A GENERALIZATION OF FINSLER GEOMETRY

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Modern differential geometry may be said to date from Riemann's famous lecture of 1854 **(9)**, in which a distance function of the form $F(x^i, dx^i) = (\gamma_{ij}(x)dx^i dx^j)^{\frac{1}{2}}$ was proposed. The applications of the consequent geometry were many and varied. Examples are Synge's geometrization of mechanics **(15)**, Riesz' approach to linear elliptic partial differential equations **(10)**, and the well-known general theory of relativity of Einstein.

Meanwhile the results of Caratheodory **(4)** in the calculus of variations led Finsler in 1918 to introduce a generalization of the Riemannian metric function **(6)**. The geometry which arose was more fully developed by Berwald **(2)** and Synge **(14)** about 1925 and later by Cartan **(5)**, Busemann, and Rund. It was then possible to extend the applications of Riemannian geometry. For example, Rund successfully generalized the dynamical results of Synge **(11)**. In certain applications, such as quasi-linear elliptic partial differential equations, however, difficulties arise from the fact that a Finsler metric tensor $\gamma_{ij}(x, \dot{x})$ must satisfy

$$\frac{\partial}{\partial \dot{x}^i}\gamma_{jk}(x, \dot{x}) = \frac{\partial}{\partial \dot{x}^k}\gamma_{ij}(x, \dot{x}) = \frac{\partial}{\partial \dot{x}^j}\gamma_{ki}(x, \dot{x}).$$

The following work is an attempt to avoid this restriction.

In § 1 a suitable invariant differential, first suggested by Moór **(7)**, is investigated and the problem of determining its coefficients in terms of the metric tensor is posed. This question is completely answered in §§ 2, 3, and 4, and in § 5 relevant uniqueness theorems are established. The remaining sections deal with the geometrical structure of the space as revealed by the theories of autoparallels, curvature, and curve deviation.

1. Fundamental Concepts. Consider a space of line-elements $(x^i, \dot{x}^i)$ $(i = 1, 2, \ldots, n)$ endowed with a second-order symmetric tensor $g_{ij}(x^k, \dot{x}^k)$. This tensor is assumed to be analytic in each of its $2n$ arguments (except when all the $\dot{x}^i$ vanish). We introduce the scalar function

$$F(x^i, \dot{x}^i) = +\,(g_{ij}(x^k, \dot{x}^k)\dot{x}^i\dot{x}^j)^{\frac{1}{2}}, \tag{1.1}$$

which in turn defines a second second-order symmetric tensor

$$\gamma_{ij}(x^k, \dot{x}^k) = \frac{1}{2}\frac{\partial^2 F^2(x, \dot{x})}{\partial \dot{x}^i \partial \dot{x}^j}. \tag{1.2}$$

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The class of functions of $(x^i, \dot{x}^i)$ which are positively homogeneous of degree p in $\dot{x}^i$ will be denoted by H_p.

The following conditions are imposed.

(a) The $g_{ij}(x, \dot{x}) \in H_0$:

$$g_{ij}(x, k\dot{x}) = kg_{ij}(x, \dot{x}) \qquad (k > 0). \tag{1.3}$$

In view of Euler's theorem on homogeneous functions it follows that

$$\frac{\partial g_{ij}}{\partial \dot{x}^k} \dot{x}^k = 0. \tag{1.3'}$$

Furthermore, relations (1.3) and (1.1) imply that $F(x, \dot{x}) \in H_1$.

(b) The quadratic form $g_{ij}(x, \dot{x})X^iX^j$ is positive definite for all line-elements $(x^i, \dot{x}^i)$. From this we readily deduce that the determinant

$$g \equiv |g_{ij}(x, \dot{x})| \neq 0. \tag{1.4}$$

(c) The quadratic form $\gamma_{ij}(x, \dot{x})X^iX^j$ is likewise positive definite. Consequently

$$\gamma \equiv |\gamma_{ij}(x, \dot{x})| \neq 0. \tag{1.4'}$$

If $g_{ij}(x, \dot{x}) = \gamma_{ij}(x, \dot{x})$, then (1.1) defines a Finsler metric function, and, by (1.2)

$$\frac{\partial g_{ij}}{\partial \dot{x}^k} = \frac{\partial g_{ik}}{\partial \dot{x}^j} = \frac{\partial g_{kj}}{\partial \dot{x}^i}.$$

In the following work *the strong condition that* $g_{ij}(x, \dot{x})$ *be a second partial derivative is dropped and hence the geometry of a Finsler space occurs as a particular case of the geometry defined by conditions* (a), (b), *and* (c).

The expression $g_{ij}(x, \dot{x})X^i(x, \dot{x})X^j(x, \dot{x})$ may be interpreted, by analogy with Finsler geometry, as the square of the length of the vector X^i defined with respect to the line-element $(x^i, \dot{x}^i)$. In particular, the unit vector $l^i(x, \dot{x})$ in the direction of the line-element is given by

$$l^i(x, \dot{x}) = [F(x, \dot{x})]^{-1}\dot{x}^i. \tag{1.5}$$

With the exception of $F(x, \dot{x})$, the vectors, connections, and other tensorial quantities which will arise will depend only on the "centre" x^i and the direction of the line-element $(x^i, \dot{x}^i)$. Thus the magnitude of the vector $\dot{x}^i$ does not affect these quantities and accordingly they $\in H_0$.

There is no intrinsic method of comparing the directions of two vectors attached to different points of a metric space. For such a comparison it is convenient to introduce an "invariant differential." This is a linear operator D, acting on vectors $X^i(x, \dot{x})$, which consists of terms resulting from

(a) a functional variation $dX^i(x, \dot{x})$ of X^i,

(b) a displacement of the centre x^i of the line-element, and

(c) a rotation of the line-element.

