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IS THE UNION-CLOSED SETS CONJECTURE THE BEST POSSIBLE?

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Abstract

A slightly strengthened version of the union-closed sets conjecture is proposed. It is shown that this version holds for a minimum set size of one or two and an examination of a boundary function shows that it holds for collections containing up to 19 sets. Some related conjectures are outlined.

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1. Introduction

A union-closed set is a non-empty finite collection of distinct non-empty finite sets, closed under union. The following conjecture (rephrased) appears in [1], and its known history is discussed in [5].

CONJECTURE 1A. Let $\mathscr{A} = \{A_1, \dots, A_n\}$ be a union-closed set. Then there exists an element which belongs to at least $\lceil n/2 \rceil$ sets in $\mathscr{A}$, where

$$\lceil n/2 \rceil = \begin{cases} n/2 & \text{if } n \text{ is even} \\ (n+1)/2 & \text{if } n \text{ is odd.} \end{cases}$$

It has been shown in [3] and [4] that this conjecture is valid up to $n = 18$ and also in all cases where the smallest set contains only one or two elements. The structure of possible counter-examples is examined in [2].

In this paper collections containing up to 17 sets are examined and it is shown that the slightly stronger conjecture given in section 2 below holds for

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all of these (and in fact also holds for $n = 18$ and $n = 19$). As in [3], this conjecture is proved valid when the smallest set size is one or two. The study up to $n = 17$ seems to indicate further conjectures may be valid.

Henceforth the term 'collection' will be used.

2. A stronger conjecture

CONJECTURE 1B. Let $\mathscr{A} = \{A_1, \dots, A_n\}$ be a union-closed collection. Then there exists an element which belongs to at least $\lfloor n/2 \rfloor + 1$ sets in $\mathscr{A}$, where $\lfloor\ \rfloor$ is the floor function defined by

$$\lfloor n/2 \rfloor + 1 = \begin{cases} n/2 + 1 & \text{if } n \text{ is even} \\ (n+1)/2 & \text{if } n \text{ is odd.} \end{cases}$$

Notice that this increases the bound only for n even, and then only by 1.

REMARK. As originally stated in [1], the conjecture assumes each set finite but does not specifically exclude the null set from the collection. If it is assumed that the null set can occur, the original conjecture is equivalent to conjecture 1B in this paper, except for the trivial collections $\mathscr{A} = \{\ \}$ and $\mathscr{A} = \{\varnothing\}$. The author thanks the referee for this observation.

For the cases where the minimum set size is 1 or 2, little modification is needed to the proofs of conjecture 1A given in [3]. These modified proofs are given below for the sake of completeness.

THEOREM 1. *Conjecture* 1B *holds whenever one of the sets has size* 1 *or* 2.

PROOF. Let $\mathscr{A} = \{A_1, \dots, A_n\}$ be a union-closed collection, ordered such that $|A_1|$ is minimum.

Case $|A_1| = 1$. Let $A_1 = \{a\}$. For each set A_j such that $a \notin A_j$ there exists the set $A_j \cup \{a\}$ in $\mathscr{A}$ and for such sets $A_j \neq A_k$ implies $A_j \cup \{a\} \neq A_k \cup \{a\}$. But a is also in A_1, hence for n even a is in at least $n/2 + 1$ sets and for n odd a is in at least $(n+1)/2$ sets in $\mathscr{A}$.

Case $|A_1| = 2$. Let $A_1 = \{a_1, a_2\}$. Suppose s sets contain neither of these elements, t sets contain both, x_1 sets contain a_1 but not a_2 and x_2 sets contain a_2 but not a_1. Then $n = s + t + x_1 + x_2$. Since for every set A_j containing neither a_1 nor a_2 there is a unique set $A_j \cup A_1$ containing both, and since both are in A_1 itself, $t \geq s + 1$ and hence $2t + x_1 + x_2 > n$. This is separable into $t + x_1 > n/2$ or $t + x_2 > n/2$ and hence one of the elements of A_1 is in more than half the sets in $\mathscr{A}$.

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3. A boundary function

In investigating the validity of the conjecture it may be of value to examine the exact bound in several cases.

