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MEASURE CONVERGENT SEQUENCES IN LEBESGUE SPACES AND FATOU'S LEMMA

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Let (f_n) be a sequence of positive P-integrable functions such that $\left(\int f_n dP\right)_n$ converges. We prove that (f_n) converges in measure to $\underline{\lim}_{n\to\infty} f_n$ if and only if equality holds in the generalised Fatou's lemma. Let f_∞ be an integrable function such that $\left(\|f_n - f_\infty\|_1\right)_n$ converges. We present in terms of the modulus of uniform integrability of (f_n) necessary and sufficient conditions for (f_n) to converge in measure to f_∞.

1. INTRODUCTION

In [**6**] we proved the following result: let (Ω, Σ, P) be a probability space and (f_n) a sequence of positive integrable functions such that $\left(\int f_n dP\right)_n$ converges. Then (f_n) converges in norm to $\underline{\lim}_{n\to\infty} f_n$ if and only if equality holds in Fatou's lemma. This is a striking example of the well known fact that under suitable extreme point conditions, weak convergence in L^1-spaces (and even much less) implies strong convergence [**1**]. By means of the modulus of uniform integrability of (f_n) (to be defined later), we proved a generalisation of Fatou's lemma [**6**, Corollary 4]. In the present paper we pose the following question: when does (f_n) converge in measure to $\underline{\lim}_{n\to\infty} f_n$? We show that this is the case if and only if for all subsequences of (f_n) equality holds in the generalised Fatou's lemma (Theorem 3). More generally we study the convergence in measure of a bounded sequence (f_n) to an arbitrary element $f_\infty \in L^1(\mathbb{R})$ (Theorem 7). Both Theorem 3 and Theorem 5 enable us to give a straightforward proof of Lebesgue's convergence Theorem [**3**, p.122].

2. PRELIMINARIES

Throughout this paper, (Ω, Σ, P) will be probability space. We shall consider the Banach space $L^1(\mathbb{R})$ of all (classes of) P-Bochner-integrable functions from Ω to $\mathbb{R}$.

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In [7] Rosenthal defined the modulus of uniform integrability $\eta(H)$ of a bounded subset $H \subseteq L^1(\mathbb{R})$: For $\varepsilon > 0$, put

$$\eta(H, \varepsilon) = \sup\Big\{ \int_A |h|\, dP : h \in H,\ A \in \Sigma,\ P(A) \leqslant \varepsilon \Big\},$$
$$\eta(H) = \lim_{\varepsilon \to 0+} \eta(H, \varepsilon).$$

Thus H is uniformly integrable if and only if $\eta(H) = 0$.

3. Results

We start with a lemma proved in [4] and extended to Banach space valued integrable functions in [5].

Lemma 1. *Let $f = (f_n)$ be a bounded sequence in $L^1(\mathbb{R}_+)$ converging in measure to an element f_∞ of $L^1(\mathbb{R}_+)$. Then the following assertions are equivalent:*

(i) $\lim_{n \to +\infty} \int f_n\, dP = \eta(f) + \int f_\infty\, dP$ *and* $\eta(f') = \eta(f)$ *for each subsequence* f' *of* f*;*

(ii) *the sequence of reals* $\left(\int f_n\, dP\right)_n$ *converges in* $\mathbb{R}_+$.

Corollary 2. *Let $f = (f_n)$ be a bounded sequence in $L^1(\mathbb{R})$ converging in measure to $f_\infty \in L^1(\mathbb{R})$. Then $(\|f_n - f_\infty\|_1)_n$ converges in $\mathbb{R}$ if and only if $\eta(f') = \eta(f)$ for each subsequence f' of f and in this case $\lim_{n \to +\infty} \|f_n - f_\infty\|_1 = \eta(f)$.*

Theorem 3. *Let $f = (f_n)$ be a bounded sequence in $L^1(\mathbb{R}_+)$ such that the sequence $\left(\int f_n\, dP\right)$ converges in $\mathbb{R}_+$. Then the following assertions are equivalent:*

(i) $\lim_{n \to +\infty} \int f_n\, dP = \eta(f) + \int \varliminf_{n \to \infty} f_n\, dP$ *and* $\eta(f') = \eta(f)$ *for each subsequence* f' *of* f*;*

(ii) *the sequence* (f_n) *converges in measure to* $\varliminf_{n \to \infty} f_n$.

