Sequences defined as minima of two Fibonacci-type relations

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If \( \{L_n\} \) is a sequence defined by

\[
L_n = \min\{L_{n-a} + L_{n-b}, L_{n-c} + L_{n-d}\},
\]

with \( a, b, c, d \) positive integers, then one can ask if necessarily \( L_n = L_{n-a} + L_{n-b} \) for all sufficiently large \( n \).

The answer is yes if \( a \) and \( b \) are relatively prime, \( L_n > 0 \) initially, and \( \lambda < \mu \), where \( \lambda^{-a} + \lambda^{-b} = 1 \), \( \mu^{-c} + \mu^{-d} = 1 \).

The answer is no if instead \( a \) and \( b \) have greatest common divisor \( k \geq 2 \), with \( \sigma \equiv 0 \pmod{k} \), \( d \equiv 0 \pmod{k} \).

Introduction. Much is known about the properties of sequences defined by a recurrence of the type \( L_n = L_{n-a} + L_{n-b} \), where \( a \) and \( b \) are fixed positive integers. In this note, we produce conditions on \( a, b, c \) and \( d \), such that if

\[
L_n = \min\{L_{n-a} + L_{n-b}, L_{n-c} + L_{n-d}\}
\]

then

\[
L_n = L_{n-a} + L_{n-b}
\]

for all sufficiently large \( n \). We concern ourselves only with the case in which all initial values are positive, so that \( L_n \) is then positive for all \( n \). For a situation in which this problem arises, see [7].

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It is well known that \( L_n = L_{n-a} + L_{n-b} \) implies \( L_n = O(\lambda^n) \), where \( \lambda \) is the positive root of
\[
\lambda^a + \lambda^b = 1.
\]
Hence, if (2) holds, we must have \( \lambda \leq \mu \) where \( \mu \) is the positive root of
\[
\mu^a + \mu^b = 1.
\]
There are examples however, to show that this condition is not sufficient. One such example is
\[ L_n = \min\{2L_{n-3}, L_{n-2} + \mu\} > n^5, \]
with the initial conditions \( L_1 = L_2 = L_3 = L_4 = 1 \).

THEOREM 1. Suppose \( a, b, c \) and \( d \) are positive integers, and \( L_1, L_2, \ldots, L_e \) are given positive real numbers, where \( e = \max\{a, b, c, d\} \). Define
\[
L_n = \min\{L_{n-a} + L_{n-b}, L_{n-c} + L_{n-d}\}
\]
for \( n > e \), and define \( \lambda > 1 \) and \( \mu > 1 \) by (3) and (4). If \( \lambda < \mu \), and if \( a \) and \( b \) are relatively prime, then there exists an integer \( n_0 \) such that
\[
L_n = L_{n-a} + L_{n-b}
\]
for all \( n \geq n_0 \).

Proof. Suppose \( N \) is an integer, \( N \geq e + 1 \). Define
\[
c_N = \max_{1 \leq k \leq e} \left\{ \frac{L_{N-k}}{\lambda^{N-k}} \right\},
\]
\[
d_N = \min_{1 \leq k \leq e} \left\{ \frac{L_{N-k}}{\lambda^{N-k}} \right\}.
\]
Since
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\[ L_N \leq L_{N-a} + L_{N-b} \]
\[ \leq \lambda^{N-a}c_N + \lambda^{N-b}c_N \]
\[ = \lambda^N c_N (\lambda^{-a} + \lambda^{-b}) \]
\[ = \lambda^N c_N , \]

it follows that \( \sigma_{n+1} \leq \sigma_N \), and hence the sequence \( \{\sigma_N\} \) is decreasing.

On the other hand

\[ L_{N-a} + L_{N-b} \geq d_N\lambda^{N-a} + d_N\lambda^{N-b} \]
\[ = d_N\lambda^N \]

and

\[ L_{N-c} + L_{N-d} \geq d_N\lambda^{N-c} + d_N\lambda^{N-d} \]
\[ = d_N\lambda^N (\lambda^{-c} + \lambda^{-d}) \]
\[ > d_N\lambda^N (\mu^{-c} + \mu^{-d}) \]
\[ = d_N\lambda^N . \]

Hence, by (1), \( L_N \geq d_N\lambda^N \), so that \( d_{N+1} \geq d_N \), and the sequence \( \{d_n\} \) is increasing.