DEFINITION. For positive integers n, define $\varphi(n)$ by $\varphi(n) = k$ where all union-closed collections containing n sets have at least one element occurring in at least k sets and there exists a union-closed collection of n sets where no element occurs in $k+1$ sets.

Two restrictions on $\varphi(n)$ will be used later, these are given in the lemmas below (inclusion of lemma 2, which shortens later proofs, was suggested by the referee).

LEMMA 1. $0 \leq \varphi(n+1) - \varphi(n) \leq 1$.

PROOF. Consider a union-closed collection of $n+1$ sets with no element occurring in more than $\varphi(n+1)$ sets. Removal of a set of minimum size forms a union-closed collection of n sets where no element occurs in more than $\varphi(n+1)$ sets, thus $\varphi(n) \leq \varphi(n+1)$.

Let $\varphi(n) = k$. Let $\mathscr{A} = \{A_1, \ldots, A_n\}$ be a union-closed collection, ordered such that $|A_n|$ is maximum, with no element occurring in more than $\varphi(n)$ sets in $\mathscr{A}$. Let z be an element which does not occur in any set in $\mathscr{A}$ and let $A_{n+1} = A_n \cup \{z\}$. Now $\mathscr{A}' = \mathscr{A} \cup \{A_{n+1}\}$ is a union-closed collection, and no element occurs in more than $k+1$ sets in $\mathscr{A}'$. Thus $\varphi(n+1) \leq \varphi(n) + 1$ and the lemma holds.

LEMMA 2. *If* $n = 2^m - i$ *for some integers* m *and* i *with* $m \geq i \geq 1$, *then* $\varphi(n) \leq 2^{m-1}$.

PROOF. In the power set on m elements (containing $2^m - 1$ non-empty sets), each element occurs in exactly 2^{m-1} sets. The removal of $i-1$ singletons gives a union-closed collection of n sets in which no element occurs in more than 2^{m-1} sets.

4. Cases to $n = 17$

In each case below assume the union-closed collection of n sets under consideration is of boundary type, with no element occurring in more than $\varphi(n)$ sets. For each value of n assume $|A_n|$ is maximum. Notice that A_j is a subset of A_n, $j = 1, \ldots, n$.

CASE $n = 1$. Trivially, $\varphi(1) = 1$.

CASE $n = 2$. This requires $A_1 \subset A_2$ and so $\varphi(2) = 2$.

CASE $n = 3$. By lemmas 1 and 2 $\varphi(3) = 2$.

CASE $n = 4$. By lemma 1, $\varphi(4) \leq 3$. Since $A_1 \subset A_4$, $A_1 \cap A_2$ or $A_1 \cap A_3$ not null implies $\varphi(4) = 3$, but both null would imply $A_1 \cup A_2 = A_1 \cup A_3 = A_4$ and hence $A_2 = A_3$, which is disallowed by the definition. Hence $\varphi(4) = 3$.

CASE $n = 5$. Since $\varphi(4) = 3$, by Lemma 1 $\varphi(5) = 3$ or $\varphi(5) = 4$. Assume $\varphi(5) = 3$ and let $x \in A_1, A_2, A_5$ only. Then $\{A_3, A_4\}$ must be union closed, with one a subset of the other; say $A_3 \subset A_4 \subset A_5$. Now elements of A_3 are in three sets; for $\varphi(5) \neq 4$ this necessitates $A_1 \cap A_3 = \varnothing$, $A_2 \cap A_3 = \varnothing$ and $A_1 \cup A_3 = A_2 \cup A_3 = A_5$ (if A_4 then x is in four sets). But then $A_1 = A_2$, which is disallowed. Thus $\varphi(5) = 4$.

CASES $n = 6$, $n = 7$. By Lemmas 1 and 2, $\varphi(6) = \varphi(7) = 4$.

CASE $n = 8$. We shall show that the assumption $\varphi(8) = 4$ leads to a contradiction and hence that $\varphi(8) = 5$ by Lemma 1.

Assume $x \in A_1, A_2, A_3, A_8$ only. Then $\{A_4, A_5, A_6, A_7\}$ is union-closed with say $|A_7|$ maximum. Since $\varphi(4) = 3$, assume $y \in A_4, A_5, A_7$ (and A_8).

