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A PARTICULAR CLASS OF SUPERSOLUBLE GROUPS

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Abstract

We consider (finite) groups in which every two-generator subgroup has cyclic commutator subgroup. Among other things, these groups are metabelian modulo their hypercentres, and in the corresponding quotient group all subgroups of the commutator subgroup are normal.

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In this note we will consider the class of finite groups G satisfying the following condition:

(∗) for all x, y in G there is $n = n(x, y)$ in $\mathbb{Z}$ such that $[x, y]^x = [x, y]^{n(x,y)}$.

Here, as usual, $[x, y]$ is the commutator $x^{-1}y^{-1}xy$, and $a^b = b^{-1}ab$.

We will see that all finite groups satisfying (∗) are supersoluble (Theorem 1), and since condition (∗) is inherited by subgroups and quotient groups, we may consider first the case of groups with Fitting subgroup a p-group. Special attention is required here if the Fitting subgroup is of index 2 in G (Lemma 4, Lemma 5).

If a finite group G satisfies condition (∗), its hypercentre $Hz(G)$ contains the second derivative G'', and all subgroups between $G'Hz(G)$ and $Hz(G)$ are normal subgroups of G (see Main Theorem).

By G^* we will denote the nilpotent residual of G, that is the intersection of all normal subgroups K of G with G/K nilpotent (this coincides with the intersection of all terms of the lower central series of G); all other notation should be standard (see for instance Huppert [3]).

This note includes and extends results of Dirscherl [1].

THEOREM 1. *If G is a finite group satisfying* (∗), *then G is supersoluble.*

PROOF. By Doerk [2] it is sufficient to prove that all two-generator subgroups of G are supersoluble. Since $\langle x, y\rangle' = \langle [x, y]\rangle$, this is obvious.

If p is any prime and G is supersoluble, if H is the maximal normal p'-subgroup of G, then the Fitting subgroup of G/H is a p-group. If p runs through all primes, the corresponding normal subgroups H have only the trivial group in common. So G can be considered a subdirect product of groups G/H. This explains why we consider first this special case.

THEOREM 2. *If G is a finite group satisfying* $(*)$ *such that its Fitting subgroup F is a p-group, then G/F is cyclic.*

PROOF. Since G is supersoluble, its Fitting subgroup F contains the commutator subgroup G', and the Hall-p'-subgroups of G are isomorphic to G/F and therefore abelian. Assume the existence of a noncyclic elementary abelian q-subgroup $\langle a, b\rangle$ in a Hall-p'-subgroup S of G. Denote by R the subgroup $[\langle a, b\rangle, F]$. We see that R is a normal subgroup of $\langle F, a, b\rangle$. Since F is the Fitting subgroup of G we obtain $C(F) \cap \langle a, b\rangle = 1$ and also $C(R) \cap \langle a, b\rangle = 1$. An element of order $q \neq p$ operating on a p-group fixes every element of it if and only if it does so with the cosets of the commutator subgroup; it suffices therefore to consider the quotient group $\langle R, a, b\rangle / R'$. By the preceding we know that $[x, R] \not\subseteq R'$ for all $x \in \langle a, b\rangle$ different from 1. Relabelling a and b if necessary we have two subgroups A and B such that $R \supset \langle A, B\rangle$ and $[a, A] \subseteq R' \subset A$, $[b, B] \subseteq R' \subset B$, $[b, A]R' = R = [a, B]R'$.

Choose two elements $x \in A$ and $y \in B$, both not in R'.

By condition $(*)$,

$$[xa, yb]^{yb} = [xa, yb]^m \qquad \text{for some } m.$$

However, considering modulo R', we have

$$[xa, yb] \equiv [x, b][a, y],$$
$$[xa, yb]^{yb} \equiv [x, b]^{yb}[a, y]^{yb} \equiv [x, b][a, y]^y \equiv [x, b][a, y]^n.$$

Since y is a p'-element, $n - 1$ is not divisible by p, and the same applies to m. This leads to a contradiction since $\langle [x, b]\rangle \cap \langle [a, y]\rangle \subseteq R'$.

