of the choice of $z_0, z_1, \ldots, z_k; z \in \mathbb{D}$.

But for $k < k(\varepsilon)$, (1.2) yields the estimate

$$|R_k[f - f_n: z_0, z_1, \ldots, z_k; z]| \leqq (1+\varepsilon)\frac{2^{k(\varepsilon)+1}}{\varepsilon^{k(\varepsilon)+2}} \max\{|f(w) - f_n(w)| : |w| = 1+\varepsilon\},$$

which approaches 0 as $n \to \infty$. We have proved

$$\limsup_{n\to\infty} \sup\{|R_k[f - f_n: z_0, z_1, \ldots, z_k; z]: z_0, z_1, \ldots, z_k; z \in \mathbb{D}\} \leqq 2\delta \quad \text{for } k = 0, 1, 2, \ldots,$$

from which the desired result follows.

Corollary. *For $f \in TC\mathbb{D}$, $f(\rho z)$ converges to $f(z)$ in the $TB\mathbb{D}$ norm as $\rho \to 1-$.*

Fix $z_0, z_1, \ldots, z_k$ in $\mathbb{D}$, and let $p_{k,\rho}$ be the kth order Lagrange interpolant to $f(\rho z)$ at these nodes. We must show that

$$R_k[f_\rho: z_0, z_1, \ldots, z_k; z] \to 0 \quad \text{as} \quad k \to \infty$$

uniformly for $0<\rho<1$ and $z_0, z_1, \ldots, z_k \in \mathbb{D}$. But this follows immediately since $p_{k,\rho}(z) = q_k(\rho z)$, where q_k interpolates f at the nodes $\{\rho z_0, \rho z_1, \ldots, \rho z_k\}$, and we have assumed that $f \in TC\mathbb{D}$. This gives condition (ii) of Lemma 1.1.

Now we know by Theorem 1.3 that $f \in H3\mathbb{D}$, that is, that f has a holomorphic extension to $3\mathbb{D}$, and we know by Proposition 1.3 that, for all g, $\|g\|_{TB\mathbb{D}} \leqq 2\|g\|_{\infty,3\mathbb{D}}$. From this, condition (i) is easily seen to be satisfied for $f(\rho_n z)$, where ρ_n is any sequence that increases to 1. This proves the corollary to Lemma 1.1. Also, since the Taylor series around zero for each $f(\rho z)$ converges uniformly in $3\mathbb{D}$ to $f(\rho z)$, it also converges in $TC\mathbb{D}$, and Theorem 1.4 follows also by the corollary.

Conjecture.** $TC\mathbb{D}^{**} = TB\mathbb{D}$.

Interpretation. We mean by this not only that $TB\mathbb{D}$ is isometrically isomorphic, via some map φ, to the second dual of $TC\mathbb{D}$, but also that we can choose φ to be the identity map on $TC\mathbb{D}$—that is, $\varphi|_{TC\mathbb{D}}$ is the canonical embedding of $TC\mathbb{D}$ into $TC\mathbb{D}^{**}$.

An approach to a Proof of Conjecture.** Let L be a bounded linear functional on $TC\mathbb{D}$, and let

$$\lambda(w) = L_z\left(\frac{1}{w-z}\right)$$

for $|w|>3$. Write

$$\lambda(w) = \sum \frac{b_n}{w^{n+1}}.$$

Now take $f \in TC\mathbb{D}$. We have just proved that $f(\rho z) \xrightarrow{TC\mathbb{D}} f(z)$ as $\rho \to 1-$. We write

$$f(z) = \sum a_n z^n.$$

The first objective is to prove that

$$L(f) = \lim_{\rho \to 1-} \sum a_n b_n \rho^n. \tag{*}$$

Again using that $f(\rho z) \xrightarrow{TC\mathbb{D}} f(z)$, it is easy to show that span $\{1/(w-z) : |w|>3\}$ is dense in $TC\mathbb{D}$. Also, $L(f(\rho z)) = \sum a_n b_n \rho^n$. Consequently, we have proved (*). Now the idea is, given $F \in TC\mathbb{D}^{**}$ to identify F with an analytic function f on $\mathbb{D}$, and then prove that $f \in TB\mathbb{D}$. Well, given $z \in \mathbb{D}$, define