Consider $A_1 \cup A_5, A_2 \cup A_5, A_3 \cup A_5$. Each contains x. If these unions are all A_8 then there exists $w \in A_7 \backslash A_5$ in A_1, A_2, A_3, A_7, A_8, that is, in five sets, contrary to the assumption. Thus one union is one of A_1, A_2, A_3, but then this set contains y and hence y is in five sets. Thus $\varphi(8) = 5$.

CASE $n = 9$. As above we shall show that the assumption $\varphi(9) = 5$ leads to a contradiction. Assume $x \in A_1, A_2, A_3, A_4, A_9$ only and hence the remaining four sets are union-closed with say $|A_8|$ maximum. Since $\varphi(4) = 3$, also assume $y \in A_5, A_6, A_8$ (and A_9).

Consider $A_1 \cup A_6, A_2 \cup A_6, A_3 \cup A_6, A_4 \cup A_6$. Each contains x. If these unions are all A_9 then there exists $w \in A_8 \backslash A_6$ in $A_1, A_2, A_3, A_4, A_8, A_9$, contrary to the assumption. Thus one union is one of A_1, A_2, A_3, A_4. This is also the case for A_5 in place of A_6. Reorder if necessary such that $A_6 \subset A_4$ with $|A_6| \geq |A_5|$. Notice that y is now in five sets.

Consider $A_1 \cup A_5, A_2 \cup A_5, A_3 \cup A_5, A_4 \cup A_5$. If these unions are all A_9 or A_4 then there exists $w \in A_6 \backslash A_5$ in $A_1, A_2, A_3, A_4, A_6, A_8, A_9$. Thus one union is one of A_1, A_2, A_3. But then y is in six sets. Lemma 1 now yields, $\varphi(9) = 6$.

CASE $n = 10$. Assume $\varphi(10) = 6$. Let $x \in A_1, \ldots, A_5, A_{10}$ only. Since $\varphi(4) = 3$, let $y \in A_6, A_7, A_9$ with $|A_9|$ maximum.

Consider $A_1 \cup A_7, \ldots, A_5 \cup A_7$. If these are all A_{10} then there exists $w \in A_9 \backslash A_7$ in $A_1, \ldots, A_5, A_9, A_{10}$. Thus one union is one of $A_1, \ldots, A_5$. This is also the case for A_6 in place of A_7. Reorder such that $A_7 \subset A_5$ with $|A_7| \geq |A_6|$. Notice that y is now in the five sets $A_5, A_6, A_7, A_9, A_{10}$.

Consider $A_1 \cup A_6, \ldots, A_5 \cup A_6$. If these are all A_5 or A_{10} then there

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exists $w \in A_7 \backslash A_6$ in $A_1, \ldots, A_5, A_7, A_9, A_{10}$. Thus one union is one of $A_1, \ldots, A_4$. Reorder such that $A_6 \subset A_4$. Now y is in six sets.

For the assumption to hold $y \notin A_1 \cup A_8, A_2 \cup A_8, A_3 \cup A_8$. These unions must be at least one of A_1, A_2, A_3. Reorder these such that $A_8 \subset A_3$.

If $A_6 \cap A_8$ and $A_7 \cap A_8$ were both null this would require $A_6 \cup A_8 = A_7 \cup A_8 = A_9$ and hence $A_6 = A_7$, which is disallowed. Assume $w \in A_7, A_8$. Now $w \subset A_3, A_5, A_7, A_8, A_9, A_{10}$ and no others by the assumption. Thus $A_1 \cup A_8 = A_2 \cup A_8 = A_3, A_1 \cup A_6 = A_2 \cup A_6 = A_4$.

Thus $A_1 \cap A_8$, $A_2 \cap A_8$ are not both null (for otherwise $A_1 = A_2$). Now there exists $z \in A_2$ (say), $A_8, A_3, A_4, A_9, A_{10}$. But $A_1 \cup A_6 = A_4$ implies $z \in A_1$ or A_6, a total of seven sets. Thus $\varphi(10) = 7$ by Lemma 1.