Thus

$$DX^i = dX^i + F^{-1}\bar{M}_j{}^i{}_k(x, \dot{x})X^j d\dot{x}^k + \bar{L}_j{}^i{}_k(x, \dot{x})X^j dx^k, \tag{1.6}$$

where $\bar{M}_j{}^i{}_k$ and $\bar{L}_j{}^i{}_k$ are functions defined with respect to the line-element.*

As indicated above we require that DX^i be independent of the length $F(x, \dot{x})$ of $\dot{x}^i$ (and *a fortiori* of dF). Now, from (1.5), we have $d\dot{x}^i = l^i dF + F dl^i$ and it follows that

$$\bar{M}_j{}^i{}_k l^k \equiv \bar{M}_j{}^i{}_0 = 0 \tag{1.7}$$

and that the functions $\bar{M}_j{}^i{}_k$, $\bar{L}_j{}^i{}_k \in H_0$. The definition (1.6) then becomes

$$DX^i = dX^i + \bar{M}_j{}^i{}_k X^j dl^k + \bar{L}_j{}^i{}_k X^j dx^k. \tag{1.8}$$

In particular, we have

$$Dl^i = dl^k(\delta^i_k + \bar{M}_0{}^i{}_k) + \bar{L}_0{}^i{}_k dx^k. \tag{1.9}$$

It is essential that we have an explicit expression for dl^k in terms of vectorial displacements Dl^i, dx^i. The simplest assumption yielding this is **(7**, p. 89**)**

$$\bar{M}_0{}^i{}_k \bar{M}_0{}^k{}_j = 0, \tag{1.10}$$

for it will then follow from (1.9] that

$$dl^i = (Dl^j - \bar{L}_0{}^j{}_k dx^k)(\delta^i_j - \bar{M}_0{}^i{}_j). \tag{1.11}$$

By substituting this value for dl^i in (1.8) we obtain

$$DX^i = dX^i + M_j{}^i{}_k X^j Dl^k + L_j{}^i{}_k X^j dx^k, \tag{1.12}$$

where

$$\begin{aligned} M_j{}^i{}_k &= \bar{M}_j{}^i{}_k(\delta^r_k - \bar{M}_0{}^r{}_k), \\ L_j{}^i{}_k &= \bar{L}_j{}^i{}_k - \bar{M}_j{}^i{}_r(\delta^r_s - \bar{M}_0{}^r{}_s)\bar{L}_0{}^s{}_k. \end{aligned} \tag{1.13}$$

It is easily seen that these relations are uniquely solvable for the $\bar{M}_j{}^i{}_k$ and $\bar{L}_j{}^i{}_k$ in terms of the $M_j{}^i{}_k$ and $L_j{}^i{}_k$ **(7**, p. 90**)**. Hence the problem of determining the invariant differential is reduced to the determination of the latter.

In order to ascertain the tensorial character of $L_j{}^i{}_k$ and $M_j{}^i{}_k$ we first note that DX^i must be a contravariant vector for arbitrary choices of X^i, Dl^i and dx^i. In particular, if the Dl^k are taken to be zero, the right-hand side of (1.12) is formally the same as in Riemannian geometry and we conclude that, under a non-singular co-ordinate change

$$x^i = x^i(u^\alpha),\ V^i_\alpha = \frac{\partial x^i}{\partial u^\alpha},\ V^\alpha_{ij} = \frac{\partial^2 u^\alpha}{\partial x^i \partial x^j}\ ;\text{etc.} \quad (\alpha = 1, 2, \ldots, n), \tag{1.14}$$

the quantitities $L_j{}^i{}_k$ transform according to the law

$$L^\alpha_{\beta\gamma} = V^\alpha_i L_j{}^i{}_k V^j_\beta V^k_\gamma - V^\alpha_{ij} V^j_\beta V^i_\gamma. \tag{1.15}$$

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*The present treatment of the invariant differential follows closely that of Moór **(7**, pp. 88–90**)**.

Quantities satisfying 1.15 are said to transform like connection parameters. Although we shall also refer to the quantities $M_j{}^i{}_k$ as connection parameters it is now clear that they transform as tensors.

It follows from the first set of equations (1.13) that conditions (1.7) and (1.10) are equivalent respectively to

$$(1.16) \qquad \text{(a)} \quad M_i{}^j{}_0 = 0, \qquad \text{(b)} \quad M_0{}^i{}_j M_0{}^j{}_k = 0.$$

If we now express dX^i in terms of Dl^k and dx^k, equations (1.12) become

$$(1.17) \qquad DX^i = X^i{}_{|k}dx^k + X^i{}_{;k}Dl^k,$$

where we have written

$$(1.18) \qquad \text{(a)} \quad X^i{}_{|k} = \frac{\partial X^i}{\partial x^k} - X^i{}_{||r}L_0{}^r{}_k + X^r L_r{}^i{}_k,$$

$$\text{(b)} \quad X^i{}_{;k} = X^i{}_{||r}(\delta^r_k - M_0{}^r{}_k) + X^r M_r{}^i{}_k,$$

and

$$(1.19) \qquad X^i{}_{||r} \equiv F\frac{\partial X^i}{\partial \dot{x}^r}.$$

The tensors (1.18) (a) and (b) are called "covariant derivatives." If we assume that the product rule holds, it is possible to extend the definition to tensors of any order. A similar remark may be made for the invariant differential operator. In particular, the invariant differential of a scalar is its ordinary differential. Also, applying the processes of covariant differentiation to g_{ij}, we obtain

$$(1.20) \qquad \text{(a)} \quad g_{ij|k} = \frac{\partial g_{ij}}{\partial x^k} - g_{ij||r}L_0{}^r{}_k - g_{rj}L_i{}^r{}_k - g_{ir}L_j{}^r{}_k$$

and

$$(1.20) \qquad \text{(b)} \quad g_{ij;k} = g_{ij||r}(\delta^r_k - M_0{}^r{}_k) - g_{rj}M_i{}^r{}_k - g_{ir}M_j{}^r{}_k.$$

An interesting property of the covariant derivative (1.18) (b) is

$$(1.21) \qquad X^i{}_{;0} \equiv X^i{}_{;k}l^k = 0,$$

whenever $X^i \in H_0$. This is an immediate consequence of the relations (1.16).

