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OBSERVATIONS ON GAUSSIAN UPPER BOUNDS FOR NEUMANN HEAT KERNELS

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Abstract

Given a domain Ω of a complete Riemannian manifold $\mathcal{M}$, define $\mathcal{A}$ to be the Laplacian with Neumann boundary condition on Ω. We prove that, under appropriate conditions, the corresponding heat kernel satisfies the Gaussian upper bound

$$h(t,x,y) \le \frac{C}{[V_\Omega(x,\sqrt{t})V_\Omega(y,\sqrt{t})]^{1/2}}\Big(1+\frac{d^2(x,y)}{4t}\Big)^{\delta} e^{-d^2(x,y)/4t} \quad \text{for } t>0,\ x,y\in\Omega.$$

Here d is the geodesic distance on $\mathcal{M}$, $V_\Omega(x,r)$ is the Riemannian volume of $B(x,r)\cap\Omega$, where $B(x,r)$ is the geodesic ball of centre x and radius r, and δ is a constant related to the doubling property of Ω. As a consequence we obtain analyticity of the semigroup $e^{-t\mathcal{A}}$ on $L^p(\Omega)$ for all $p\in[1,\infty)$ as well as a spectral multiplier result.

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1. Introduction and main results

This short note is devoted to the Gaussian upper bound for the heat kernel of the Neumann Laplacian. Let us start with the Euclidean setting in which Ω is a bounded Lipschitz domain of $\mathbb{R}^n$. Let Δ_N be the Neumann Laplacian. It is well known that the corresponding heat kernel $h(t,x,y)$ satisfies

$$0 \le h(t,x,y) \le Ct^{-n/2}e^t e^{-c|x-y|^2/t}, \quad t>0,\ x,y\in\Omega. \tag{1.1}$$

One can replace the extra term e^t by $(1+t)^{n/2}$ but the decay $h(t,x,y)\le Ct^{-n/2}$ cannot hold for large t since $e^{t\Delta_N}1=1$. We refer to the monographs [5] or [14] for more details.

In applications, for example when applying the Gaussian bound to obtain spectral multiplier results, one can apply (1.1) to $-\Delta_N+I$ (or ϵI for any $\epsilon>0$) and not to $-\Delta_N$. It is annoying to add the identity operator especially as it is not clear how the functional calculus for $-\Delta_N$ can be related to that of $-\Delta_N+I$. The same problem occurs for

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analyticity of the semigroup $e^{t\Delta_N}$ on $L^p(\Omega)$ for $p \in [1, \infty)$. One obtains from (1.1) analyticity of the semigroup but not a bounded analytic semigroup. This boundedness (on sectors of the right half plane) is important in order to obtain appropriate estimates for the resolvent or for the time derivatives of the solution to the corresponding evolution equation on L^p. In this note we will show in an elementary way how one can resolve this question. The idea is that (1.1) can be improved to a Gaussian upper bound of the type

$$h(t, x, y) \leq \frac{C}{[V_\Omega(x, \sqrt{t})V_\Omega(y, \sqrt{t})]^{1/2}} e^{-c|x-y|^2/t}, \quad t > 0, \ x, y \in \Omega, \tag{1.2}$$

where $V_\Omega(x, r)$ denotes the volume of $\Omega \cap B(x, r)$ and $B(x, r)$ is the open ball of centre x and radius r. There is no extra factor in (1.2) and one can use this estimate in various applications of Gaussian bounds instead of (1.1).

We shall state most of the results for Lipschitz domains of general Riemannian manifolds.

Let $(\mathcal{M}, g)$ be a complete Riemannian manifold of dimension n without boundary. Let Ω be a subdomain of $\mathcal{M}$ with Lipschitz boundary Γ. That is, Γ can be described in an appropriate local coordinate system by means of graphs of Lipschitz functions. Specifically, for any $p \in \Gamma$, there exist a local chart (U, ψ), $\psi : U \to \mathbb{R}^n$ with $\psi(p) = 0$ and a Lipschitz function $\lambda : \mathbb{R}^{n-1} \to \mathbb{R}$ with $\lambda(0) = 0$ and $\epsilon > 0$, such that

$$\psi(U \cap \Omega) = \{(x', \lambda(x') + t); 0 < t < \epsilon, x' \in \mathbb{R}^{n-1}, |x'| < \epsilon\},$$
$$\psi(U \cap \Gamma) = \{(x', \lambda(x')); x' \in \mathbb{R}^{n-1}, |x'| < \epsilon\}.$$