CASE $n = 11$. Consider the power set on four elements, with all singleton sets removed. Eleven sets remain, and each of the four elements occurs in exactly seven sets. Thus $\varphi(11) = 7$.

CASE $n = 12$. Assume $\varphi(12) = 7$. Let $x \in A_1, \ldots, A_6, A_{12}$ only. Since $\varphi(5) = 4$, let $y \in A_7, A_8, A_9, A_{11}$ with $|A_{11}|$ maximum.

Consider $A_1 \cup A_9, \ldots, A_6 \cup A_9$. If these were all A_{12} there would exist some $z \in A_{11} \backslash A_9$ in the eight sets $A_1, \ldots, A_6, A_{11}, A_{12}$. This also holds for A_7 and A_8 in place of A_9. Thus each is a subset of one of $A_1, \ldots, A_6$. Reorder if necessary, selecting $|A_9|$ maximum with $A_9 \subset A_6$.

Consider now $A_1 \cup A_8, \ldots, A_6 \cup A_8$. If these were all A_6 or A_{12} there would exist some $z \in A_9 \backslash A_8$ in the nine sets $A_1, \ldots, A_6, A_9, A_{11}, A_{12}$. This also holds for A_7 in place of A_8. Thus each is a subset of one of $A_1, \ldots, A_5$. Reorder if necessary, selecting $|A_8|$ maximum with $A_8 \subset A_5$. Notice y is now in the seven sets $A_5, A_6, A_7, A_8, A_9, A_{11}, A_{12}$. By the above, A_7 is a subset of one of $A_1, \ldots, A_5$, which requires $A_7 \subset A_5$.

If $A_1 \cup A_7, \ldots, A_6 \cup A_7$ were all A_5 or A_{12} there would exist some $z \in A_8 \backslash A_7$ in the nine sets $A_1, \ldots, A_6, A_8, A_{11}, A_{12}$. Thus at least one union is A_6 and hence $A_7 \cup A_8 \subset A_5, A_7 \cup A_9 \subset A_6$.

Suppose $A_8 \subset A_6$. Then since $A_1 \cup A_7, \ldots, A_6 \cup A_7$ are all A_5 or A_6 or A_{12} there exists $z \in A_8 \backslash A_7$ in the nine sets $A_1, \ldots, A_6, A_8, A_{11}, A_{12}$. Thus A_8 cannot be a subset of $A_7 \cup A_9$, this union must now be A_9 and hence $A_7 \subset A_9$. Similarly, $A_7 \subset A_8$. But then A_7 is a subset of the seven sets $A_5, A_6, A_7, A_8, A_9, A_{11}, A_{12}$. By the assumption this requires the intersection of A_7 with each of $A_1, \ldots, A_4$ to be null. But the union of A_7 with any of these four sets must be one of A_5, A_6, A_{12}, thus two unions are equal, with corresponding intersections null. This implies two of $A_1, \ldots, A_4$ are equal, a contradiction. Thus $\varphi(12) = 8$ by Lemma 1.

CASES $n = 13, 14, 15$. By Lemmas 1 and 2, $\varphi(13) = \varphi(14) = \varphi(15) = 8$.

CASE $n = 16$. Assume $\varphi(16) = 8$. Let $x \in A_1, \ldots, A_7, A_{16}$ only. Since $\varphi(8) = 5$ let $y \in A_8, \ldots, A_{11}, A_{15}$ with $|A_{15}|$ maximum. As before consider $A_1 \cup A_{11}, \ldots, A_7 \cup A_{11}$. If these were all A_{16} there would exist $z \in A_{15} \backslash A_{11}$ in the nine sets $A_1, \ldots, A_7, A_{15}, A_{16}$. Thus A_{11} is a subset of one of $A_1, \ldots, A_7$; this also holds for $A_8, \ldots, A_{10}$. Reorder if necessary such that $A_{11} \subset A_7$, $|A_{11}|$ maximum. Notice y is in the seven sets $A_7, A_8, A_9, A_{10}, A_{11}, A_{15}, A_{16}$.