In the sequel g_{ij} and its inverse g^{ij}, the unique solution of

$$(1.22) \qquad g^{ij}g_{jk} = \delta^i_k$$

(which exists, by (1.4)), will be used to lower and raise tensor indices. If T_{ijk} is any three-index symbol, we write $\frac{1}{2}(T_{ijk} + T_{jik}) = T_{(ij)k}$ etc. Also we adopt the notation

$$(1.23) \qquad \tfrac{1}{2}g_{ij||k} = A_{ij,k}.$$

Note, that by (1.3′) and (1.19) we have

$$(1.23') \qquad A_{ij,0} = 0.$$

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The process of invariant differentiation is called metric if Dg_{ij} vanishes identically. This assumption represents the generalization of the Ricci lemma of Riemannian geometry. It is equivalent to the assumption that the invariant differential of the length of a vector vanishes whenever the invariant differential of the vector itself does. In this case, several important identities follow. Firstly, applying D to the relation $g_{ij}l^il^j \equiv 1$, and, noting the symmetry of g_{ij}, we obtain

$$2g_{ij}l^iDl^j \equiv 2l_jDl^j = 0. \tag{1.24}$$

Thus the displacements Dl^i are not independent. However, (1.24) is the only relation connecting the Dl^i since further ones would imply the interdependence of the $\dot{x}^i$. Combining equations (1.17) (with $X^i = l^i$) and (1.24) we conclude that, for arbitrary λ^i,

$$l^i{}_{|k}dx^k + (l^i{}_{;k} - \delta^i_k + \lambda^i l_k)Dl^k \equiv 0.$$

Now for each fixed i we can choose λ^i so that the coefficient of one of the Dl^k vanishes. Since the remaining Dl^k and the dx^k are independent it follows that

$$l^i{}_{|k} = 0 \tag{1.25}$$

and $l_{;k}{}^i = \delta_k{}^i - \lambda^i l_k$. To find λ^i explicitly, we multiply these equations by l^k and sum over k, noting (1.21) and the fact that l^i is a unit vector. This yields $l^i = \lambda^i$ and hence

$$l^i_{;k} = \delta^i_k - l^i l_k. \tag{1.26}$$

A similar argument applied directly to the condition for a metric invariant differential yields $g_{ij|k} \equiv 0$, $g_{ij;k} \equiv \lambda_{ij}l_k$, for some λ_{ij}. In view of (1.21) we then find that the λ_{ij} must be zero. From (1.20) (a), (b), and (1.23) these conditions may be expressed in the form

$$\text{(1.27)}\qquad \begin{aligned} &\text{(a)}\quad \frac{1}{2}\frac{\partial g_{ij}}{\partial x^k} - A_{ij,r}L_0{}^r{}_k - L_{(ij)k} = 0 = g_{ij|k},\\ &\text{(b)}\quad A_{ij,k} - A_{ij,r}M_0{}^r{}_k - M_{(ij)k} = 0 = g_{ij;k}. \end{aligned}$$

The problem of determining a metric invariant differential therefore depends on the solution of equations (1.16) (a), (b) and (1.27) (a), (b) for the $L_j{}^i{}_k$ and the $M_j{}^i{}_k$ as functions of the g_{ij} and their derivatives. Thus we are led to the following questions: *To what extent, if at all, do these conditions determine* $L_j{}^i{}_k$ *and* $M_j{}^i{}_k$?

Moór has produced implicit solutions in **(7)**. His treatment of explicit solutions and of uniqueness is, however, rather cursory. Certain conclusions drawn about uniqueness seem to be invalidated by the interdependence of the defining equations.

In the following four sections we shall attempt to clarify the position as regards the uniqueness of the quantities $L_j{}^i{}_k$ and $M_j{}^i{}_k$ as well as to determine new explicit solutions.

2. A basic equation. Both of the equations (1.27) (a), (b) have the form

$$X_{(ij)k} = Y_{ij,k} - A_{ij,r}X_0{}^r{}_k, \tag{2.1}$$

where the X_{ijk} are the quantities to be determined and $Y_{ij,k}$, $A_{ij,k}$ are given quantities, symmetric in i and j, the latter of which satisfies $A_{ij,0} = 0$. We further assume that the unknowns X_{ijk} transform either as tensors or as connection parameters (cf. (1.15)). In either case we choose a quantity X^*_{ijk} with the same transformation law as X_{ijk} so that $\bar{X}_{ijk}$, defined by

$$X_{ijk} = X^*_{ijk} + \bar{X}_{ijk}, \tag{2.2}$$

is a tensor. Substitution of (2.2) in (2.1) yields

$$\bar{X}_{(ij)k} = Z_{ij,k} - A_{ij,r}\bar{X}_0{}^r{}_k, \tag{2.3}$$

where $Z_{ij,k}$ is a tensor symmetric in i and j:

$$Z_{ij,k} \equiv Y_{ij,k} - X^*_{(ij)k} - A_{ij,r}X^*_0{}^r{}_k. \tag{2.4}$$

If we now denote the part of $\bar{X}_{ijk}$ which is skew-symmetric in i and j by ξ_{ijk}, we have

$$\xi_{ijk} = \tfrac{1}{2}(\bar{X}_{ijk} - \bar{X}_{jik}) \tag{2.5}$$

and it follows from (2.3) that

$$\bar{X}_{ijk} = Z_{ij,k} - A_{ij,r}\bar{X}_0{}^r{}_k + \xi_{ijk}. \tag{2.6}$$

In order to deal with these equations, we shall find it convenient to introduce a set of mutually orthogonal unit vectors spanning the contravariant tangent space at a point $P(x)$ of our n-dimensional space. Two vectors X^i, Y^i defined with respect to the line-element $(x^k, \dot{x}^k)$ are called orthogonal with respect to this line-element if