$$f(z) = \langle \Delta_z, F \rangle,$$

where Δ_z is point evaluation at z of functions in $TC\mathbb{D}$. It is easy to show that f must be analytic in $\mathbb{D}$ and that

$$f^{(n)}(z_0)=\langle \Delta_{z_0}^{(n)}, F\rangle,$$

where $\Delta_{z_0}^{(n)}$ is point evaluation of the nth derivative, at z_0, of functions in $TC\mathbb{D}$. Indeed, $f\in TB\mathbb{D}$. This can be seen on twice applying Theorem 1.3, or more directly as follows. From formulas (1.0) and (1.0′), for $\xi\in\mathbb{D}$ and $p\in L(f)$, we have

$$p(\xi)=\sum_{j=0}^{n} l_j(\xi)f(z_j)=\sum_{j=0}^{n} c_j f(z_j).$$

Hence the bounded linear functional $L\in TC\mathbb{D}^*$ given by $L(f)=p(\xi)$ is just a linear combination of point evaluations Δ_{z_j}. Hence

$$L(f)=\langle L, F\rangle$$

and

$$\sup|L(f)|=\sup|\langle L, F\rangle|\leqq\sup\|L\|\,\|F\|,$$

where the suprema range over all $\xi; z_0, z_1, \ldots, z_n\in\mathbb{D}$, and where L depends on $\xi; z_0,\ldots,z_n$. But for any such L, $\|L\|\leqq 1$, and therefore

$$\sup\{|p(\xi)|:\xi\in\mathbb{D}, p\in L(f)\}\leqq\|F\|.$$

Hence $f\in TB\mathbb{D}$ and $\|f\|_{TB\mathbb{D}}\leqq\|F\|$.

Let us now choose some definite Banach limit, LIM, on $[0,1)$. Given $f\in TB\mathbb{D}$, to describe the $F\in TC\mathbb{D}^{**}$ that corresponds, we let, for $L\in TC\mathbb{D}^*$,

$$F(L)=\operatorname*{LIM}_{\rho\to 1-}[L(f(\rho z))].$$

Thus, we have produced a linear map α from $TC\mathbb{D}^{**}$ into $TB\mathbb{D}$ and a linear map β from $TB\mathbb{D}$ into $TC\mathbb{D}^{**}$. It is not hard to prove that the map α is onto. What remains to be proved is that α and β are inverses of each other (and consequently that the Banach limit above is an ordinary limit), and that α and β are isometries. The main part we are unable to prove is that α is injective, i.e. that if $F\neq 0$ then $f\neq 0$. This would follow if we could prove that the span of the point evaluations is norm-dense in $TC\mathbb{D}^*$, but we are still unable to do this.

2. Almost totally bounded functions

Throughout this section, G will be an arbitrary open set in $\mathbb{C}$. We will call a function $f:G\to\mathbb{C}$ "almost totally bounded on G", written $f\in ATBG$, if for each compact subset K, there is a uniform bound on the Lagrange interpolants as the variable and all the interpolation nodes range over K. More precisely:

Definition 2.1. The space $ATBG$ of almost totally bounded functions on G is defined via the seminorms $\rho_K(f)$ (where K ranges over the compact subsets of G) defined by

$$\rho_K(f)=\sup\{|p_n(f:z_0,z_1,\dots,z_n;z)|:z_0,z_1,\dots,z_n;z\in K\}.$$

Remark. As with TBG in Proposition 1.2, it is easy to prove that if $f\in ATBG$, then f must be analytic on G.

Definition 2.2. For $z\in G$, define

$$e(z)=e_G(z)=\sup\{|w-z|:w\in G\}.$$

Definition 2.3. Define $E(G)$, the envelope of G, as

$$E(G)=\bigcup_{z\in G}\{\xi\in\mathbb{C}:|\xi-z|<e_G(z)\}.$$

Note. Geometrically, the envelope of G is the union of all those discs with centres in G whose boundaries contain some points of G.

Example. $E(\mathbb{D})=3\mathbb{D}$.