Since \( a \) and \( b \) are relatively prime, the set \( S \) consisting of all integers of the form \( sa + tb \), where \( s \) and \( t \) are positive integers, contains a smallest element with the property that all greater integers also belong to \( S \). Denote this smallest element by \( f \).

Suppose \( 0 < \varepsilon < 1 \), and \( r \) is an integer, \( r \geq N - 1 + f \). We claim that

\[ L_r/\lambda^r \geq (1-\varepsilon)\sigma_N \]

implies

\[ L_{r-q}/\lambda^{r-q} \geq (1-\varepsilon\lambda^q)\sigma_N \]
for all \( q \) in \( S \), \( q < r \).

For, (7) implies that

\[
(1-\varepsilon)\sigma_N \leq \left( \frac{L_{r-a} + L_{r-b}}{\lambda^r} \right) \lambda^r
\]

\[
= \lambda^a \left( \frac{L_{r-a}}{\lambda^{r-a}} \right) + \lambda^b \left( \frac{L_{r-b}}{\lambda^{r-b}} \right)
\]

so that \( \lambda^a (1-\varepsilon - \lambda^{-b}) \sigma_N \leq L_{r-a} / \lambda^{r-a} \) or \( (1-\varepsilon \lambda^a) \sigma_N \leq L_{r-a} / \lambda^{r-a} \) by (3).

Similarly

\[
(1-\varepsilon \lambda^b) \sigma_N \leq L_{r-b} / \lambda^{r-b} .
\]

Successively repeating the argument yields (8).

Since \( r \geq N - 1 + f \), each member of the set \( \{N-1, N-2, \ldots, N-e\} \) is of the form \( r - q \) for \( q \) in \( S \). Thus by (6) and (8), the inequality (7) implies \( d_N \geq \inf (1-\varepsilon \lambda^q) \sigma_N \), where the infimum is taken over those \( q \) for which \( N - 1 \geq r - q \geq N - e \); that is, \( r + 1 - N \leq q \leq r + e - N \). Thus (7) implies

\[
d_N \geq (1-\varepsilon \lambda^{r+e-N}) \sigma_N .
\]

By reversing the argument, if \( \varepsilon \) is now chosen such that

\[
(1-\varepsilon \lambda^{r+e-N}) > d_N / \sigma_N ,
\]

then

\[
L_{r} / \lambda^{r} < (1-\varepsilon) \sigma_N .
\]

It follows, since this implication is valid for all \( r \) in \( R = \{ r : N-1+f \leq r \leq N+f+e-2 \} \), that

\[
1 - \varepsilon \lambda^{f+2e-2} > d_N / \sigma_N
\]

implies

\[
\sup_{r \in R} L_{r} / \lambda^{r} < (1-\varepsilon) \sigma_N ,
\]
that is, (10) implies

\[(11) \quad \sigma_{N+f+e-2} < (1-\varepsilon)\sigma_N.\]

Put \(\phi_N = \sigma_N/d_N\), and choose \(\varepsilon = \left(1-\phi_N^{-1}\right)\lambda^{-f-2e+2}/2\) so that (10) holds. It follows from (11), with this choice of \(\varepsilon\), and the fact that \(d_N\) is increasing, that

\[\phi_{N+f+e-2} < \left[1-\left(1-\phi_N^{-1}\right)\lambda^{-f-2e+2}/2\right]\phi_N,\]

whence

\[\phi_{N+f+e-2} - 1 < \left[1-\lambda^{-f-2e+2}/2\right](\phi_N-1) .\]

Since \(\{\phi_N\}\) is decreasing, and the factor in the square brackets is a fixed constant between 0 and 1, we have

\[(12) \quad \lim_{N\to\infty} \phi_N = 1.\]

To complete the proof, suppose

\[L_{n-a} + L_{n-b} > L_{n-c} + L_{n-d}\]

for some \(n > \max(N+a, N+b)\). Then, since

\[\lambda^n d_N \leq L_n \leq \lambda^n \sigma_N ,\]

we have

\[
\sigma_N\lambda^{n-a} + \sigma_N\lambda^{n-b} > d_N\lambda^{n-c} + d_N\lambda^{n-d}
\]

or

\[\phi_N(\lambda^{-a}+\lambda^{-b}) > \lambda^{-c} + \lambda^{-d}\]

or

\[\phi_N > \lambda^{-c} + \lambda^{-d} > 1.\]

This contradicts (12) if \(N\) is big enough.