$$g_{ij}(x, \dot{x})X^iY^j \equiv X_jY^j = 0.$$

Thus if we denote the basis vectors by $l_{(\mu)}{}^i$ $(\mu = 1, 2, \ldots, n)$, it follows that

$$l^i_{(\mu)}l_{(\gamma)i} = \delta_{\mu\gamma}. \tag{2.7}$$

Arbitrary tensors may be expressed as a linear combination of products of these vectors. For example, we have (repeated indices denote summation)

$$\delta^i_j = l^i_{(\mu)}l_{(\mu)j}, \qquad g_{ij} = l_{(\mu)i}l_{(\mu)j}, \tag{2.8}$$

by the definition of the Kronecker delta tensor $\delta_j{}^i$, and (2.7). In particular, the tensor $\bar{X}_{ijk}$ may be written as $\bar{X}_{ijk} = x_{\lambda\mu\nu}l_{(\lambda)i}l_{(\mu)j}l_{(\nu)k}$, where $x_{\lambda\mu\nu}$ are a set of n^3 scalars given by $x_{\lambda\mu\nu} = \bar{X}_{ijk}l_{(\lambda)}{}^il_{(\mu)}{}^jl_{(\nu)}{}^k$.

We will assume that the vector $l_{(1)}{}^i$ coincides with the vector defined by (1.5.)

It is also useful to introduce a basis for second order skew-symmetric covariant tensors. This basis has elements

$$\epsilon_{(\theta)ij} = l_{(\rho)i}l_{(\sigma)j} - l_{(\sigma)i}l_{(\rho)j} \qquad (\theta = 2, 3, \ldots, \tfrac{1}{2}n(n-1) + 1 \equiv N), \tag{2.9}$$

where ρ, σ are related by θ to the formula

$$\theta = n(\rho - 1) + \sigma - \tfrac{1}{2}(\rho + 2)(\rho - 1).$$

From this and the relations (2.7) we conclude that

$$\epsilon_{(\theta)\,ij}l_{(1)}{}^i = \begin{cases} l_{(\theta)\,j}, & (\theta = 2, 3, \ldots, n) \\ 0, & (\theta = n+1, \ldots, N). \end{cases} \tag{2.10}$$

If T_{ij} is any skew-symmetric tensor, its components $t_{\rho\sigma}$ satisfy $t_{(\rho\sigma)} = 0$ and hence it may be written $t_{(\theta)}\epsilon_{(\theta)\,ij}$, by (2.9).

Returning now to (2.6) we note that ξ_{ijk} is skew-symmetric in i and j, by (2.5), and therefore there exist tensors $a_{(\mu)\,ij}$ such that $\xi_{ijk} = a_{(\mu)\,ij}l_{(\mu)k}$, $a_{(\mu)\,ij} + a_{(\mu)\,ji} = 0$. Accordingly $\xi_{ijk} = b^\theta{}_\mu\epsilon_{(\theta)\,ij}l_{(\mu)k}$, where the $b_\mu{}^\theta$ are the scalar coefficients of $\epsilon_{(\theta)\,ij}$ in the expansion of $a_{(\mu)\,ij}$. If we substitute this expression for ξ_{ijk} in (2.6), multiply by $l_{(1)}{}^i$, sum over i and apply (2.10), we obtain $\bar{X}_0{}^r{}_k(g_{rj} + A_{0j,r}) - Z_{0j,k} = b_\mu{}^\theta l_{(\theta)\,j}l_{(\mu)k}$ $(\theta = 2, \ldots, n)$. This equation may be used to determine the scalars $b^\theta{}_\mu$ $(\mu = 1, 2, \ldots, n, \theta = 2, 3, \ldots, n)$ in terms of the $\bar{X}_0{}^r{}_k$. Having done so we find

$$b^\theta_\mu\epsilon_{(\theta)\,ij}l_{(\mu)k} = l_i[\bar{X}_0{}^r{}_k(g_{rj} + A_{0j,r}) - Z_{0j,k}] - l_j[\bar{X}_0{}^r{}_k(g_{ri} + A_{0i,r}) - Z_{0i,k}]. \tag{2.11}$$

A combination of equations (2.2), (2.6), and (2.11) then yields

$$X_{ijk} = \overset{*}{X}_{ijk} + (Z_{ij,k} - l_iZ_{0j,k} + l_jZ_{0i,k}) - T^r_k(A_{ij,r} - l_iA_{0j,r} + l_jA_{0i,r} - l_ig_{rj} + l_jg_{ri}) + \sigma_{ijk} \tag{2.12}$$

where we have set

$$\bar{X}_0{}^r{}_k = T^r_k,\ \sigma_{ijk} = b^\theta_\mu\epsilon_{(\theta)\,ij}l_{(\mu)k} \qquad (\theta = n+1, \ldots, N). \tag{2.12$'$}$$