We use the Einstein summation convention for repeated indices. We recall that, in local coordinates $x = (x_1, \ldots, x_n)$,

$$g(x) = g_{ij} dx_i \otimes dx_j.$$

If $f \in C^\infty(\mathcal{M})$, the gradient of f is the vector field given by

$$\nabla f = g^{ij} \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_j}$$

and the Laplace–Beltrami operator is the operator acting as follows:

$$\Delta f = |g|^{-1/2} \frac{\partial}{\partial x_i}\Big(|g|^{1/2} g^{ij} \frac{\partial f}{\partial x_j}\Big),$$

where (g^{ij}) is the inverse of the metric g and $|g|$ is the determinant of g.

Let μ be the Riemannian measure induced by the metric g. That is

$$d\mu = |g|^{1/2} dx_1 \ldots dx_n.$$

We set $L^2(\Omega) = L^2(\Omega, d\mu)$. Let $H^1(\Omega)$ be the closure of $C_0^\infty(\overline{\Omega})$ with respect to the norm

$$\|f\|_{H^1(\Omega)} = \Big(\int_\Omega f(x)^2 \, d\mu(x) + \int_\Omega |\nabla f(x)|^2 \, d\mu(x)\Big)^{1/2}.$$

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Here

$$|\nabla f|^2 = \langle \nabla f, \nabla f \rangle,$$

where

$$\langle \nabla f, \nabla g \rangle = g^{ij} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}.$$

We consider on $L^2(\Omega) \times L^2(\Omega)$ the unbounded bilinear form

$$\mathfrak{a}(f, g) = \int_\Omega \langle \nabla f, \nabla g \rangle \, d\mu(x)$$

with domain $D(\mathfrak{a}) = H^1(\Omega)$.

Since Γ is Lipschitz, the unit conormal $\nu \in T^*\mathcal{M}$ is defined almost everywhere with respect to the surface measure $d\sigma$. Let $\partial_\nu f = \langle \nabla f, \nu \rangle = g^{ij} \nu_i (\partial f / \partial x_j)$ and

$$H_\Delta(\Omega) = \{ f \in L^2(\Omega); \Delta f \in L^2(\Omega) \}.$$

We recall the Green's formula

$$\int_\Omega \langle \nabla f, \nabla g \rangle \, d\mu = -\int_\Omega \Delta f g \, d\mu + \int_\Gamma \partial_\nu f g \, d\sigma, \qquad f \in C_0^\infty(\overline{\Omega}), \; g \in H^1(\Omega).$$

In light of this formula, we define $\partial_\nu f$, $f \in H_\Delta(\Omega)$, as an element of $H^{-1/2}(\Gamma)$, the dual space of $H^{1/2}(\Gamma)$, by the formula

$$(\partial_\nu f, g)_{1/2} := \int_\Omega \Delta f g \, d\mu + \int_\Omega \langle \nabla f, \nabla g \rangle \, d\mu, \qquad g \in H^1(\Omega).$$

Here $(\cdot, \cdot)_{1/2}$ is the duality pairing between $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$.

We define the operator $\mathcal{A}u = -\Delta u$ with domain

$$D(\mathcal{A}) = \{ u \in H_\Delta(\Omega); \partial_\nu u = 0 \}.$$

Then it is straightforward to see that $\mathcal{A}$ is the operator associated to the form $\mathfrak{a}$.

Let d be the geodesic distance and $B(x, r)$ be the geodesic ball with respect to d of centre $x \in \mathcal{M}$ and radius $r > 0$, and set $V(x, r) = \mu(B(x, r))$.

We assume in what follows that $\mathcal{M}$ satisfies the volume doubling property (abbreviated to VD from here on): there exists $C > 0$ so that

$$V(x, 2r) \le C V(x, r), \qquad x \in \mathcal{M}, \; r > 0.$$

We shall assume that the heat kernel $p(t, x, y)$ of the Laplacian on $\mathcal{M}$ satisfies the Gaussian upper bound

$$p(t, x, y) \le \frac{C}{[V(x, \sqrt{t}) V(y, \sqrt{t})]^{1/2}} e^{-c d^2(x,y)/t}, \qquad t > 0, \; x, y \in \mathcal{M} \tag{1.3}$$

in which C and c are positive constants.