We consider briefly what can happen if \(a\) and \(b\) are not relatively
prime. Let \( k \) be the highest common factor of \( a \) and \( b \). It is immediate, by considering the subsequences of the form \( L_{n_0 + mk} \), that the result of Theorem 1 still holds if \( c \equiv 0 \pmod{k} \) and \( d \equiv 0 \pmod{k} \).

**THEOREM 2.** If \( \lambda < \mu \), if \( k \) is the greatest common divisor of \( a \) and \( b \), with \( k \geq 2 \), if \( c \equiv 0 \pmod{k} \), and \( d \not\equiv 0 \pmod{k} \), then there is a set of positive values for \( L_n \), \( 1 \leq n \leq e \), such that (1) holds for \( n > e \), and \( L_n < L_{n-a} + L_{n-b} \) for an infinite set of integers \( n \).

**Proof.** Define (for convenience) \( L_{\gamma k} = 1 \) for integer \( \gamma \), \( 0 \leq \gamma k < \max(a, b) \). This determines \( L_n \) for all \( n \equiv 0 \pmod{k} \) by \( L_n = L_{n-a} + L_{n-b} \). Next define \( L_n \) for \( n \equiv -d \pmod{k} \) by the equation \( L_n = L_{n-a} + L_{n-d} \) for \( n \equiv 0 \pmod{k} \), that is, \( L_n \equiv L_{n+d} - L_{n+d-c} \) for \( n \equiv -d \pmod{k} \). It is easy to check that one then has \( L_n = L_{n-a} + L_{n-b} \) for \( n \equiv -d \pmod{k} \), at least for \( n \geq c - d + \max(a, b) \). In a similar manner define \( L_n \) successively for \( n \equiv -2d, n \equiv -3d, n \equiv -4d, \ldots, n \equiv -(k-2)d \). \( L_n \) is then determined for all \( n \) larger than some fixed integer \( n_0 \), \( n \not\equiv d \pmod{k} \), and, for such \( n \), \( L_n = L_{n-a} + L_{n-b} = L_{n-c} + L_{n-d} \).

Now define \( L_n = L_{n-c} + L_{n-d} \) for \( n \equiv d \). Since then \( n - d \equiv 0 \), \( L_{n-d} = L_{n-a-d} + L_{n-b-d} \), so the equation \( (L_n - L_{n-a} - L_{n-b}) = (L_{n-c} - L_{n-a} - L_{n-b}) \) holds for all \( n \equiv d \). Thus, suitable initial conditions can ensure that if this value is initially a negative constant, then by induction, \( L_n = L_{n-c} + L_{n-d} < L_{n-a} + L_{n-b} \) for all \( n \equiv d \pmod{k} \).

The author has been unable to obtain similar general results for the case when \( k \geq 2 \) and both \( c \not\equiv 0 \) and \( d \not\equiv 0 \pmod{k} \). We cite two examples to show what may or may not occur.
If \( a = b = k = 3 \), \( c = 1 \), and \( d = 4 \), then \( L_n = 2L_{n-3} \) for all sufficiently large \( n \). It is worth noting that this result cannot be established by the method of proof of Theorem 1, since the quotient \( \sigma_n^d \) need not converge. The proof however is straightforward after observing that

(a) one cannot have \( L_n = L_{n-1} + L_{n-4} \) for three consecutive values of \( n \);

(b) if \( L_n = 2L_{n-3} \) for four consecutive values of \( n \), then
\[ L_n = 2L_{n-3} \] for all larger \( n \).

On the other hand, if \( a = b \), \( k = 3 \), \( c = 1 \) and \( d = 5 \), and if \( L_1, L_2, L_3, L_4, L_5 \) respectively equal 16, 16, 11, 6, 1; then

\[ L_n = 2L_{n-3} \] if \( n \equiv 0, 1, 2 \) or \( 5 \) (mod 6)

\[ L_n = L_{n-1} + L_{n-5} < 2L_{n-3} \] if \( n \equiv 3 \) or \( 4 \) (mod 6).

Theorems 1 and 2 generalize immediately to sequences of the form

\[ L_n = \min_{1 \leq m} \left\{ L_{n-a} + L_{n-b} \right\} \]

Clearly too, one can establish analogous results for maxima.

Reference