Hence we have shown that any solution of equation (2.1) must be of the form (2.12). Conversely one may readily verify that *any tensor of the form* (2.12) *satisfies equation* (2.1) *for arbitrary* $T_k{}^r$, *provided only that* σ_{ijk} *satisfies*

$$\sigma_{ijk} + \sigma_{jik} = 0, \qquad \sigma_{0jk} = 0. \tag{2.13}$$

The most general tensor which satisfies these conditions is given by the second part of (2.12$'$), where the scalars $b_\mu{}^\theta$ are arbitrary $(\theta = n+1, \ldots, N, \mu = 1, 2, \ldots, n)$. Since it requires n^2 scalars to form the arbitrary tensor $T_k{}^r$, it would appear that the most general solution of equations (2.1) contains $n^2 + n(N - n)$ scalars. However, not all of the components of $T_k{}^r$ contribute to X_{ijk}. To prove this consider the expression $l_{(\mu)}{}^rl_{(\nu)k}(A_{ij,r} - l_iA_{0j,r} + l_jA_{0i,r} - l_ig_{rj} + l_jg_{ri})$ which would occur in the third term of the right member of (2.12) if $T_k{}^r$ were expanded in terms of the basis vectors. It is obvious that when $\mu = 1$, this is zero, since $A_{ij,0}$ vanishes. If it were to vanish for $\mu \neq 1$, we would find, on multiplication by $l^il_{(\nu)}{}^k$ and summation over i, k, that $0 = l_j(A_{00,r}l_{(\mu)}{}^r) - l_{(\mu)\,j}$. Since the basis vector $l_{(\mu)\,j}$ is not co-directional with l_j this is impossible. Thus the coefficient of l^r in the expansion of $T_k{}^r$ contributes nothing to X_{ijk} and we may assume that

$$T_{0k} = 0. \tag{2.14}$$

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Hence the number of unspecified scalars in a solution of (2.1) is

$$N_0 = n(n-1) + n[\tfrac{1}{2}n(n-1) + 1 - n] = \tfrac{1}{2}n^2(n-1) \tag{2.15}$$

and we can, in general, impose this number of new conditions on such a solution.

It should be emphasized, however, that, in specifying a solution of (2.1) by means of (2.12), we may choose the tensor $T_k{}^r$ arbitrarily. This situation is clearly advantageous from a practical standpoint and it would be desirable to have the solution (2.12) expressed entirely in terms of arbitrary tensors. In particular, it would be rather difficult to specify the tensor σ_{ijk} by means of the latter relation of (2.12′) since the basis vectors would first have to be constructed. This difficulty may be avoided by the use of the following lemma.

LEMMA 1. *Suppose that* $\mathfrak{F}(x, \dot{x})$ *is any scalar* $\in H_1$ *and suppose that the matrix* $((B_{ij}))$, *where*

$$B_{ij} \equiv \mathfrak{F}\frac{\partial^2 \mathfrak{F}}{\partial \dot{x}^i \partial \dot{x}^j} \equiv \mathfrak{F}\mathfrak{F}_{\dot{x}^i\dot{x}^j}, \tag{2.16}$$

is of rank $n-1$. *Then, a tensor* $\sigma_{ijk} \in H_0$ *satisfies* (2.13), *if and only if it has the form*

$$\sigma_{ijk} = (B_{ir}B_{js} - B_{is}B_{jr})S_k^{rs}. \tag{2.17}$$

Proof. That the right-hand side of (2.17) satisfies conditions (2.13) is evident from the homogeneity of $\mathfrak{F}$. To prove the converse we may assume that $S_k{}^{rs} + S_k{}^{sr} = 0$, $S_k{}^{0s} = 0$, since these parts of the tensor $S_k{}^{rs}$ will contribute nothing to the sum. But then (2.17) reduces to

$$\tfrac{1}{2}\sigma_{ijk} = B_{ir}B_{js}S_k^{rs}. \tag{2.17′}$$

These relations are solvable for $S_k{}^{rs}$. Indeed let us assume that $b_{\alpha\beta}$ are the scalar components of the tensor B_{ij} under expansion in terms of the basis, that is, $B_{ij} = b_{\alpha\beta}l_{(\alpha)i}l_{(\beta)j}$. In view of the homogeneity of $\mathfrak{F}$, we have $B_{0j} = b_{1\beta}$ $l_{(\beta)j} = 0$ and, since the basis vectors are linearly independent, it follows that the $b_{1\beta}$ all vanish. By the symmetry of B_{ij}, a similar result holds for $b_{\alpha 1}$. Furthermore, the matrix $((l_{(\alpha)i}))$ is non-singular. In fact, from the second part of equations (2.8), we have

$$g = |l_{(\mu)i}|^2 \tag{2.18}$$

and, in the light of (1.4), the left-hand side here does not vanish. Consequently, the matrix $((b_{\alpha\beta}))$ has the same rank as the matrix $((B_{ij}))$, namely $n-1$. Since the entries in the first row and column of $((b_{\alpha\beta}))$ are all zero, it follows that the cofactor of b_{11} is non-singular and therefore, considered as a matrix, it has an inverse, say $((\bar{b}_{\rho\sigma}))$ $(\rho, \sigma = 2, 3, \ldots, n)$. A new tensor $\bar{B}^{ij}$ may now be defined by $\bar{B}^{ij} \equiv \bar{b}_{\rho\sigma}l_{(\rho)}{}^i l_{(\sigma)}{}^j$. From the definition of the $\bar{b}_{\rho\sigma}$ and (2.7) it then follows that

$$B_{ij}\bar{B}^{jk} = \delta_i^k - l^k l_i. \tag{2.19}$$

With these remarks in mind (as well as the conditions (2.13)), we find that the solution of equations (2.17′) is $S_k^{hm} = \frac{1}{2}\sigma_{ijk}\bar{B}^{hi}\bar{B}^{mj}$, which proves the lemma.

In order that the rank of $((B_{ij}))$ be $n-1$ it is sufficient that the matrix of second derivatives of $\mathfrak{F}^2$ be non-singular. For it can be shown (**12**, pp. 8–9) that

$$(2.20)\qquad |(\mathfrak{F}^2)_{\dot{x}^i\dot{x}^j}| = -2^n\mathfrak{F}^{n-1}\begin{vmatrix} \mathfrak{F}_{\dot{x}^i\dot{x}^j} & \mathfrak{F}_{\dot{x}^i} \\ \mathfrak{F}_{\dot{x}^j} & 0 \end{vmatrix},$$

where the determinant on the right-hand side has $(n+1)^2$ entries. Now, if $((B_{ij}))$ has rank less than $n-1$, then there exists a non-trivial solution X^i of the corresponding homogeneous linear equations other than $\lambda\dot{x}^i$. Therefore there must exist a solution say z^i, such that

$$z^i\mathfrak{F}_{\dot{x}^i} = 0,$$

as well. But then the $(n+1)^2$ determinant of (2.20) vanishes which is a contradiction.