A typical example of a manifold which satisfies both properties is a manifold with nonnegative Ricci curvature. The volume doubling property is then an immediate

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consequence of the Gromov–Bishop theorem. The Gaussian upper bound can be found in [11].

We define V_Ω by

$$V_\Omega(x,r) = \mu(B(x,r) \cap \Omega), \quad r > 0, \ x \in \Omega.$$

The main assumption on Ω is the following variant of the VD property: there exist two constants $K > 0$ and $\delta > 0$ so that

$$V_\Omega(x,s) \leq K\Big(\frac{s}{r}\Big)^{\delta} V_\Omega(x,r), \quad 0 < r \leq s, \ x \in \Omega. \tag{1.4}$$

Note that this doubling property holds for all bounded Lipschitz domains of $\mathbb{R}^n$ (with $\delta = n$). We shall discuss this in Section 3.

Most of the results we will refer to are valid for metric measure space with Borel measure. In our case this metric measure space is just (Ω, d, μ). Here, we keep the notation d and μ for the distance and measure induced on Ω by d on $\mathcal{M}$ and μ on $\mathcal{M}$.

Now we state our main results, which we formulate as the following theorem and the subsequent corollaries.

THEOREM 1.1.

(1) *The operator $-\mathcal{A}$ generates a symmetric Markov semigroup $e^{-t\mathcal{A}}$ with kernel $h \in C^\infty((0,\infty) \times \Omega \times \Omega)$.*
(2) *Suppose that $\mathcal{M}$ satisfies VD and (1.3) and Ω satisfies the VD property (1.4) and diam $(\Omega) < \infty$. Then h has the Gaussian upper bound*

$$h(t,x,y) \leq \frac{C}{[V_\Omega(x,\sqrt{t})V_\Omega(y,\sqrt{t})]^{1/2}}\Big(1 + \frac{d^2(x,y)}{4t}\Big)^{\delta} e^{-d^2(x,y)/4t}, \quad t > 0, \ x,y \in \Omega.$$

Since $(1+\rho)^\delta e^{-\rho/2}$ is a bounded function on $[0,\infty)$, the following corollary is an immediate consequence of Theorem 1.1.

COROLLARY 1.2. *Suppose that $\mathcal{M}$ satisfies VD and (1.3) and Ω satisfies the VD property (1.4) and diam $(\Omega) < \infty$. Then*

$$h(t,x,y) \leq \frac{C}{[V_\Omega(x,\sqrt{t})V_\Omega(y,\sqrt{t})]^{1/2}} e^{-d^2(x,y)/8t}, \quad t > 0, \ x,y \in \Omega.$$

We note that for unbounded domains, Gaussian upper bounds for the Neumann heat kernel are proved in [8].

Theorem 1.1 (2) or its corollary has several consequences.

COROLLARY 1.3. *Suppose that $\mathcal{M}$ satisfies VD and (1.3) and Ω satisfies the VD property (1.4) and diam $(\Omega) < \infty$. Then:*

(1) *the semigroup $e^{-t\mathcal{A}}$ extends to a bounded holomorphic semigroup of $\mathbb{C}^+$ on $L^p(\Omega,\mu)$ for all $p \in [1,\infty)$;*
(2) *the spectrum of $\mathcal{A}$, viewed as an operator acting on $L^p(\Omega)$, $p \in [1,\infty)$, is independent of p.*

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Assertion (1) is a consequence of Corollary 1.2 combined with [14, Corollary 7.5, page 202]. It was originally proved in [13]. Assertion (2) follows from a result in [6] which asserts that a Gaussian upper bound implies p-independence of the spectrum. See also [14, Theorem 7.10, page 206] for the general form needed here.

Let (E_λ) be the spectral resolution of the nonnegative self-adjoint operator $\mathcal{A}$. We recall that for any bounded Borel function $f : [0,\infty) \to \mathbb{C}$, the operator $f(\mathcal{A})$ is defined by

$$f(\mathcal{A}) = \int_0^\infty f(\lambda)\, dE_\lambda.$$

An operator T on the measure space (Ω, μ) is said to be of weak type $(1, 1)$ if

$$\|T\|_{L^1(\Omega)\to L^1_w(\Omega)} := \sup\{\lambda\mu(\{x \in \Omega; |T\varphi(x)| > \lambda\}); \lambda > 0, \|\varphi\|_{L^1(\Omega)} = 1\} < \infty.$$

In light of [7, Theorem 1.3, page 450 and Remark 1, page 451], the following corollary is another consequence of Corollary 1.2.