Proof: The implication (ii) $\Rightarrow$ (i) is a consequence of Lemma 1. Suppose now that (i) is true. Let $f' = (f'_n)$ be a subsequence of f. On account of the generalised Fatou's lemma [**6**, Corollary 4], we have

$$\lim_{n \to +\infty} \int f_n\, dP \geqslant \eta(f) + \int \varliminf_{n \to \infty} f'_n\, dP.$$

By comparing this inequality with the hypothesis, we obtain the following relation:

$$\int \varliminf_{n \to \infty} f_n\, dP \geqslant \int \varliminf_{n \to \infty} f'_n\, dP.$$

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It follows that $\underline{\lim}_{n\to\infty} f_n = \underline{\lim}_{n\to\infty} f'_n$ P-almost everywhere. Hence

$$\lim_{n\to+\infty}\int f'_n\,dP = \eta(f') + \int \underline{\lim}_{n\to\infty} f'_n\,dP.$$

So Theorem 10 of [**6**] applies to the sequence (f'_n) and says that there is a further subsequence (f''_n) of (f'_n) converging in measure to $\underline{\lim}_{n\to\infty} f'_n$, which equals $\underline{\lim}_{n\to\infty} f_n$ P-almost everywhere.

The proof is complete. $\square$

Proposition 4. *Let $f = (f_n)$ be a bounded sequence in $L^1(\mathbb{R}_+)$ and let $f' = (f'_n)$ a subsequence of f such that $\lim_{n\to+\infty} \int f'_n\,dP = \underline{\lim}_{n\to\infty} \int dP$. Then the following assertions are equivalent:*

(i) *$\underline{\lim}_{n\to\infty} \int f_n\,dP = \eta(f') + \int \underline{\lim}_{n\to\infty} f_n\,dP$ and $\eta(f') = \eta(f'')$ for each subsequence f'' of f';*

(ii) *the sequence (f'_n) converges in measure to $\underline{\lim}_{n\to\infty} f_n$.*

Proof: Suppose that (i) is true. Let $f' = (f'_n)$ be a subsequence of (f_n) satisfying the hypothesis of Proposition 4. It follows that

$$\lim_{n\to+\infty}\int f'_n\,dP = \eta(f') + \int \underline{\lim}_{n\to\infty} f_n\,dP \leqslant \eta(f') + \int \underline{\lim}_{n\to\infty} f'_n\,dP. \tag{1}$$

By the generalised Fatou's lemma [**6**, Corollary 4] we obtain

$$\lim_{n\to+\infty}\int f'_n\,dP \geqslant \eta(f') + \int \underline{\lim}_{n\to\infty} f'_n\,dP.$$

Thus we have two equalities in (1). Since all subsequences of f' have the same modulus of uniform integrability, Theorem 3 applies to the sequence f'. Consequently (f'_n) converges in measure to $\underline{\lim}_{n\to\infty} f'_n$. Now $\underline{\lim}_{n\to\infty} f'_n$ and $\underline{\lim}_{n\to\infty} f_n$ are comparable and their integrals coincident because of the second equality in (1). This means that $\underline{\lim}_{n\to\infty} f'_n(\omega) = \underline{\lim}_{n\to\infty} f_n(\omega)$ P-almost everywhere.

Conversely, suppose that (ii) is true and let $f' = (f'_n)$ be a subsequence of f such that

$$\lim_{n\to+\infty}\int f'_n\,dP = \underline{\lim}_{n\to\infty}\int f_n\,dP.$$

As (f'_n) converges in measure to $\underline{\lim}_{n\to\infty} f_n$, we can apply the implication (ii) $\Rightarrow$ (i) of Lemma 1 to the sequence (f'_n), and the proof is done. $\square$

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Let us consider a special case of Theorem 3. If $\eta(f) = 0$, then we obtain a result which was the starting point of our investigation. Note that it was used in the proof of Theorem 3.

THEOREM 5. *Let $f = (f_n)$ be a bounded sequence in $L^1(\mathbb{R}_+)$. Then the following assertions are equivalent:*

(i) $\left(\int f_n \, dP\right)$ *converges in* $\mathbb{R}_+$ *and* $\lim\limits_{n\to+\infty} \int f_n \, dP = \int \underline{\lim}_{n\to\infty} f_n \, dP$;

(ii) (f_n) *converges in norm to* $\underline{\lim}_{n\to\infty} f_n$.

PROOF: Suppose that (i) is true. By the generalised Fatou's lemma we have

$$\lim_{n\to+\infty} \int f_n \, dP \geqslant \eta(f) + \int \underline{\lim}_{n\to\infty} f_n \, dP.$$

It follows that $\eta(f) = 0$. We know from Theorem 3 that (f_n) converges in measure to $\underline{\lim}_{n\to\infty} f_n$. Note that a measure convergent and uniformly integrable sequence converges in norm. ∎

REMARK. As pointed out in [**6**], the combination of Theorem 5 and Fatou's lemma yields Lebesgue's convergence theorem [**3**, p.122].

LEMMA 6. *Let $f = (f_n)$ be a bounded sequence in $L^1(\mathbb{R})$ converging in measure to an element f_∞ belonging to $L^1(\mathbb{R})$. Then the sequence $(\|f_n\|_1)$ converges if and only if $(\|f_n - f_\infty\|)$ does and in this case we have* $\lim\limits_{n\to+\infty} \|f_n - f_\infty\|_1 = \eta(f) = \lim\limits_{n\to+\infty} (\|f_n\|_1 - \|f_\infty\|_1)$.

