

IMMERSION OF MANIFOLDS WITH UNBOUNDED IMAGE AND A MODIFIED MAXIMUM PRINCIPLE OF YAU

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Abstract

Let N be a complete Riemannian manifold isometrically immersed into a Hadamard manifold M . We show that the immersion cannot be bounded if the mean curvature of the immersed manifold is small compared with the curvature of M and the Laplacian of the distance function on N grows at most linearly. The latter condition is satisfied if the Ricci curvature of N does not approach $-\infty$ too fast. The main tool in the proof is a modification of Yau's maximum principle.

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1. Introduction

Let N be a complete Riemannian manifold isometrically immersed into a Hadamard manifold M ; in other words, M is a complete, simply connected manifold with nonpositive curvature. We are looking for conditions that force the immersion to be unbounded. We impose two conditions. First, we require the immersion to be close to a minimal immersion; that is, the mean curvature of the immersion must be small. Second, we require that the Laplacian of the distance function on N grows at most linearly.

It is perhaps unusual to impose an analytical condition on the immersed manifold, rather than a geometrical one (involving bounds on the curvatures); but our reasons are two-fold.

First of all, there is a strong connection between the Laplacian of the distance function and the Ricci curvature. For instance, a lower bound on the Ricci curvature would imply, via standard comparison arguments, an upper bound on the Laplacian of the distance function for distances larger than one. By the same token, a decay condition on the Ricci curvature (which says that the Ricci curvature cannot approach $-\infty$ faster than a certain rate) yields a growth condition on the Laplacian of the distance function.

The second reason is that this analytical condition on the immersed manifold is the property that will play an especially important role in the proofs.

It is also clear that we must impose some condition on the immersed manifold, either in the form of a curvature condition or in terms of the Laplacian of the distance function, because there are examples [1] of complete immersed minimal surfaces into \mathbb{R}^3 that are bounded.

Before we proceed to the theorem, it will be convenient to formulate in a precise manner the analytical condition we need.

CONDITION (Δ). We say that the Riemannian manifold N^n satisfies Condition (Δ) if there exist a point $p \in N$ and a positive number $C > 0$ such that $\Delta r(x) < Cr(x)$ for all $x \in N$, where r , the distance function from p on N , is smooth with $r(x) > 1$.

Our result can now be formulated as follows.

THEOREM. *Let N^n be a complete Riemannian manifold satisfying Condition (Δ). Let M^m be a complete simply connected manifold with sectional curvatures $K \leq -a^2$, where $a \geq 0$ and $m > n$. Let $f : N^n \rightarrow M^m$ be an isometric immersion such that the norm of the mean curvature vector of f satisfies $\|H_f\| \leq a(n-1)/n$. Then $f(N^n)$ is not bounded in M^m .*

The important tool in the proof of this theorem is the following modified version of Yau's maximum principle [3, Theorem 1].

LEMMA. *Let N^n be a complete Riemannian manifold satisfying Condition (Δ) and let $g : N^n \rightarrow \mathbb{R}$ be a smooth function that is bounded from above. Then, for every $\varepsilon > 0$, there is a point $x_\varepsilon \in N^n$ such that $|g(x_\varepsilon) - \sup g| < \varepsilon$, $\|\nabla g(x_\varepsilon)\| < \varepsilon$ and $\Delta g(x_\varepsilon) < \varepsilon$.*

The point x_ε in the lemma plays the role of an 'approximate maximum'.

Let us remark that the condition $\Delta r = O(r)$ is quite optimal. For manifolds with $\Delta r > r^{1+\varepsilon}$ for large r , it is easy to construct a bounded function for which the lemma does not apply. The details can be found at the end of this paper.

2. Proofs

PROOF OF THE THEOREM. Let us assume, on the contrary, that $f(N)$ is bounded in M .

Let $O \in M - N$ be any fixed point, and denote by $r : M \rightarrow \mathbb{R}^+$ the distance function from O on M . Let $r|_N : N \rightarrow \mathbb{R}^+$ be the restriction of r to N , that is, $r|_N(x) = r(f(x))$. Our goal is to compute $\Delta r|_N$.