Theorem 2.1. *$ATBG=HE(G)$, the space of all holomorphic functions on the envelope of G, in the topology of uniform convergence on compact subsets of $E(G)$.*

Sketch of Proof. Suppose $f\in ATBG$ and $z_0\in G$. Let $S_n(z:z_0)$ be the nth partial sum of the Taylor series for $f(z)$ around z_0. We think of $S_n(z:z_0)$ as a Lagrange interpolant of f, with all its nodes at z_0. Choose $w\in G$. Then $|S_n(w:z_0)|\leqq A\rho_K(f)$ for a suitable compact subset K of G and constant $A>0$. It follows that this Taylor series converges in $\{z\in\mathbb{C}:|z-z_0|<|w-z_0|\}$, and thus f extends to be analytic in $E(G)$.

In the converse direction, if f is analytic in $E(G)$, if K is a compact subset of G, and if $z_0,z_1,\dots,z_n;z\in K$, then (recalling 1.2)

$$R_n[f:z_0,z_1,\dots,z_n;z]=\frac{1}{2\pi i}\int_\gamma\prod_{i=0}^{n}\frac{z-z_i}{t-z_i}f(t)\frac{dt}{t-z} \tag{2.1}$$

where (essentially), γ is the boundary of the *union* of all discs D with centres in K, whose boundaries pass through points of K, i.e. $\gamma=\partial E(K)$. (First, though, one may need to enlarge K slightly so that $E(K)$ now has a smooth boundary.) Now for

$$z_0,z_1,\dots,z_n;z\in K \quad\text{and}\quad t\in\gamma,$$

$$\left|\frac{z-z_i}{t-z_i}\right|\leqq 1 \quad\text{for}\quad i=0,1,\dots,n,$$

so that from (2.1), we get the required bound on the remainder R_n. It is left to the reader to verify that the seminorms ρ_K on $ATBG$ define the same topology on $HE(G)$ as the usual seminorms on this space do, and vice versa.

3. Comparison with the Bloch spaces

We change our focus to $TC\frac{1}{3}\mathbb{D}$ and $TB\frac{1}{3}\mathbb{D}$, i.e. the spaces of totally convergent and totally bounded functions in $1/3\mathbb{D}=\{z\in\mathbb{C}:|z|<1/3\}$. By Theorem 1.3, functions f in these spaces all extend to be analytic in $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$; indeed, they are each derivatives of Bloch functions in $\mathbb{D}$, i.e. they satisfy the growth condition

$$\|f\|_{B'\mathbb{D}}=\sup\{|f(z)|(1-|z|^2):z\in\mathbb{D}\}<\infty. \tag{3.1}$$

For extended discussions of Bloch functions and Bloch spaces, see **[1]**, **[10]** and **[14, 15]**.

We need here a detailed description of the dual $(B'_0\mathbb{D})^*$ of the Bloch space $B'_0\mathbb{D}$ (the closure of the polynomials in the norm in (3.1)), and we adapt it from **[1]**. Each bounded linear functional L on $B'_0\mathbb{D}$ has the form L_g for a suitable analytic function g on $\mathbb{D}$, where

$$L_g(f)=\lim_{\rho\to 1-}\sum a_n b_n \rho^n,$$

$$f(z)=\sum a_n z^n,\quad g(z)=\sum b_n z^n$$

and

$$\|L_g\|_{(B'_0\mathbb{D})^*}\sim|g'(0)|+\iint_{|t|<1}|g''(t)|\,dA(t),$$

$dA(t)$ being the element of *area* on $\mathbb{D}$.

Theorem 3.1. *There is a constant K, $0<K<\infty$, such that*

$$\frac{1}{K}\|f\|_{B'\mathbb{D}}\leqq\|f\|_{TB\frac{1}{3}\mathbb{D}} \tag{3.2}$$

and such that, for any given positive integer n, and any $f\in B'_0\mathbb{D}$

$$\|L_n f\|_{H^\infty\frac{1}{3}\mathbb{D}}\leqq Kn^{1/2}\|f\|_{B'\mathbb{D}}, \tag{3.2'}$$

for any Lagrange interpolation operator L_n with nodes at $z_0,z_1,\ldots,z_n\in\frac{1}{3}\mathbb{D}$. Moreover, the factor $n^{1/2}$ cannot be improved.

Corollary 3.1. (Immediate) $TB\frac{1}{3}\mathbb{D}\subsetneqq B'\mathbb{D}$.