THEOREM 3. *If G is a finite group satisfying* $(*)$ *such that its Fitting subgroup F is a p-group and $|G : F| \neq 1$ or 2, then G^* is abelian.*

PROOF. Choose x such that $G = \langle x, F\rangle$ and $\langle x\rangle \cap F = 1$. The subgroup $[x, F] = R$ is normal in G and contained in F. By condition $(*)$, the element x acts on R/R' by conjugation as a power automorphism, so

$$x^{-1}uxR' = u^m R'$$

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for all $u \in R$. Assume that R' is different from 1, and N is a normal subgroup of G that contains R_3 such that $R' \supset N$ and R'/N is cyclic. By construction, $R' = [u, v]N$ for some u, v in R. Now x does not operate as a power automorphism on $\langle u, [u, v], N\rangle/N$ and does not fix $[u, v]$ mod N since $m^2 \neq 1$ (x is not of order 2). Accordingly, $(*)$ is not satisfied for $\langle xN, u[u, v]N\rangle$ in $\langle x, u, [u, v], N\rangle/N$. This contradiction shows $R' = 1$. This proves Theorem 3 since $R = G^*$.

LEMMA 4. *If G is a finite group satisfying $(*)$ such that its Fitting subgroup F is a p-group and $[G : F] = 2$, then G^* is nilpotent of class 2 if $p > 3$ and of class 3 if $p = 3$.*

PROOF. Assume that x is an element or order 2 in G. Denote by C the elements of F that are centralized by x, and by I the set of elements inverted by x by conjugation and contained in F. While C is clearly a subgroup, we have the following closure property for I: if a and b belong to I, so does aba. We prove first an interrelation between C and I.

(1) $\quad [a, b] \in C$ if $\{a, b\} \subset I$ or $\{a, b\} \subset C$

(2) $\quad [a, b] \in I$ if $a \in C$ and $b \in I$ or vice versa.

To prove (1) and (2) we remember that in groups satisfying $(*)$ we know that $[a^k, b^h]$ is a power of $[a, b]$. So if $\{a, b\} \subset I \cup C$ we find

$$x^{-1}[a, b]x = [a, b]^m$$

for some m, which in this case can only be 1 or -1, so $[a, b] \in I \cup C$.

In (1), $[a, b] \in I$ and $\{a, b\} \subset I$ leads to

$$[a^{-1}, b^{-1}] = [a, b]^{-1}$$

and

$$ab[a, b]b^{-1}a^{-1} = [a, b]^{-1}$$

which is impossible for ab of odd order.

In (2), $[a, b] \in C$ with $a \in C$ and $b \in I$ leads to

$$[a, b^{-1}] = [a, b]$$

and

$$[a, b^2] = 1.$$

Now $(*)$, (1) and (2) yield

(3) $\quad$ if $\{a, b\} \subset I$, then $[[a, b], b] = 1$.

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To see this, we have $[a, b] \in C$ by (1) and $[[a, b], b] \in I$ by (2). On the other hand, by $(*)$, $[[a, b], b]$ is a power of $[a, b]$ and belongs to C. So $[[a, b], b] = 1$ since $I \cap C = \{1\}$.

By analogous argument we have

(4) $$\text{if } a \in C \text{ and } b \in I, \text{ then } [[a, b], b] = 1.$$

Now we consider $[x, F]$. This is a p-group which can be generated by elements of I since each commutator $[x, w]$ is inverted by conjugation with x. We fix a basis of $[x, F]$ which is contained in I, and we consider $D = [x, F]/[x, F]_5$. For images of basis elements a, b, c, d we have

$$[[[a, b], c], c] = [[a, b], b] = 1 \text{ and so } [[[a, b], dcd], dcd] = 1.$$

Since all commutators of length 5 are trivial we have

$$[[[a, b], c], c][[[a, b], c], d]^2[[[a, b], d], c]^2[[[a, b], d], d]^4 = 1$$

which yields

$$[[[a, b], c], d] = [[[a, b], d], c]^{-1},$$

and from

$$[[[a, cbc], cbc], d] = 1$$

we obtain in the same way

$$[[[a, b], c], d] = [[[a, c], b], d]^{-1}.$$

Now

$$\begin{aligned}[[a, b], [c, d]] &= [[[a, b], c], d]^2 = [[[a, c], b], d]^{-2} = [[[a, c], d], b]^2 \\ &= [[[c, a], d], b]^{-2} = [[[c, d], a], b]^2 = [[c, d], [a, b]] = [[a, b], [c, d]]^{-1},\end{aligned}$$

we obtain

$$[[[a, b], c], d] = 1$$

and $D_4 = 1$.