PROOF: We know from Brezis and Lieb [**2**] that

$$\lim_{n\to+\infty} (\|f_n\|_1 - \|f_n - f_\infty\|_1)_n = \|f_\infty\|_1 \, .$$

Suppose that $\lim\limits_{n\to+\infty} \|f_n\|_1$ exists. As $(|f_n|)_n$ converges in measure to $|f_\infty|$, it follows from Lemma 1 of [4] that

$$\lim_{n\to+\infty} \|f_n\|_1 = \eta(f) + \|f_\infty\|_1 \, .$$

The combination of the last two equalities yields the first implication. To prove the opposite implication, suppose that the sequence $(\|f_n - f_\infty\|_1)_n$ converges. We know from Lemma 1 of [4] that its limit is $\eta(f)$. An application of Brezis' and Lieb's equality completes the proof. ∎

THEOREM 7. *Let $f = (f_n)$ be a bounded sequence in $L^1(\mathbb{R})$ and let f_∞ be an element of $L^1(\mathbb{R})$. Suppose that $(\|f_n - f_\infty\|_1)_n$ converges in $\mathbb{R}$. Then the following assertions are equivalent:*

(i) (f_n) *converges in measure to* f_∞;

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(ii) $\lim_{n\to+\infty}\|f_n - f_\infty\|_1 \leqslant \eta(f)$ *and* $\eta(f) = \eta(f')$ *for each subsequence* f' *of* f.

(iii) $\lim_{n\to+\infty}\|f_n - f_\infty\|_1 = \eta(f)$ *and* $\eta(f) = \eta(f')$ *for each subsequence* f' *of* f.

PROOF: We know from Corollary 2 that (i) implies (iii). Suppose now that (ii) is true and let $f' = (f'_n)$ be any subsequence of f. Note that

$$\lim_{n\to+\infty}\|f'_n - f_\infty\|_1 \leqslant \eta(f').$$

Hence Theorem 6 of [4] applies to the subsequence (f'_n) and says that there is a further subsequence (f''_n) of (f'_n) which converges in measure to f_∞. Consequently assertion (i) follows. ☐

PROPOSITION 8. *Let* $f = (f_n)$ *be a bounded sequence in* $L^1(\mathbb{R}_+)$ *such that*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n}\int f_k\,dP = \int \varliminf_{n\to\infty} f_n\,dP.$$

Then the following statements hold:

(i) $\left(1/n\sum_{k=0}^{n} f_k\right)_n$ *converges in norm to* $\varliminf_{n\to\infty} f_n$;

(ii) *Let* $f' = (f'_n)$ *be any subsequence of* (f_n) *satisfying* $\lim_{n\to+\infty}\int f'_n\,dP = \varliminf_{n\to\infty}\int f_n\,dP$. *Then* (f'_n) *converges in norm to* $\varliminf_{n\to\infty} f_n$.

PROOF: Put $m(f) = \left(1/n\sum_{k=0}^{n}\int f_k\,dP\right)_n$. Note that

$$\int \varliminf_{n\to\infty} f_n\,dP = \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n}\int f_k\,dP \geqslant \eta(m(f)) + \int \varliminf_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n} f_k\,dP$$
$$\geqslant \eta(m(f)) + \int \varliminf_{n\to\infty} f_n\,dP.$$

The first of the preceding inequalities comes from the generalised Fatou's lemma. The second one is obvious. It follows that $\eta(m(f)) = 0$ and that $\varliminf_{n\to\infty} f_n(\omega) = \varliminf_{n\to\infty} 1/n\sum_{k=0}^{n} f_k(\omega)$ P-almost everywhere. Now the hypothesis can be written as follows:

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n}\int f_k\,dP = \int \varliminf_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n} f_k\,dP.$$

Theorem 5 applies and yields the assertion (i).

Let $f' = (f'_n)$ be as in (ii). Note that

$$\underline{\lim}_{n\to\infty} f_n \int f_n\, dP \leqslant \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n}\int f_k\, dP = \int \underline{\lim}_{n\to\infty} f_n\, dP.$$

In particular we have $\underline{\lim}_{n\to\infty} \int f_n\, dP = \int \underline{\lim}_{n\to\infty} f_n\, dP$. On the other hand, we know that

$$\underline{\lim}_{n\to\infty} \int f_n\, dP = \lim_{n\to+\infty} \int f'_n\, dP \geqslant \eta(f') + \int \underline{\lim}_{n\to\infty} f'_n\, dP \geqslant \eta(f') + \int \underline{\lim}_{n\to\infty} f_n\, dP.$$

So the preceding inequalities reduce to equalities and it follows that $\eta(f') = 0$. Proposition 4 or Theorem 5 enable us to say that (f'_n) converges in norm to $\underline{\lim}_{n\to\infty} f_n$. ∎

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