To simplify notation and to make the calculations easier to follow, we shall identify the manifold N with its image $f(N)$. This has the advantage that we do not need to push forward (and pull back) tangent vectors of N to tangent vectors of M via $df : TN \rightarrow TM$. Since $f : N \rightarrow M$ is an isometric immersion, this identification is justified.

Let ∇ denote the connection on M , $\tilde{\nabla}$ the connection on N and $\langle \cdot, \cdot \rangle$ the Riemannian metric on M .

Let $q \in N$; near q , we can decompose ∇r into tangential and normal components of TN ,

$$\nabla r = (\nabla r)^T + (\nabla r)^N,$$

and we have the relation

$$\nabla r|_N = (\nabla r)^T.$$

For tangent vectors $X, Y \in T_q N$,

$$\begin{aligned} \text{Hess}(r|_N)(X, Y) &= \langle \tilde{\nabla}_X \nabla(r|_N), Y \rangle = \langle \nabla_X (\nabla r - (\nabla r)^N), Y \rangle \\ &= \text{Hess } r(X, Y) - \langle \nabla_X (\nabla r)^N, Y \rangle. \end{aligned}$$

Since the normal component $(\nabla r)^N$ of the gradient is orthogonal to N , it follows that

$$\langle \nabla_X (\nabla r)^N, Y \rangle = -\langle (\nabla r)^N, \nabla_X Y \rangle = \langle (\nabla r)^N, A(X, Y) \rangle,$$

where $A(X, Y) = -(\nabla_X Y)^N$ denotes the vector-valued second fundamental form of N . Combining this with the equality above, we obtain

$$\text{Hess}(r|_N)(X, Y) = \text{Hess } r(X, Y) - \langle (\nabla r)^N, A(X, Y) \rangle. \tag{1}$$

Let λ be a nonzero eigenvalue of r at $x \in M$. Since M is a Hadamard manifold with sectional curvature $K \leq -a^2$, a standard comparison argument [2, Ch. 6] gives us that $\lambda \geq a \coth(ar(x))$ if $a > 0$, and $\lambda \geq 1/r(x)$ if $a = 0$. Since we assumed that $f(N)$ is bounded in M , there is a $\delta > 0$, depending only on the bound of $f(N)$, such that for all points of N , $\lambda > a + \delta$. This implies that for any unit vector $X \in T_q N$ orthogonal to ∇r^T , we have $\text{Hess } r(X, X) > a + \delta$.

Taking the trace in (1) with respect to $T_q N$, we get

$$\Delta(r|_N) > (n - 1)(a + \delta) - \langle nH_f(q), \nabla r^N \rangle \geq (n - 1)(a + \delta) - n\|H_f(q)\|,$$

where $H_f = (1/n)\text{Trace}(A)$ denotes the mean curvature vector of N .

Since we assumed that $\|H_f\| \leq a(n - 1)/n$, it follows that

$$\Delta(r|_N) > (n - 1)\delta,$$

for every point of N . But this clearly contradicts the lemma, as $r|_N$ is a bounded smooth function on N . The proof of the theorem is thus complete. \square

PROOF OF THE LEMMA. Set $L = \sup g$. We can assume that $g < L$ at every point of N ; otherwise, g must assume its maximum at some point, and that point would trivially satisfy the conditions of the lemma for all $\varepsilon > 0$.

Let $p \in N$ be the point and $C > 0$ the positive constant in Condition (Δ) ; also let r be the distance function from p on N .

For any $\varepsilon < L - \sup\{g(x) \mid r(x) \leq 1\}$, define the function $h_\lambda : N \rightarrow \mathbb{R}$ by

$$h_\lambda(x) = \lambda r(x) + L - \varepsilon.$$

Clearly,

$$h_\lambda(x) > g(x) \quad \text{if } r(x) \leq 1 \text{ and } \lambda \geq 0. \tag{2}$$

Therefore, if $\lambda > \varepsilon$, then $h_\lambda(x) > g(x)$ for all $x \in N$.