Similarly if $A_1 \cup A_{10}, \ldots, A_7 \cup A_{10}$ were all A_7 or A_{16} there would exist $z \in A_{11} \backslash A_{10}$ in the ten sets $A_1, \ldots, A_7, A_{11}, A_{15}, A_{16}$. Thus A_{10} is a subset of one of $A_1, \ldots, A_6$; this also holds for A_8, A_9. Reorder if necessary such that $A_{10} \subset A_6$, $|A_{10}|$ maximum. Now y is in eight sets and by the assumption in no more. Thus A_8 and A_9 are also subsets of A_6 and $A_8 \cup A_9 \cup A_{10} \subset A_6$.

If $A_1 \cup A_9, \ldots, A_7 \cup A_9$ were all A_6 or A_{16} there would exist $z \in A_{10} \backslash A_9$ in the ten sets $A_1, \ldots, A_7, A_{10}, A_{15}, A_{16}$. Thus $A_9 \subset A_7$ and similarly $A_8 \subset A_7$. Reorder if necessary such that $|A_9| \geq |A_8|$. But if $A_1 \cup A_8, \ldots, A_7 \cup A_8$ were all A_6 or A_7 or A_{16} then there exists $z \in A_9 \backslash A_8$ in the ten sets $A_1 \ldots, A_7, A_9, A_{15}, A_{16}$. Thus $\varphi(16) = 9$ by Lemma 1.

CASE $n = 17$. Assume $\varphi(17) = 9$. Let $x \in A_1, \ldots, A_8, A_{17}$ only. Since $\varphi(8) = 5$, let $y \in A_9, \ldots, A_{12}, A_{16}$ with $|A_{16}|$ maximum; also $y \in A_{17}$. As in case $n = 16$ each of $A_9, \ldots, A_{12}$ is a subset of at least one of $A_1, \ldots, A_8$ and these can be reordered such that $A_{12} \subset A_8$ with $|A_{12}|$ maximum; subsequently each of A_9 to A_{11} is a subset of one of A_1 to A_7 and reordering leads to $A_{11} \subset A_7$, $|A_{11}|$ maximum. So far, y is in eight sets.

Reorder such that $|A_{10}| \geq |A_9|$. If the unions of A_{10} with each of A_1 to A_8 were all A_8 or A_{17} there would exist $z \in A_{12} \backslash A_{10}$ in eleven sets; if these unions were all A_7 or A_{17} there would exist $z \in A_{11} \backslash A_{10}$ in eleven sets. Suppose then that A_{10} is a subset of A_7 and of A_8. But then if the unions of A_9 with each of A_1 to A_8 were A_7 or A_8 or A_{17} there would exist $z \in A_{10} \backslash A_9$ in eleven sets. Hence one at least of A_9, A_{10} is a subset of one of A_1 to A_6. Reorder such that $A_{10} \subset A_6$. Now y is in nine sets and by the assumption in no others; hence A_9 is a subset of A_6 or A_7.

If $A_9 \subset A_7$ the unions of A_1 to A_8 with A_9 cannot all be A_7 or A_{17} for otherwise $A_{11} \backslash A_9$ is in eleven sets and hence A_9 is also a subset of A_6 or A_8. If $A_9 \subset A_6$ we can reorder A_{10}, A_9 such that A_{10} is maximum in size and again show A_9 must also be a subset of one of A_7, A_8. Thus A_9 is a subset of at least two of A_6, A_7, A_8.

By the assumption $\varphi(17) = 9$, $\{A_{13}, A_{14}, A_{15}\}$ is union-closed and no set contains x or y. Let $|A_{15}| > |A_{14}| \geq |A_{13}|$. Now A_{15} is a subset of A_{16} and of A_{17} and also its union with A_1 to A_5 does not contain y and hence we may assume A_{15} is a subset of A_5.

If the unions of A_{14} with each of A_{10} to A_{12} were A_{16} then $A_{15} \backslash A_{14}$ would be in A_{15}, A_{16}, A_{17}, each of A_{10} to A_{12}, each of A_6 to A_8 and A_5, that is, in ten sets. Thus A_{14} is a subset of one of A_{10} to A_{12}. Reorder again and assume $A_{14} \subset A_{12} \subset A_8$. Note that A_{14} is a subset of the seven sets $A_5, A_8, A_{12}, A_{14}, A_{15}, A_{16}, A_{17}$.