Corollary 1.4. *Suppose that $\mathcal{M}$ satisfies VD and (1.3) and Ω satisfies the VD property (1.4) and* $\operatorname{diam}(\Omega) < \infty$. *Let $s > \delta/2$, where δ is as in (1.4), $\varphi \in C_0^\infty((0,\infty))$ not identically equal to zero and $f : [0,\infty) \to \mathbb{C}$ a Borel function satisfying*

$$\sup_{t>0} \|\varphi(\cdot) f(t\cdot)\|_{W^{s,\infty}} < \infty.$$

Then $f(\mathcal{A})$ is of weak type $(1, 1)$ and bounded on $L^p(\Omega)$ for any $p \in (1,\infty)$. Additionally,

$$\|f(\mathcal{A})\|_{L^1(\Omega)\to L^1_w(\Omega)} \le C_s\Big(\sup_{t>0} \|\varphi(\cdot) f(t\cdot)\|_{W^{s,\infty}} + |f(0)|\Big).$$

A particular case of this corollary concerns the imaginary powers of $\mathcal{A}$. Precisely, $\mathcal{A}^{ir}$, $r \in \mathbb{R}$, extends to a bounded operator on $L^p(\Omega)$, $p \in (1,\infty)$, and, for any $\epsilon > 0$, there is a constant $C_\epsilon > 0$ so that

$$\|\mathcal{A}^{ir}\|_{\mathcal{B}(L^p(\Omega))} \le C_\epsilon(1 + |r|)^{\delta|1/2-1/p|+\epsilon}. \tag{1.5}$$

Indeed, an application of the previous corollary with $f(\lambda) = \lambda^{ir}$ shows that

$$\|\mathcal{A}^{ir}\|_{L^1(\Omega)\to L^1_w(\Omega)} \le C_\epsilon(1 + |r|)^{\delta/2+\epsilon}.$$

On the other hand, the standard functional calculus for self-adjoint operators gives

$$\|\mathcal{A}^{ir}\|_{\mathcal{B}(L^2(\Omega))} \le 1.$$

Therefore, (1.5) follows by interpolation. We refer to [14, Corollary 7.24, page 239] for more details.

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2. Proof of the main theorem

PROOF OF THEOREM 1.1. (1) We first recall that $-\mathcal{A}$ generates on $L^2(\Omega)$ an analytic semigroup $e^{-t\mathcal{A}}$. Note that

$$e^{-t\mathcal{A}} = \int_0^{+\infty} e^{-t\lambda}\, dE_\lambda, \quad t \geq 0.$$

PROPOSITION 2.1.

(a) *The semigroup $e^{-t\mathcal{A}}$ is positivity preserving.*

(b) *The semigroup $e^{-t\mathcal{A}}$ is a contraction on $L^p(\Omega) = L^p(\Omega, d\mu)$ for all $1 \leq p \leq \infty$ and $t \geq 0$.*

PROOF. (a) We recall that if $u \in H^1(\Omega)$, then $u^+, u^- \in H^1(\Omega)$ and $\nabla|u| = \nabla u^+ + \nabla u^-$. Hence

$$\mathfrak{a}(|u|, |u|) = \mathfrak{a}(u, u), \quad u \in H^1(\Omega).$$

In light of [5, Theorem 1.3.2, page 12], we deduce that $e^{-t\mathcal{A}}$ is positivity preserving.

(b) If $0 \leq u \in H^1(\Omega)$, then one can check in a straightforward manner that $u \wedge 1 = \min(u, 1) \in H^1(\Omega)$ and

$$\nabla(u \wedge 1) = \begin{cases} \nabla u & \text{in } [u > 1], \\ 0 & \text{in } [u \leq 1]. \end{cases}$$

Therefore $e^{-t\mathcal{A}}$ is a contraction semigroup on $L^p(\Omega)$ for all p with $1 \leq p \leq \infty$ by [5, Theorem 1.3.3, page 14]. □

This proposition says that $e^{-t\mathcal{A}}$ is a symmetric Markov semigroup.