Corollary 3.2. *$TB\mathbb{D}$ is not an algebra.*

Proof. From Theorem 1.1, say, $(3-z)^{-1} \in TB\mathbb{D}$. However $(3-z)^{-2} \notin B'3\mathbb{D}$. (See also the Remark following Theorem 1.3.)

Proof of Theorem 3.1. The first estimate in (3.2) follows easily by expanding $f(t)$ in a Taylor series around a point t_0, with $|t_0|=1/3$, and using Theorem 1.3, which provides just the right bounds on the Taylor coefficients.

In the other direction, let

$$\tilde{\lambda}_n = 2^n \iint_{|t|<1} \left|\frac{t}{3-t}\right|^n dA(t), \qquad \lambda_n = n^2 \tilde{\lambda}_n,$$

and take $f \in B_0'\mathbb{D}$ and let $R_n[f:z_0, z_1, \dots, z_n; z]$ be a remainder, where $z_0, z_1, \dots, z_n; z \in \frac{1}{3}\mathbb{D}$. We assert that for some constant K, independent of n,

$$|R_n[f:z_0, \dots, z_n; z]| \leqq K(1+\lambda_n) \sup\{|f(t)|(1-|t|^2): t \in \mathbb{D}\}.$$

We further assert that

$$\alpha n^{1/2} \leqq \lambda_n \leqq \beta n^{1/2}$$

for some positive finite constants α and β. Later we will show that the factor λ_n is sharp—i.e. is achieved for a suitable L_n.

Recalling (1.2) again

$$R_n = \frac{1}{2\pi i} \int_{|t|=R} \prod \frac{z-z_i}{t-z_i} f(t) \frac{1}{t-z}\, dt, \quad \tfrac{1}{3} < R < 1, \tag{3.3}$$

let us write, for $0<s<1$,

$$R_n[s] = \frac{1}{2\pi i} \int_{|t|=1} \prod \frac{z-z_i}{t-z_i} f(st) \frac{1}{t-z}\, dt. \tag{3.4}$$

Writing

$$f(t) = \sum a_m t^m,$$

$$b_m = \frac{1}{2\pi i} \int_{|t|=1} \prod \frac{z-z_i}{t-z_i} t^m \frac{dt}{t-z},$$

$$g(t) = \sum b_m t^m = \frac{1}{2\pi i} \int_{|w|=1} \prod \frac{z-z_i}{w-z_i} \frac{1}{1-tw} \frac{dw}{w-z},$$

we then have

$$R_n[s] = \sum a_m b_m s^m.$$

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Now

$$g'(0)=b_1=\frac{1}{2\pi i}\int\prod\frac{z-z_i}{w-z_i}\,w\,\frac{dw}{w-z},$$

so that

$$|g'(0)|\leqq\frac{(2/3)^{n+1}}{(2/3)^{n+1}}\frac{1}{2/3}=3/2.$$

We refer now to the description of $(B_0'\mathbb{D})^*$ given at the beginning of this section, so that it remains to estimate $\iint_{|t|<1}|g''(t)|\,dA(t)$. Now $g(t)$ is exactly the remainder of the interpolant, at z, to $(1-tw)^{-1}$, knotted at $z_0, z_1, \ldots, z_n$. It is a polynomial in z. So

$$g(t)=\left[\prod\frac{z-z_i}{(1/t)-z_i}\right]\frac{1}{1-tz}. \tag{3.5}$$

(We get this from $(1-tz)g(t)=A\prod(z-z_i)$, where we set $z=1/t$ to evaluate the constant A; see the proof of Theorem 1.1.)

Write

$$g(t)=\frac{1}{1-tz}\prod(z-z_i)\prod\left(\frac{t}{1-tz_i}\right),$$

and take the logarithmic derivative to get

$$g'(t)=g(t)\left[\frac{z}{1-tz}+\frac{n+1}{t}+\sum\frac{z_i}{1-tz_i}\right].$$

Now take the ordinary derivative to get

$$g''(t)=g(t)\left\{\left[\frac{z}{1-tz}+\frac{n+1}{t}+\sum\frac{z_i}{1-tz_i}\right]^2+\left[\frac{z^2}{(1-tz)^2}-\frac{n+1}{t^2}+\sum\frac{z_i^2}{(1-tz_i)^2}\right]\right\}.$$

Thus, for $|t|>1/2$, say,

$$K^{-1}n^2|g(t)|\leqq|g''(t)|\leqq Kn^2|g(t)|,$$

where K is an absolute constant.