If the unions of A_{13} with each of A_{10} to A_{12} were A_{12} or A_{16} then $A_{14} \backslash A_{13}$ would additionally be in A_{10}, A_{11}, A_6, A_7, that is, in eleven sets. Thus assume A_{13} is a subset of A_{11} and of A_7. Note this implies that $A_{13} \cap A_{14} = \varnothing$ and $A_{13} \cup A_{14} = A_{15}$.

Now $A_1 \cup A_{14}, \ldots, A_4 \cup A_{14}$ cannot all be A_5 for otherwise A_{13} (= $A_{15} \backslash A_{14}$) is in eleven sets, and hence we can assume $A_{14} \subset A_4$. Similarly we can assume $A_{13} \subset A_3$. Each is thus a subset of eight sets.

Now $A_{14} \cap A_{10}$ not null would also lead to elements in A_{10} and A_6, that is, ten sets; similarly for $A_{13} \cap A_{10}$. Now A_9 is a subset of two of the sets A_6 to A_8, and hence null intersection with A_{13} and A_{14} is again required. But then $A_{13} \cup A_9$ and $A_{13} \cup A_{10}$ must both be A_{11}, and then $A_9 = A_{10}$, which is disallowed. Thus $\varphi(17) = 10$ by Lemma 1.

The results so far are

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$\lfloor n/2 \rfloor + 1$	1	2	2	3	3	4	4	5	5	6	6	7	7	8	8	9	9
$\varphi(n)$	1	2	2	3	4	4	4	5	6	7	7	8	8	8	8	9	10

Notice that this verifies the strengthened conjecture to $n = 19$. Note also that the bounds given in this conjecture are attained only at $n = 1, 2, 3, 4, 6, 7, 8, 14, 15, 16$.

5. Further conjectures

Examination of the values of $\varphi(n)$ above as well as the arguments for various values of n indicates that there may be a close link between reduced power-sets and φ-values. This can be formulated as follows.

CONJECTURE 2A. For $2^m - 1 \geq n \geq 2^{m-1}$, there exists a union-closed reduced power-set on m elements (containing n sets) such that no element occurs in more than $\varphi(n)$ sets.

This can also be strengthened.

CONJECTURE 2B. The subset lattice structure of any boundary union-closed collection containing n sets is isomorphic to that of a reduced power set on m elements, where $2^m - 1 \geq n \geq 2^{m-1}$.

These conjectures, if valid, enable one to evaluate $\varphi(n)$ up to quite large values of n but do not give an explicit formula for $\varphi(n)$ except for rather restricted values of n, near powers of two.

Examination of the above table can lead to several minor conjectures on $\varphi(n)$, one of which is

CONJECTURE 3. The integer-valued function $\varphi(n)$ is greater than $n/2$. $\varphi(n) = n/2 + 1$ only when n has form 2^m or $2^m - 2$; $\varphi(n) = (n+1)/2$ only when n has form $2^m - 1$.

Note added in proof

Fred Galvin has informed the author that the conjecture was originally proposed in 1979 by Peter Frankl. It appears on page 525 in I. Rival (Ed), *Graphs and Order* (Reidel, 1984) and on pages 161 and 186 of Volume 1 in R. P. Stanley, *Enumerative Combinatorics* (Wadsworth and Brooks, 1986).

References

[1] 'A much-travelled conjecture', *Austral. Math. Soc. Gaz.* **14** (3) (1987), 63.
[2] R. M. Norton and D. G. Sarvate, 'Union-closed sets conjecture: some observations', (manuscript, College of Charleston, September 1989).
[3] D. G. Sarvate and J-C. Renaud, 'On the union-closed sets conjecture', *Ars Combin.* **27** (1989), 149–154.
[4] ——, 'Improved bounds for the union-closed sets conjecture', *Ars Combin.* **29** (1990), 181–185.
[5] P. Winkler, 'Union-closed sets conjecture', *Austral. Math. Soc. Gaz.* **14** (4) (1987), 99.

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