We have for any integer k,

$$\mathcal{A}^k e^{-t\mathcal{A}} = \int_0^{+\infty} \lambda^k e^{-t\lambda}\, dE_\lambda. \tag{2.1}$$

Therefore, $e^{-t\mathcal{A}} f \in D(\mathcal{A})$, for all $f \in L^2(\Omega)$ and $t > 0$.

On the other hand, we get from the usual interior elliptic regularity

$$\bigcap_{k \in \mathbb{N}} D(\mathcal{A}^k) \subset C^\infty(\Omega).$$

Hence, $x \to e^{-t\mathcal{A}} f(x)$ belongs to $C^\infty(\Omega)$ for any fixed $t > 0$. But, $t \to e^{-t\mathcal{A}} f$ is analytic on $(0, \infty)$ with values in the Hilbert space $D(\mathcal{A}^k)$. Consequently, $(t, x) \to e^{-t\mathcal{A}} f(x)$ is in $C^\infty((0, \infty) \times \Omega)$.

From now on, the scalar product of $L^2(\Omega)$ will be denoted by $(\cdot, \cdot)_{2,\Omega}$ and the norm of $L^p(\Omega)$, $1 \leq p \leq \infty$, by $\|\cdot\|_{p,\Omega}$. The norm of $L^p(\mathcal{M})$ is simply denoted by $\|\cdot\|_p$, $1 \leq p \leq \infty$.

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We fix $t > 0$. Since $\lambda \to \lambda^k e^{-t\lambda}$ attains its maximum value at $\lambda = k/t$, we obtain from (2.1) for $f \in L^2(\Omega)$

$$\begin{aligned} \|\mathcal{A}^k e^{-t\mathcal{A}} f\|_{2,\Omega}^2 &= \int_0^\infty [\lambda^k e^{-\lambda t}]^2 \, d\|E_\lambda f\|_{2,\Omega}^2 \\ &\le \sup_{\lambda>0} [\lambda^k e^{-\lambda t}]^2 \int_0^\infty d\|E_\lambda f\|_{2,\Omega}^2 \\ &\le \Big(\frac{k}{t}\Big)^{2k} e^{-2k} \|f\|_{2,\Omega}^2. \end{aligned} \tag{2.2}$$

Again by the interior elliptic regularity, $D(\mathcal{A}^k)$ is continuously embedded in $C(\Omega)$ when k is sufficiently large. This and (2.2) entail: for any $\omega \Subset \Omega$, there exists $C = C(\Omega, \omega, k)$ so that

$$\sup_{\overline{\omega}} |e^{-t\mathcal{A}} f| \le \frac{C^2}{t^k} \|f\|_{2,\Omega}.$$

In particular, for any fixed $x \in \Omega$ and $t > 0$, the (linear) mapping $f \to e^{-t\mathcal{A}} f(x)$ is continuous. We can then apply the Riesz representation theorem to deduce that there exists $\ell(t, x) \in L^2(\Omega)$ so that

$$e^{-t\mathcal{A}} f(x) = (\ell(t, x), f)_{2,\Omega}, \quad x \in \Omega, \ t > 0.$$

Therefore, $(t, x) \to \ell(t, x) \in L^2(\Omega)$ is weakly C^∞ on $(0, \infty) \times \Omega$ and hence norm C^∞ by [4, Section 1.5].

Let $h(t, x, y) = (\ell(t/2, x), \ell(t/2, y))$. Then $h \in C^\infty((0, \infty) \times \Omega \times \Omega)$ and

$$(e^{-t\mathcal{A}} f, g)_{2,\Omega} = (e^{-(t/2)\mathcal{A}} f, e^{-(t/2)\mathcal{A}} g)_{2,\Omega} = \int_\Omega \int_\Omega h(t, x, y) f(x) g(y) \, d\mu(x) \, d\mu(y)$$

for $f, g \in C_0^\infty(\Omega)$. By the density of $C_0^\infty(\Omega)$ in $L^2(\Omega)$, the last identity yields

$$e^{-t\mathcal{A}} f(x) = \int_\Omega h(t, x, y) f(x) \, d\mu(x), \quad t > 0, \ x \in \Omega, \ f \in L^2(\Omega).$$

(2) We start with the following proposition.