This yields

(5) $$[x, F] \text{ is nilpotent of class 3 at most.}$$

For $p \neq 3$ we obtain from

$$[[a, cbc], cbc] = 1$$

that

$$[[a, b], c] = [[a, c], b]^{-1}, \qquad [[a, b], c]^3 = 1$$

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and

(6) $[x, F]$ is nilpotent of class 2 at most if $p \neq 3$.

Since $[x, F]$ is the nilpotent residual G^*, Lemma 4 is proved.

We want to remove the restriction in Lemma 4 regarding the prime 3. For this we begin with a special case.

LEMMA 5. *Let $G = \langle a, b, c, x\rangle$ with $x^2 = (ax)^2 = (bx)^2 = (cx)^2 = 1$ and a, b, c elements of order a power of 3. Suppose that G satisfies* $(*)$. *Then* $[[a, b], c] = 1$.

PROOF. Assume to the contrary. Then $[x, c] = c^2$ is a power of $[([a, b]x), c]$ since x is a power of $[a, b]x$.

Likewise $[c^2, [a, b]]$ is a power of $[c^2, ([a, b]x)]$ which in turn is a power of c^2. So we have deduced

$$[[a, b], c] \in \langle c\rangle.$$

By the proof of Lemma 4 we also know that $[[a, b], c]$ is of order 3, and we have $[[a, b], c] = [[b, c]a] = [[c, a], b]$. Using the same argument as before we have

$$[[a, b], c] \in \langle c\rangle \cap \langle b\rangle \cap \langle a\rangle.$$

We will show that we can choose a, b, c in such a way that this inclusion does not hold, and this will be the contradiction needed. Assume that a, b, c are chosen such that the product of their orders is minimal, and assume further that the orders of b and c are not smaller than that of a. Let $a^k = [[a, b], c]$, and choose $v \in \langle b\rangle$ such that $v^k = a^k$. Then

$$(vav)^k = v^k a^k v^k [a, v]^{2s},$$

where $s = k(k-1)/2$.

Since $[[a, v], a] = 1$ and $a^k = [[a, b], c] \in Z(\langle a, b, c\rangle)$, we have

$$(vav)^k = 1.$$

Now $\langle vav, b, c\rangle = \langle a, b, c\rangle$ and $(xvav)^2 = 1$. Also $[[vav, b], c] = [[a, b], c]$, and the inclusion of $[[a, b], c]$ in $\langle a\rangle \cap \langle b\rangle \cap \langle c\rangle$ leads to a contradiction to minimality. So $[[a, b], c] = 1$ as stated.

Now we are able to conclude

THEOREM 6. *If G is a finite group satisfying* $(*)$ *such that its Fitting subgroup F is a p-group and $[G : F] = 2$, then G^* is nilpotent of class* 2.

The proof follows directly from Lemma 4 and Lemma 5.

THEOREM 7. *If G is a finite group satisfying $(*)$ such that its Fitting subgroup F is a p-group and $G \neq F$, then the Carter subgroups of G are abelian.*

PROOF. Let $G = \langle x, F\rangle$ and $\langle x\rangle \cap F = 1$. It is clear that $C(x)$ is a Carter subgroup of G. It is therefore sufficient to show that $F \cap C(x)$ is abelian since $C(x) = F \cap C(x) \times \langle x\rangle$. We assume the contrary and choose two elements a, b of $F \cap C(x)$ which do not commute. Without loss of generality we may assume $[[a, b], b] = 1$. Furthermore we choose an element u contained in G^* but not in $(G^*)'$ such that x normalizes $\langle u\rangle$ and $[u, F]$ is contained in $(G^*)'$.