Define λ_0 as

$$\lambda_0 = \inf\{\lambda \mid h_\lambda(x) > g(x) \text{ for all } x \in N\}.$$

Since $\sup g = L$, it is easy to see that $\lambda_0 > 0$ and $h_{\lambda_0}(x) \geq g(x)$ for all $x \in N$.

Next, we claim that there is a point $x_\varepsilon \in N$ such that $h_{\lambda_0}(x_\varepsilon) = g(x_\varepsilon)$. Let us assume this is not true, that is, $h_{\lambda_0}(x) > g(x)$ for all $x \in N$. Set $r_\varepsilon = 2\varepsilon/\lambda_0$ and denote by $B_{r_\varepsilon} = \{x \in N \mid r(x) \leq r_\varepsilon\}$ the closed ball of radius r_ε . From the definition of h_λ it follows that $h_{\lambda_0}(x) \geq L + \varepsilon$ for all $x \notin B_{r_\varepsilon}$. Using the fact that B_{r_ε} is compact and our assumption that $h_{\lambda_0}(x) > g(x)$ for all $x \in N$, one concludes that

$$\inf_{x \in B_{r_\varepsilon}} (h_{\lambda_0}(x) - g(x)) > 0.$$

Since $g < L$ on N ,

$$\inf_{x \in N} (h_{\lambda_0}(x) - g(x)) = \eta > 0.$$

Now let $0 < \delta < \min\{\eta, \varepsilon\}$ be any number and set $\lambda' = \lambda_0 - \delta/(2r_\varepsilon)$. We shall show that $h_{\lambda'}(x) > g(x)$ for all $x \in N$, which would contradict the definition of λ_0 .

If $r(x) \leq r_\varepsilon$, then

$$\begin{aligned} h_{\lambda'}(x) &= \left(\lambda_0 - \frac{\delta}{2r_\varepsilon}\right)r(x) + L - \varepsilon \\ &= h_{\lambda_0}(x) - \frac{\delta r(x)}{2r_\varepsilon} \geq g(x) + \eta - \delta \frac{r(x)}{2r_\varepsilon} > g(x). \end{aligned}$$

On the other hand, taking into consideration that $\lambda_0 - \delta/(2r_\varepsilon) > 0$ because $\lambda_0 r_\varepsilon = 2\varepsilon > \delta$, we deduce, for $r(x) > r_\varepsilon$, that

$$\begin{aligned} h_{\lambda'}(x) &= \left(\lambda_0 - \frac{\delta}{2r_\varepsilon}\right)r(x) + L - \varepsilon > \left(\lambda_0 - \frac{\delta}{2r_\varepsilon}\right)r_\varepsilon + L - \varepsilon \\ &= L + \varepsilon - \frac{\delta}{2} > L > g(x). \end{aligned}$$

As the assumption has led to a contradiction, we conclude that there is a point $x_\varepsilon \in N$ such that $h_{\lambda_0}(x_\varepsilon) = g(x_\varepsilon)$. Moreover, from (2) it follows that $r(x_\varepsilon) \geq 1$.

Finally, we have to show that h_{λ_0} is smooth at x_ε . Since $h_\lambda(x) = \lambda r(x) + L - \varepsilon$, it is enough to show that r is smooth at x_ε . If not, then x_ε must be on the cut locus of p , in which case we have two possibilities: either there are two distinct minimizing geodesic segments $\gamma_1, \gamma_2 : [0, t_0] \rightarrow N$ joining p to x_ε , or there is a geodesic segment $\gamma : [0, t_0] \rightarrow N$ from p to x_ε along which x_ε is conjugate to p .

Let us start with the first case. Let $w = \gamma_1'(t_0)$ and $v = \gamma_2'(t_0)$. Since γ_1 and γ_2 are distinct segments, we have $w \neq v$. The functions $t \rightarrow r(\gamma_i(t))$ are differentiable on $(0, t_0)$ (for $i = 1, 2$) and they have a left-derivative at t_0 .