Proposition 2.2. *The semigroup $e^{-t\mathcal{A}}$ satisfies the Davies–Gaffney property (abbreviated to DG from here on). That is, for any $t > 0$, U_1, U_2 open subsets of Ω, $f \in L^2(U_1, d\mu)$ and $g \in L^2(U_2, d\mu)$,*

$$|(e^{-t\mathcal{A}} f, g)_{2,\Omega}| \le e^{-r^2/4t} \|f\|_{2,\Omega} \, \|g\|_{2,\Omega}.$$

Here

$$r = \operatorname{dist}(U_1, U_2) = \inf_{x \in U_1, y \in U_2} d(x, y).$$

Proof. We omit the proof, which is similar to that of [3, Theorem 3.3, page 515]. □

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We now observe that Ω has the 1-extension property (see for instance [12, Theorem C]). In other words, there exists $\mathcal{E} \in \mathcal{B}(H^1(\Omega), H^1(\mathcal{M}))$ satisfying $(\mathcal{E}u)_{|\Omega} = u$, $u \in H^1(\Omega)$.

On the other hand, since $\mathcal{M}$ has the volume doubling property and the Gaussian bound (1.3), it follows from [1, Theorem 1.2.1] that the following Gagliardo–Nirenberg type inequality holds: for $2 < q \leq +\infty$, there exists a constant $C > 0$ so that

$$\|fV^{1/2-1/q}(\cdot, r)\|_q \leq C(\|f\|_2 + r\||\nabla f|\|_2^2), \quad r > 0,\ f \in C_0^\infty(\mathcal{M}). \tag{2.3}$$

In light of (2.3) and using $V_\Omega(\cdot, r) \leq V(\cdot, r)$ in Ω, we obtain for $r > 0$, $f \in H^1(\Omega)$ and fixed $2 < q \leq \infty$,

$$\begin{aligned}\|fV_\Omega^{1/2-1/q}(\cdot, r)\|_{q,\Omega} &\leq \|fV^{1/2-1/q}(\cdot, r)\|_{q,\Omega}\\ &\leq \|(\mathcal{E}f)V^{1/2-1/q}\|_q\\ &\leq C(\|\mathcal{E}f\|_2 + r\||\nabla(\mathcal{E}f)|\|_2)\\ &\leq C\|\mathcal{E}\|((1+r)\|f\|_{2,\Omega} + r\||\nabla f|\|_{2,\Omega}).\end{aligned}$$

Here $\|\mathcal{E}\|$ is the norm of $\mathcal{E}$ in $\mathcal{B}(H^1(\Omega), H^1(\mathcal{M}))$. Hence

$$\|fV_\Omega(\cdot, r)^{1/2-1/q}\|_{q,\Omega} \leq C(\|f\|_{2,\Omega} + r\||\nabla f|\|_{2,\Omega}), \quad r > 0,\ f \in H^1(\Omega),$$

where we used the fact that $V_\Omega(\cdot, r) = V_\Omega(\cdot, r_0) = \mu(\Omega)$, for all $r \geq r_0 = \operatorname{diam}(\Omega)$.

We then apply [1, Theorem 1.2.1] to show that h possesses a diagonal upper bound. In other words, there exists a constant $C > 0$ so that

$$h(t, x, x) \leq \frac{C}{[V_\Omega(x, \sqrt{t})V_\Omega(x, \sqrt{t})]^{1/2}}, \quad t > 0,\ x \in \Omega. \tag{2.4}$$

Since $e^{-t\mathcal{A}}$ has the *DG* property by Proposition 2.2, from [3, Corollary 5.4, page 524],

$$h(t, x, y) \leq \frac{eC}{[V_\Omega(x, \sqrt{t})V_\Omega(y, \sqrt{t})]^{1/2}}\Big(1 + \frac{d^2(x, y)}{4t}\Big)^\delta e^{-(d^2(x,y)/4t)}, \quad t > 0,\ x, y \in \Omega.$$

The proof is then complete. □

3. Domains with volume doubling property

3.1. Flat case. It is known that any bounded Lipschitz domain of $\mathbb{R}^n$ satisfies the volume doubling property. We discuss this again here. We consider $\mathbb{R}^n$ equipped with its Euclidean metric $g = (\delta_{ij})$. Let

$$\mathscr{C}(y, \xi, \epsilon) = \{z \in \mathbb{R}^n; (z - y) \cdot \xi \geq (\cos \epsilon)|z - y|, 0 < |y - z| < \epsilon\},$$

where $y \in \mathbb{R}^n$, $\xi \in \mathbb{S}^{n-1}$ and $0 < \epsilon$. That is, $\mathscr{C}(y, \xi, \epsilon)$ is the cone of dimension ϵ, with vertex y, aperture ϵ and directed by ξ.