By Cauchy's formula (changing K, perhaps),

$$\iint_{|t|<1}|g''(t)|\,dA(t)\leqq K\iint_{\frac{1}{2}<|t|<1}|g''(t)|\,dA(t).$$

Thus, we may ignore t with $|t|\leqq 1/2$, and we have

$$\iint_{\frac{1}{2}<|t|<1}|g''(t)|\,dA(t)\leqq Kn^2\iint_{\frac{1}{2}<|t|<1}|\tilde{g}(t)|\,dA(t).$$

So our task is to estimate

$$I=\iint_{\frac{1}{2}<|t|<1}\prod\left|\frac{z-z_i}{(1/t)-z_i}\right|dA(t).$$

For convenience, we replace $n+1$ by n. Surely,

$$I\leqq(\tfrac{2}{3})^n\iint_{\frac{1}{2}<|t|<1}\prod\left|\frac{t}{1-tz_i}\right|dA(t).$$

Now by the extended Hölder inequality (see [4], p. 22, formula 2.7.2), we get

$$I\leqq(\tfrac{2}{3})^n\left\{\left[\iint\left|\frac{t}{1-tz_1}\right|^n dA(t)\right]^{1/n}\cdots\left[\iint\left|\frac{t}{1-tz_n}\right|^n dA(t)\right]\right\}^{1/n}.$$

By subharmonicity and rotation invariance,

$$I\leqq(\tfrac{2}{3})^n\iint_{\frac{1}{2}<|t|<1}\left|\frac{t}{1-(1/3)t}\right|^n dA(t)=2^n\iint_{\frac{1}{2}<|t|<1}\left|\frac{t}{3-t}\right|^n dA(t).$$

We must estimate this last integral. Writing it in polar coordinates and taking standard estimates in terms of incomplete Beta functions, we arrive at the desired estimate. Some of the details are given below, where we have replaced n by $2n$ for convenience of notation. We write

$$J_n(r)=2^{2n}\int_{-\pi}^{\pi}\left|\frac{1}{3-r\,e^{i\theta}}\right|^{2n}d\theta.$$

Away from $\theta=0$, there is no action, so look at $\int_0^a$, for a fixed. We have

$$2^{2n}\int_0^a\left(\frac{1}{9+r^2-6r\cos\theta}\right)^n d\theta\leqq 2^{2n}\int_0^a\left(\frac{1}{9+r^2-6r(1-\alpha\theta^2)}\right)^n d\theta$$

$$=2^{2n}\int_0^a\left(\frac{1}{(3-r)^2+6\alpha r\theta^2}\right)^n d\theta,$$

where we choose $\alpha=10^{-2}$, say. Going further, we get

$$J_n(r)\leqq\frac{2^{2n}}{(3-r)^{2n}}\int_0^1(1-\gamma r\theta^2)^n\,d\theta$$

for a suitable small γ, and, writing $x=\gamma r\theta^2$, we get

$$J_n(r)\leqq K\frac{2^{2n}}{(3-r)^{2n}}\int_0^{\gamma r}x^{\frac{1}{2}-1}(1-x)^{n-1}\,dx.$$

This is the incomplete Beta function we mentioned, and it is asymptotically the same as

$$B(\tfrac{1}{2}, n) = \int_0^1 x^{\frac{1}{2}-1}(1-x)^{n-1}\,dx = \frac{\Gamma(\frac{1}{2})\Gamma(n)}{\Gamma(n+\frac{1}{2})} \sim n^{-1/2}$$

by Stirling's formula.

So we must obtain the asymptotic estimate

$$\int_{\frac{1}{2}}^{1} \frac{2^{2n} r^{2n+1}}{(3-r)^{2n}}\,dr \sim C/n.$$

But this follows on making the change of variables $S = r/(3-r)$. It is a simple matter to reverse these estimates, so we finally get

$$\bar{K} n^{-3/2} \leqq \int_0^1 J_n(r) r^{2n+1}\,dr \leqq \tilde{K} n^{-3/2},$$

which leads directly to the desired estimate for λ_n.