From $h_{\lambda_0} \geq g$ and $h_{\lambda_0}(x_\varepsilon) = g(x_\varepsilon)$ it follows that

$$\liminf_{s \rightarrow 0} \frac{h_{\lambda_0}(\gamma_2(t_0 + s)) - h_{\lambda_0}(\gamma_2(t_0))}{s} \geq D_v g(x_\varepsilon),$$

where $D_v g(x_\varepsilon)$ denotes the directional derivative of g at the point x_ε in the direction of v . Moreover, since g is smooth and h_{λ_0} has a directional derivative at x_ε in the direction of $-v$, we also have

$$-\lambda_0 = D_{-v} h_{\lambda_0}(x_\varepsilon) \geq D_{-v} g(x_\varepsilon) = -D_v g(x_\varepsilon). \tag{3}$$

Combining this with the previous inequality, we obtain

$$\liminf_{s \rightarrow 0} \frac{h_{\lambda_0}(\gamma_2(t_0 + s)) - h_{\lambda_0}(\gamma_2(t_0))}{s} \geq \lambda_0.$$

Taking into account the special form of h_{λ_0} , we find that

$$\liminf_{s \rightarrow 0} \frac{r(\gamma_2(t_0 + s)) - r(\gamma_2(t_0))}{s} \geq 1. \tag{4}$$

This will lead to a contradiction.

Since $v \neq w$, there is a $0 < c < 1$, depending only on the angle of v and w , such that

$$r(\gamma_2(t_0 + s)) < t_0 + cs \tag{5}$$

for a small enough $s > 0$. One can see this by connecting the point $\gamma_1(t_0 - s)$ to the point $\gamma_2(t_0 + s)$ by a geodesic segment; as γ_1 and γ_2 are different, there is a $0 < c_1 < 1$ such that for a small enough $s > 0$ we have $\text{dist}(\gamma_1(t_0 - s), \gamma_2(t_0 + s)) < c_1 2s$, which implies (5). But since $r(\gamma_2(t_0)) = t_0$, it is easy to see that (4) and (5) are in direct contradiction.

We now turn our attention to the second case. Since γ is distance-minimizing between p and x_ε , the distance function r is smooth at $\gamma(t)$ for $0 < t < t_0$. Set $m(t) = \Delta r(\gamma(t))$; then $m(t)$ is also smooth on the interval $(0, t_0)$ and, since $\gamma(t_0)$ is conjugate to $p = \gamma(0)$ along γ , it follows that

$$\lim_{t \rightarrow t_0^-} m(t) = -\infty. \tag{6}$$

Because $\lambda_0 > 0$ from (3), we conclude that $D_v g(x_\varepsilon) > 0$, that is, $\nabla g(x_\varepsilon) \neq 0$. This implies that the level surface $F = \{x \in N \mid g(x) = g(x_\varepsilon)\}$ is a smooth hypersurface near x_ε . Denote by F_s the surface parallel to F and passing through the point $\gamma(t_0 - s)$, for some $s > 0$. Again, since F is smooth near x_ε , the surface F_s will also be smooth near $\gamma(t_0 - s)$ for a small enough $s > 0$.

It is now clear from (6) that, for some small $s > 0$,

$$m(t_0 - s) < \text{trace of the second fundamental form of } F_s \text{ at } \gamma(t_0 - s),$$

where the second fundamental form of F_s at $\gamma(t_0 - s)$ is taken in the direction of $\gamma'(t_0 - s)$.