We say that Ω has the ϵ-cone property if

$$\text{for any } x \in \Gamma, \text{ there exists } \xi_x \in \mathbb{S}^{n-1} \text{ so that, for all } y \in \overline{\Omega} \cap B(x, \epsilon), \mathscr{C}(y, \xi_x, \epsilon) \subset \Omega.$$

Let Ω be a bounded Lipschitz domain of $\mathbb{R}^n$. Then, by [9, Theorem 2.4.7, page 53], Ω has the ϵ-cone property, for some $\epsilon > 0$. As in [2], this implies that there exist $c_0 > 0$ and $\rho > 0$ so that

$$V_\Omega(x, r) = |B(x, r) \cap \Omega| \geq c_0 r^n, \quad x \in \Omega,\ 0 < r \leq \rho. \tag{3.1}$$

An immediate consequence is that Ω (equipped with its euclidean metric) satisfies the volume doubling property. Indeed, let $r_0 = \operatorname{diam}(\Omega)$ and $0 < r \leq s$. Then (3.1) entails

$$V_\Omega(x, s) \leq c_1 s^n = c_1 \Big(\frac{s}{r}\Big)^n r^n \leq \frac{c_1}{c_0}\Big(\frac{s}{r}\Big)^n V_\Omega(x, r), \quad 0 < r \leq \rho, \tag{3.2}$$

where $c_1 = |B(0, 1)|$.

Also, when $\rho < r_0$,

$$V_\Omega(x, s) \leq \frac{c_1}{c_0}\Big(\frac{s}{\rho}\Big)^n V_\Omega(x, \rho) \leq \frac{c_1}{c_0}\Big(\frac{r_0}{\rho}\Big)^n \Big(\frac{s}{r}\Big)^n V_\Omega(x, r), \quad \rho < r \leq r_0. \tag{3.3}$$

Finally, it is obvious that

$$V_\Omega(x, s) = |\Omega| = V_\Omega(x, r_0) \leq \Big(\frac{s}{r}\Big)^n V_\Omega(x, r), \quad r > r_0. \tag{3.4}$$

Estimates (3.2)–(3.4) show the volume doubling property.

3.2. Manifold with sectional curvature bounded from above. Let $T_x\mathcal{M}$ be the tangent space at $x \in \mathcal{M}$, $\mathbb{S}_x \subset T_x\mathcal{M}$ the unit tangent sphere and $S\mathcal{M}$ the unit tangent bundle. Let Φ_t be the geodesic flow with phase space $S\mathcal{M}$. That is, for any $t \geq 0$,

$$\Phi_t : S\mathcal{M} \to S\mathcal{M} : (x, \xi) \in S\mathcal{M} \to \Phi_t(x, \xi) = (\gamma_{x,\xi}(t), \dot{\gamma}_{x,\xi}(t)).$$

Here $\gamma_{x,\xi} : [0, \infty) \to \mathcal{M}$ is the unit speed geodesic starting at x with tangent unit vector ξ and $\dot{\gamma}_{x,\xi}(t)$ is the unit tangent vector to $\gamma_{x,\xi}$ at $\gamma_{x,\xi}(t)$ in the forward t direction.

If $(x, \xi) \in S\mathcal{M}$, we denote by $r(x, \xi)$ the distance from x to the cutlocus in the direction of ξ:

$$r(x, \xi) = \inf\{t > 0; d(x, \Phi_t(x, \xi)) < t\}.$$

We fix $\delta \in (0, 1]$ and $r > 0$. Following [15], a (δ, r)-cone at $x \in \mathcal{M}$ is the set of the form

$$\mathscr{C}(x, \omega_x, r) = \{y = \gamma_{x,\xi}(s); \xi \in \omega_x, 0 \leq s < r\},$$

where ω_x is a subset of $\mathbb{S}_x$ so that $r < r(x, \xi)$ for all $\xi \in \omega_x$ and $|\omega_x| \geq \delta$ (here $|\omega_x|$ is the volume of ω_x with respect to the normalised measure on the sphere $\mathbb{S}_x$).

A domain D which contains a (δ, r)-cone at x for any $x \in D$ is said to satisfy the interior (δ, r)-cone condition.