To see that the factor $n^{1/2}$ in our theorem is sharp, take $z_0 = z_1 = \cdots = z_n = 1/3$, and $z = -1/3$. (This extremal situation illustrates again the principle that the Taylor interpolation around a boundary point, evaluated at the diametrically opposite boundary point, is the worst case of Lagrange interpolation.) Looking over the estimates we have just made, we see that in this case, they are sharp and this observation demonstrates the final assertion of the theorem.

Remark. It follows from the theorem just proved that there is no function $\beta(r)$, continuous and increasing on $[0, 3)$, with $\beta(3-) = +\infty$, such that $TB\mathbb{D}$ is exactly those functions f analytic in $3\mathbb{D}$ with $\sup_{-\pi \leqq \theta \leqq \pi} |f(re^{i\theta})|(\beta(r))^{-1}$ bounded. For $\beta(r)$ could go to ∞ no faster than $\beta_1(r) = c(3-r)^{-1}$ as $r \to 3-$, but the space corresponding to this β_1 is exactly $B'(3\mathbb{D})$ which we have seen *properly* contains $TB\mathbb{D}$. It would be interesting to exhibit a specific function $f \in B'3\mathbb{D} \setminus TB\mathbb{D}$.

4. Open problems

Problem 1. Is it true that

$$TC\mathbb{D} \approx c_0 \quad \text{and} \quad TB\mathbb{D} \approx l^\infty,$$

where $\approx$ denotes linear isomorphism of Banach spaces? Note that it was shown in [13] and [15] that for the Bloch spaces, $B_0\mathbb{D} \approx c_0$ and $B\mathbb{D} \approx l^\infty$. More generally, are there descriptions of $TC\mathbb{D}$ and $TB\mathbb{D}$ in terms of some of the classical Banach spaces?

Problem 2. Is there a natural *geometrical* characterization of totally bounded functions in the unit disc, similar in spirit to the characterization in [1], via the size of schlicht discs in their range, of Bloch functions?

Problem 3. Does $TB\mathbb{D}$ (or $TB\frac{1}{3}\mathbb{D}$) have any kind of Möbius invariance?

Problem 4. Can anything more be said about (radial) boundary values in $\mathbb{D}$ of primitives of functions in $TB\frac{1}{3}\mathbb{D}$ than is said about boundary values of Bloch functions in Section 4 of [1]?

Problem 5. If $g \in TB\mathbb{D}$ and f is analytic in $3\mathbb{D}$, with $|f(z)| \leqq |g(z)|$ for all $z \in 3\mathbb{D}$, must $f \in TB\mathbb{D}$?

Catchall Problem. Let P be a property that a function f on a set S may or may not have. We say that f is "totally P" if every Lagrange interpolant p to f has property P. This scheme opens many interesting avenues of inquiry which we only briefly delineate here.

For analytic functions in $\mathbb{D}$, how about totally univalent (exclude the constant interpolants, of course), totally convex, and totally starlike?

On a general set $S \subseteq \mathbb{C}$, how about totally continuous, and totally Lipschitz? To say that f is totally continuous is to say that there exists a continuous function $\omega: \mathbb{R}^+ \to \mathbb{R}^+$, with $\omega(0+) = 0$ (a "modulus of continuity"), such that $|p(x) - p(y)| \leqq \omega(|x - y|)$ for all x and y in S. If we take $\omega(t) = kt$, with k a generic positive constant, we get "totally Lipschitz".

How about totally normal, where we assume now that the set of all Lagrange interpolants to f forms a normal family on S?

For the unit disc $\mathbb{D}$ again, choose a number $p > 0$ and look at $TH^p\mathbb{D}$, where

$$\|f\|_{TH^p\mathbb{D}} = \sup \|p_n\|_{H^p},$$

where the p_n range over all the Lagrange interpolants to f, with all nodes in $\mathbb{D}$. Similarly we could have $TB^p\mathbb{D}$ (totally Bergman), or totally $BMO\mathbb{D}$, where the H^p norm above is replaced by the Bergman or BMO norm. Similarly, we could use the Bloch or Dirichlet norm to get "totally Bloch" and "totally Dirichlet". The possibilities are endless!

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