Since $[u, t] \in (G^*)'$ for all t in $F \cap C(x)$ and $(G^*)'$ is contained in $C(x)$, we deduce

$$[u, t] = 1 \text{ for all } t \in F \cap C(x).$$

By $(*)$ there is an integer m such that

$$[au, bx]^{bx} = [au, bx]^m$$

and we have

$$[au, bx] = [a, bx]^u[u, bx] = [a, x]^u[a, b]^{xu}[u, x] = [a, b][u, x].$$

This yields

$$[au, bx]^{bx} = [a, b]^{bx}[u, x]^{bx} = [a, b][u, x]^x = [a, b][u, x]^n$$

for some integer n. From the original equation we deduce

$$[au, bx]^{bx} = [a, b]^m[u, x]^m$$

and finally

$$[a, b]^{m-1} = [u, x]^{m-n}.$$

By construction we have $[a, b] \in C(x)$ while $\langle [u, x]\rangle \cap C(x) = 1$. Now $[a, b]^{m-1} = [u, x]^{m-n} = 1$; and $m-1$ and $m-n$ are divisible by p. But $n - 1 = (m-1) - (m-n)$ is not divisible by p since x is of order prime to p, and this contradiction shows that the non-commuting pair a, b of elements does not exist: $F \cap C(x)$ is abelian, and so is $C(x)$. Theorem 7 is true.

Now we are ready for another structural statement.

THEOREM 8. *If G is a finite group satisfying $(*)$ such that the Fitting subgroup F is a p-group and $F \neq G$, then $G/Z(G)$ is the extension of an abelian group by a cyclic group, and every subgroup satisfying $G'Z(G) \supseteq L \supseteq Z(G)$ is a normal subgroup of G.*

PROOF. Assume first that $(G^*)' = 1$ and choose an element a of $C(x) \cap F$. Since $(ax)^r = x$ for suitable integer r, we have $G^* \supseteq [G^*, ax] \supseteq [G^*, x] = G^*$ so that

$$G^* = [G^*, ax]$$

and every element of G^* can be written in the form $[t, ax]$. By $(*)$ we see now that ax normalizes all subgroups of G^*, so ax induces by conjugation a power automorphism in G^*. This shows that $G/C(G^*)$ is cyclic (of order $p^n - p^{n-1}$, if p^n is the exponent of G^*). Now

$$C(G^*) = C(G^*) \cap (G^*C(x)) = G^* \times (C(x) \cap C(G^*)) = G^* \times Z(G),$$

and Theorem 8 follows for $(G^*)' = 1$.

If $(G^*)' \neq 1$, let $L/(G^*)' = Z(G/(G^*)')$. Then $L \subseteq C(x)$ is abelian, and for all $u \in L$ and $v \in G^*$ we have $[u, v] = 1$. Now $L = Z(G)$, and Theorem 8 is true also in this case.

Now the general statement on the structure can be proved.

MAIN THEOREM. *If G is a finite group satisfying $(*)$, the following two statements are true.*

(i) *$G/Hz(G)$ is metabelian and all subgroups W with $G'Hz(G) \supseteq W \supseteq Hz(G)$ are normal subgroups of G.*

(ii) *The orders of the quotient groups $G'Hz(G)/Hz(G)$ and $Hz(G)/Z(G)$ are relatively prime.*

PROOF. By Theorem 1, G is supersoluble; in particular we know that the Fitting subgroup F of G contains the commutator subgroup G' of G. Let the prime p be a divisor of the order of G. Denote by S some p-Sylow subgroup of G and by R the maximal normal p'-subgroup of G. We distinguish two cases.

Case 1: $SR/R \not\subseteq Hz(G/R)$.

In this case we have by Theorem 8

$$Hz(G/R) = Z(G/R) \qquad \text{and} \qquad (G/R)'' \subseteq Z(G/R)$$

and we deduce

$$S \cap Hz(G) = S \cap Z(G).$$

Also by Theorem 8 we obtain: all subgroups A/R satisfying

$$(G/R)'Z(G/R) \supseteq A/R \supseteq Z(G/R)$$

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are normal in G/R, consequently all subgroups A with

$$G'(Z(G) \cap S)R \supseteq A \supseteq (Z(G) \cap S)R$$

are normal in G.

Case 2: $SR/R \subseteq Hz(G/R)$.

In this case G/R is a p-group, and

$$G'R \subseteq Hz(G)R/R.$$

Now the primes of Case 1 are the divisors of $G'Hz(G)/Hz(G)$, while the order of $Hz(G)/Z(G)$ is divisible only by primes of Case 2, and this proves statement (ii). On the other hand, statement (i) follows since W is the intersection of all WR, where R is defined as above and p runs through all primes dividing the order of G. Since these products WR are normal in G, so is their intersection.

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