Let

$$s_\kappa(r) = \begin{cases} \Big(\dfrac{\sin(\sqrt{\kappa}r)}{\sqrt{\kappa}}\Big)^{n-1} & \text{if } \kappa > 0, \\ r^{n-1} & \text{if } \kappa = 0, \\ \Big(\dfrac{\sinh(\sqrt{-\kappa}r)}{\sqrt{-\kappa}}\Big)^{n-1} & \text{if } \kappa < 0. \end{cases}$$

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We assume that the sectional curvature of $\mathcal{M}$ is bounded above by a constant κ, $\kappa \in \mathbb{R}$, and Ω satisfies the interior (δ, r)-cone condition. Let $J(x, \xi, t)$ be the density of the volume element in geodesic coordinates around x. That is

$$dV(y) = J(x, \xi, t) d_{\mathbb{S}_x}\, dt, \quad y = \gamma_{x,\xi}(t),\ t < r(x, \xi).$$

By an extension of Günther's comparison theorem (see for instance [10]), J satisfies the uniform lower bound

$$J(x, \xi, t) \geq s_\kappa(t).$$

Consequently, for some $r_0 > 0$,

$$V_\Omega(x, r) \geq V(\mathscr{C}(x, \omega_x, r)) \geq c_0 r^n, \quad x \in \Omega,\ 0 < r \leq r_0.$$

We proceed similarly to the flat case to prove the following lemma.

Lemma 3.1. *Assume that $\mathcal{M}$ has sectional curvature bounded from above and satisfies the volume growth condition*

$$V(x, r) \leq c_1 r^n, \quad 0 < r \leq r_1,$$

for some constants c_1 and r_1. If Ω is of finite diameter and satisfies the (δ, r)-cone condition, then V_Ω is doubling.

References

[1] S. Boutayeb, T. Coulhon and A. Sikora, 'A new approach to pointwise heat kernel upper bounds on doubling metric measure spaces', *Adv. Math.* **270** (2015), 302–374.

[2] M. Choulli and L. Kayser, 'Gaussian lower bound for the Neumann Green function of a general parabolic operator', *Positivity*, doi:10.1007/s11117-014-0319-z.

[3] T. Coulhon and A. Sikora, 'Gaussian heat kernel bounds via Phragmèn–Lindelöf theorem', *Proc. Lond. Math. Soc.* (3) **96**(3) (2008), 507–544.

[4] E. B. Davies, *One-Parameter Semigroups* (Academic, London, 1980).

[5] E. B. Davies, *Heat Kernels and Spectral Theory*, Cambridge Tracts in Mathematics, 92 (Cambridge University Press, London, 1989).

[6] E. B. Davies, 'L^p spectral independence and L^1 analyticity', *J. Lond. Math. Soc.* (2) **52**(2) (1995), 177–184.

[7] X. T. Duong, E. M. Ouhabaz and A. Sikora, 'Plancherel-type estimates and sharp spectral multipliers', *J. Funct. Anal.* **196** (2002), 443–485.

[8] P. Gyrya and L. Saloff-Coste, 'Neumann and Dirichlet heat kernels in inner uniform domains', *Astérisque* (No. 336) (2011).

[9] A. Henrot and M. Pierre, *Variation et optimisation de formes*, Mathématiques et Applications, 48 (Springer, Berlin, 2005).

[10] B. R. Kloeckner and G. Kuperberg, 'A refinement of Günther's candle inequality', Preprint, 2012, arXiv:1204.3943.

[11] P. Li and S.T. Yau, 'On the parabolic kernel of the Schrödinger operator', *Acta Math.* **156** (1986), 153–201.

[12] D. Mitrea, M. Mitrea and M. C. Shaw, 'Traces of differential forms on Lipschitz domains, the boundary De Rham complex, and Hodge decompositions', Preprint.

[13] E. M. Ouhabaz, 'Gaussian estimates and holomorphy of semigroups', *Proc. Amer. Math. Soc.* **123**(5) (1995), 1465–1474.

[14] E. M. Ouhabaz, *Analysis of Heat Equations on Domains*, London Mathematical Society Monographs, 31 (Princeton University Press, Princeton, NJ, 2004).

[15] L. Saloff-Coste, 'Pseudo-Poincaré inequalities and applications to Sobolev inequalities', in: *Around the Research of Vladimir Maz'ya. I*, International Mathematical Series (New York), 11 (Springer, New York, 2010), 349–372.

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