In the light of (1.2) and (1.4′) it follows that the function $F(x, \dot{x})$ is a suitable choice for $\mathfrak{F}$. For future reference we note that the associated space whose metric tensor is given by γ_{ij} is of the Finsler type and we shall refer to it in the sequel as the *associated Finsler space.**

On application of Lemma 1 it follows that

THEOREM 1. *Any solution of equations* (2.1) *can be expressed in the form*

$$(2.21)\quad \begin{aligned} X_{ijk} = \overset{*}{X}_{ijk} &+ (Z_{ij,k} - l_iZ_{0j,k} + l_jZ_{0i,k}) - \\ &- T_k^r(A_{ij,r} - l_iA_{0j,r} + l_jA_{0i,r} - l_ig_{jr} + l_jg_{ir}) + \\ &+ (B_{ir}B_{js} - B_{jr}B_{is})S_k^{rs}, \end{aligned}$$

and, conversely, any expression of this form, where $T_k{}^r$ and $S_k{}^{rs}$ are completely arbitrary tensors, is a solution of (2.1).

If, in addition to conditions (2.1), we demand that the X_{ijk} be symmetric in i and k we obtain $\frac{1}{2}n^2(n-1)$ new conditions. In view of (2.15), then, the solution would appear to be unique. We shall obtain necessary and sufficient conditions for the uniqueness of a symmetric solution. Equations (2.1), together with

$$(2.22)\qquad X_{ijk} = X_{kji},$$

may be explicitly solved for X_{ijk} in terms of $X_0{}^r{}_i$:

$$(2.23)\quad X_{ijk} = (Y_{ij,k} + Y_{kj,i} - Y_{ik,j}) - (A_{ij,r}X_0{}^r{}_k + A_{kj,r}X_0{}^r{}_i - A_{ik,r}X_0{}^r{}_j).$$

If we multiply this equation by l^i, sum over i, and collect the coefficients of X_{0rs}, we find

$$(2.23')\quad X_{0rs}(\delta_j^r\delta_k^s + A_{0j,}{}^r\delta_k^s + A_{jk,}{}^rl^s - A_{0k,}{}^r\delta_j^s) = (Y_{0j,k} + Y_{kj,0} - Y_{0k,j}).$$

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*See (**12**, chapter I) for the set of conditions defining a Finsler space.

Adopting the notation of Moór, we call the tensor in the left member of (2.23′) $H_{jk}{}^{rs}$ and state

THEOREM 2. *In order that equations* (2.1) *and* (2.22) *should define a unique solution* X_{ijk}, *it is necessary and sufficient that there exists a tensor* $K_{mn}{}^{jk}$ *such that*

$$H^{rs}_{jk}K^{jk}_{mn} = \delta^r_m\delta^s_n. \tag{2.24}$$

This result follows directly from equations (2.23′), for, under the stated condition, these equations can be solved for X_{0rs} which, in view of (2.23), yields X_{ijk} uniquely.*

3. Determination of $L_j{}^i{}_k$. We are now in a position to determine the parameters $L_j{}^i{}_k$ (and hence L_{ijk}) since the defining equations (1.27) (a) are of the form (2.1) with $Y_{ij,k} = \frac{1}{2}\partial g_{ij}/\partial x^k$ and $A_{ij,k}$ given by (1.23). The right-hand side of equation (2.21) then yields L_{ijk} in its most general form. A more precise specification involves the choice of a suitable X^*_{ijk} and this choice should be governed by the particular problem under consideration.

Problems in which the extremals of $F(x, \dot{x})$ occur suggest the use of the Berwald connection parameters $G_j{}^i{}_k$ which are defined by†

$$G_j{}^i{}_k = \frac{\partial^2 G^i}{\partial \dot{x}^j \partial \dot{x}^k}, \; G^i = \frac{1}{4}\gamma^{ih}\left(\frac{\partial^2 F^2}{\partial \dot{x}^h \partial x^m}\dot{x}^m - \frac{\partial F^2}{\partial x^h}\right). \tag{3.1}$$

The extremals of $F(x, \dot{x})$ are the solutions of the differential equations

$$\frac{d^2x^i}{ds^2} + 2G^i(x, \dot{x}) = 0. \tag{3.2}$$

Having formed the $G_j{}^i{}_k$ we may choose $X_{ijk}{}^*$ to be $g_{jh}G_i{}^h{}_k$.

The Cartan connection parameters $\Gamma_j{}^i{}_k$ for a Finsler space (**3**, equation (2.8)) are defined by

$$\Gamma_j{}^i{}_k = \tfrac{1}{2}\gamma^{ih}(\gamma_{jh\{k\}} + \gamma_{kh\{j\}} - \gamma_{jk\{h\}}), \tag{3.3}$$

where, for arbitrary ϕ,

$$\phi_{\{h\}} = \frac{\partial\phi}{\partial x^h} - \frac{\partial\phi}{\partial\dot{x}^r}\frac{\partial G^r}{\partial\dot{x}^h}.^{**} \tag{3.3′}$$

If a comparison with the associated Finsler space is desired, then we may use $g_{jh}\Gamma_i{}^h{}_k$ for the $X_{ijk}{}^*$.

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*Cf. (**7**, pp. 92–93). Moór does not appear to have realized that the assumption of the existence of $K_{mn}{}^{jk}$ insures the uniqueness of a symmetric $L_j{}^i{}_k$.