Taking into account the fact that $m(t_0 - s)$ is the trace of the second fundamental form of the geodesic ball $B_p(t_0 - s)$ around p at the point $\gamma(t_0 - s)$ (with respect to the same normal vector $\gamma'(t_0 - s)$), we conclude that there has to be a point $q_s \in F_s$, sufficiently close to $\gamma(t_0 - s)$, that lies inside $B_p(t_0 - s)$. This means that

$$r(q_s) < t_0 - s.$$

Since F_s is parallel to F , we have a point $q \in F$ such that $\text{dist}(q_s, q) = s$. Combining this with the above inequality gives

$$r(q) < t_0.$$

From the above, we obtain

$$h_{\lambda_0}(q) = \lambda_0 r(q) + L - \varepsilon < \lambda_0 t_0 + L - \varepsilon = h_{\lambda_0}(x_\varepsilon) = g(x_\varepsilon) = g(q),$$

which leads to a contradiction since $h_{\lambda_0} \geq g$ on N .

Once we have established the smoothness of h_{λ_0} at x_ε , the rest of the argument is straightforward. Since

$$h_{\lambda_0}(x) > g(x) \quad \text{and} \quad h_{\lambda_0}(x_\varepsilon) = g(x_\varepsilon),$$

we have

$$\nabla g(x_\varepsilon) = \nabla h_{\lambda_0}(x_\varepsilon) = \lambda_0 \nabla r(x_\varepsilon) \quad \text{and} \quad \Delta h_{\lambda_0}(x_\varepsilon) \geq \Delta g(x_\varepsilon). \tag{7}$$

From (2) and the fact that $g(x_\varepsilon) = \lambda_0 r(x_\varepsilon) + L - \varepsilon < L$, we deduce that

$$r(x_\varepsilon) > 1 \quad \text{and} \quad \lambda_0 < \frac{\varepsilon}{r(x_\varepsilon)}. \tag{8}$$

Taking $|\nabla r| = 1$ and combining it with the first equality in (7) yields

$$\|\nabla g(x_\varepsilon)\| = \lambda_0 < \varepsilon. \tag{9}$$

From (7), (8) and Condition (Δ) we find that

$$\Delta g(x_\varepsilon) \leq \Delta h_{\lambda_0}(x_\varepsilon) = \lambda_0 \Delta r(x_\varepsilon) < \varepsilon \frac{\Delta r(x_\varepsilon)}{r(x_\varepsilon)} < \varepsilon C. \tag{10}$$

Since $L - \varepsilon < h_{\lambda_0}(x_\varepsilon) = g(x_\varepsilon) < L$, inequalities (9) and (10) show that the point x_ε satisfies the conditions of the lemma (if $C > 1$, then just replace ε by εC in the statement of the lemma); therefore the proof is complete. \square

3. An example

In this section we sketch an example which shows that the condition in the lemma, namely Condition (Δ) , is quite optimal. Let N be a complete unbounded manifold, that is, a manifold where the distance function is unbounded. Let r be the distance function from some point, and assume that $\Delta r(x) > r(x)^{1+s}$ for those points $x \in N$ where $r(x)$ is large enough and $0 < s < 1$. Then there is a bounded function $h : N \rightarrow \mathbb{R}$ for which the lemma does not apply.

Define $h(x) = 1 - 1/r(x)^s$ for $r(x) > 2$ and extend it smoothly over the closed ball $B_2 = \{x \in N \mid r(x) \leq 2\}$ so that $h \leq 1/2$ over B_2 . Clearly, h is a bounded function with $\sup h = 1$.

A simple computation shows that, outside the ball B_2 ,

$$\Delta h = -s(1+s) \frac{|\nabla r|^2}{r^{2+s}} + s \frac{\Delta r}{r^{1+s}}.$$

Since $|\nabla r| = 1$, there is an $r_0 > 0$ such that if $r(x) > r_0$, then $\Delta h > s/2$. Let us choose $\varepsilon > 0$ such that $\varepsilon < 1 - 1/r_0^s$ and $\varepsilon < s/2$. Now, if $x_\varepsilon \in N$ is the point in N satisfying the conditions of the lemma, then from the condition $|h(x_\varepsilon) - 1| < \varepsilon$ one concludes that $r(x_\varepsilon) > r_0$. In this case, $\Delta h > s/2$, which contradicts the condition that $\Delta h < \varepsilon$.

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