†See, for example, (**3**, equation (2.1) *et seq.*). The equations (3.2) above are equivalent to the Euler-Lagrange equations, namely,

$$\frac{d}{ds}(\partial F/\partial\dot{x}^i) - \frac{\partial F}{\partial x^i} = 0.$$

**This type of derivative was introduced by Berwald. It is obtained as the coefficient of $l^i = dx^i/ds$ in the derivative of ϕ with respect to the arc-length of the extremal through the line-element $(x, \dot{x})$.

On the assumption that the tensor $K_{mn}{}^{jk}$ of equation (2.24) exists, we may obtain a third choice for X^*_{jjk}, namely, the result of solving (2.23′) for X_{0rs} and substituting in (2.23). (Note that here

$$(3.4)\qquad Y_{ij,k} + Y_{jk,i} - Y_{ik,j} = \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ik}}{\partial x^j}\right) \equiv [ijk],$$

the Christoffel symbol of the first kind.) Following Moór (**7**, p. 92,) we denote this quantity by Γ^*_{ijk}. It has the required transformation law, as may be seen by a somewhat lengthy calculation based on the transformation law of the Christoffel symbols. It is noteworthy that not all solutions of (1.27) (a) do possess the proper transformation properties, a fact which is exemplified by the solution $[ijk] - l_i[0jk] + l_j[0ik]$.

For a general discussion it is most convenient to choose $X^*_{ijk} = g_{jh}\Gamma_i{}^h{}_k$ since the Cartan parameters have well-known properties and are capable of explicit expression in terms of the g_{ij} by means of (1.1), (1.2), and (3.3). Condition (c) of § 1 shows that γ^{ij} (the inverse tensor of γ_{ij}) is well defined. We have adopted this assumption in preference to the existence of $K_{hk}{}^{ij}$ in view of its importance for the associated Finsler space.

Having chosen X^*_{ijk}, we see from (2.4) that $Z_{ij,k}$ represents one-half of the covariant derivative of g_{ij} with respect to the connection parameters $X^*{}_i{}^j{}_k$. For example, if $X^*{}_j{}^i{}_k = \Gamma_j{}^i{}_k$, we have

$$2Z_{ij,k} = \frac{\partial g_{ij}}{\partial x^k} - \frac{\partial}{\partial \dot{x}^r}(g_{ij})\Gamma^r_{sk}\dot{x}^s - g_{rj}\Gamma^r_{ik} - g_{ir}\Gamma^r_{jk} \equiv g_{ij[k},$$

the Cartan covariant derivative of g_{ij} (**12**, p. 70). Also we note that if $X^*{}_i{}^j{}_k = \Gamma^*{}_i{}^j{}_k$, then $Z_{ij,k}$ vanishes identically.

The discussion following Lemma 1 of § 2 indicates that a suitable choice for B_{ij} will be given by (2.16) with $\mathfrak{F} \equiv F$. In view of these remarks and the identity

$$(3.5)\qquad l^i_{[j} = 0,^*$$

we see that equation (2.21) can be written

$$(3.6)\quad L_{ijk} = g_{jh}\Gamma^h_{ik} + (\tfrac{1}{2}g_{ij[k} - A_{ij,r}T^r_k) - l_i[\tfrac{1}{2}l_{j[k} - (g_{jr} + A_{0j,r})T^r{}_k]$$
$$+ l_j[\tfrac{1}{2}l_{i[k} - (g_{ir} + A_{0i,r})T^r_k] + (B_{ir}B_{js} - B_{jr}B_{is})S_k^{rs}.$$

This is the most general connection parameter for which the Ricci lemma (1.27) (a) *holds.* It follows from (2.14) and (3.5) that

$$(3.6')\qquad L_0{}^j{}_k = \Gamma^j_0{}_k + T^r{}_k(\delta^j_r - l^jA_{00,r}).$$

Particular cases are immediately obtainable by specializing the (arbitrary) tensors $T_k{}^r$ and $S_k{}^{rs}$. The simplest example is

$$(3.7)\qquad L_{ijk} = g_{jh}\Gamma^h_{ik} + \tfrac{1}{2}(g_{ij[k} - l_il_{j[k} + l_jl_{i[k}),$$

*See, for example, (**12**, p. 74, equation (2.12)). Compare also with (1.29) in the present work.

where both $T_k{}^r$ and $S_k{}^{rs}$ have been taken to be zero. Although this expression differs from Γ_{ijk} in that it is not, in general, symmetric in i and k, it does have the property that $L_0{}^j{}_k = \Gamma_0{}^j{}_k$. Thus we expect that a theory based on this connection would bear some resemblance to the theory of Finsler spaces.

Another simple solution may be obtained when the matrix $((g_{ij} + A_{0i,j}))$ is non-singular, for then there exists a tensor J^{ki} satisfying

$$(3.8) \qquad (g_{ij} + A_{0i,j}) J^{jk} = \delta_i^k,^*$$

and hence, choosing $T_k{}^r = \frac{1}{2} J^{rs} l_{s[k}$, $S_k{}^{rs} = 0$, we find that equation (3.6) becomes

$$(3.9) \qquad L_{ijk} = g_{jh}\Gamma^h_{ik} + \tfrac{1}{2}(g_{ij[k} - A_{ij,r}J^{rs} l_{s[k}).$$

It follows from (3.6′) and (3.8) that $L_0{}^j{}_k = \Gamma_0{}^j{}_k + \frac{1}{2}J^{jr}l_{r[k}$. This relation is slightly more complicated than the previous one but it is still manageable.

If we assume the existence of $K_{hk}{}^{ij}$ then, as was previously remarked, when $X^*{}_{ijk}$ is set equal to $\Gamma^*{}_{ijk}$, the $Z_{ij,k}$ vanish and equation (2.21) becomes

$$(3.10) \quad L_{ijk} = \overset{*}{\Gamma}_{ijk} - (A_{ij,k} - l_i A_{0j,r} + l_j A_{0i,r} - l_i g_{jr} + l_j g_{ir}) T^r_k + \\ + (B_{ir}B_{js} - B_{is}B_{jr}) S^{rs}_k.$$

Particular solutions similar to those described above may again be found from this form for L_{ijk}.

The advantage of solutions of the form exhibited in equation (3.6) over solutions in the form (3.10) lies in the fact that the former are determinable explicitly from the g_{ij} and its derivatives while the latter are obtained implicitly by the inversion of matrices.

4. Determination of $M_j{}^i{}_k$. In view of §3 it is only necessary to specify M_{ijk} to complete the evaluation of the invariant differential (1.12). This tensor must satisfy equations (1.16) and (1.27) (b) which we now proceed to solve. The latter equations may be solved by the theory developed in § 2 if X_{ijk}, $Y_{ij,k}$ and $A_{ij,k}$ of (2.1) are interpreted as M_{ijk}, $A_{ij,k}$, and $A_{ij,k}$ respectively. As in § 3 we must now choose $X^*{}_{ijk}$. In this case, however, $X^*{}_{ijk}$ is a tensor and so may be taken to be zero. We then find that $Z_{ij,k}$, as given by (2.4), is $A_{ij,k}$ and hence, by (2.21), *the most general solution of* (1.27) (b) *may be written*

$$(4.1) \quad M_{ijk} = (A_{ij,r} - l_i A_{0j,r} + l_j A_{0i,r})(\delta^r_k - T^r_k) + l_i T_{jk} - l_j T_{ik} + \sigma_{ijk},$$

where σ_{ijk} is defined by (2.13) with the same value for B_{ir} as in § 3.

The tensors T_{jk} and $S_k{}^{rs}$ may be chosen in an arbitrary manner, the simplest choice being zero for both:

$$(4.2) \qquad M_{ijk} = A_{ij,k} - l_i A_{0j,k} + l_j A_{0i,k}.$$

*An equivalent tensor $J_k{}^i$ was introduced by Moór (7, §2, equation (2.15)), but for an entirely different purpose. In fact, this tensor occurs in Moór's solution for the connection parameters.

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Other tensors may of course be used for $X^*{}_{ijk}$ in (2.21) but in view of the relative simplicity of (4.1) we shall adopt the above choice in the sequel.

We now consider the effect of superimposing conditions (1.16). Note that (1.27) (b) implies that $A_{ij,k} = M_{(ij)k} + (M_{(ij)r} + A_{ij,s}M_0{}^s{}_r)M_0{}^r{}_k$, the last term of which vanishes, by (1.16) (b). Inner multiplication with the tensor l^il^j then yields

$$A_{00,k} = M_{00k}, \tag{4.3}$$

and hence $A_{00,}{}^rM_0{}^r{}_k = 0$. Now the most general solution of (1.27) (b) has the form (4.1), where T_{ij} is arbitrary, while σ_{ijk} satisfies (2.13). Condition (1.16) (a) is therefore equivalent to

$$-(A_{ij,r} - l_iA_{0j,r} + l_jA_{0i,r})T_0^r + l_iT_{j0} - l_jT_{i0} + \sigma_{ij0} = 0, \tag{4.4}$$

by virtue of (1.23′). If we multiply this by l^i, summing over i, and then repeat the process with l^j, we obtain successively $-l_jA_{00,r}T_0{}^r + T_{j0} - l_jT_{00} = 0$, $-A_{00,r}T_0{}^r = 0$, in view of (2.13). Substitution from the latter relation in the former then yields

$$T_{j0} = l_jT_{00}, \tag{4.5}$$

and hence, by (1.23′), equation (4.4) reduces to

$$\sigma_{ij0} = 0. \tag{4.6}$$

Turning again to (4.1), we multiply by l^i and sum, thus deriving

$$M_{0jk} = l_jA_{00,r}(\delta_k^r - T_k^r) + T_{jk} - l_jT_{0k}, \tag{4.7}$$

again by (2.13). Consequently, we have $M_{00k} = A_{00,r}(\delta_k{}^r - T_k{}^r) = A_{00,k}$, by (4.3). It therefore follows that

$$A_{00,r}T_k^r = 0. \tag{4.8}$$

Combining (4.7) and (4.8) we find $M_{0jk} = l_j(A_{00,k} - T_{0k}) + T_{jk}$, to which we apply conditions (1.16) (b). In the light of (4.5) and (4.8), this leads to

$$(T_j^i - l^iT_{0j})T_k^j = 0. \tag{4.9}$$

The above results yield

THEOREM 1. *In order that a tensor M_{ijk} satisfy conditions* (1.27) (b) *and* (1.16), *it is necessary and sufficient that M_{ijk} be given by* (4.1) *where T_{ij} is a tensor satisfying* (4.5), (4.8), *and* (4.9) *while σ_{ijk} is a tensor satisfying* (2.13) *and* (4.6).

Proof. We have just seen the necessity of these conditions. Their sufficiency follows by direct substitution.

The problem of determining the connection parameters M_{ijk} is therefore reduced to the problem of representing tensors T_{ij} and σ_{ijk} which satisfy the above conditions. This problem, as regards σ_{ijk}, is completely resolved by

LEMMA 1. *The class of tensors σ_{ijk} satisfying* (2.13) *and* (4.6) *is identical to the class of tensors of the form*

$$(B_{ir}B_{js} - B_{jr}B_{is})V^{rst}B_{tk},$$

where V^{rst} is arbitrary and B_{ij} is any tensor which fulfils the conditions of Lemma 1, § 2.

Proof. It is evident from the homogeneity properties of such B_{ij} that tensors of the above form satisfy equations (2.13) and (4.6). Conversely if a tensor σ_{ijk} is given, subject to these conditions, there exists a corresponding tensor V^{rst}, namely $\frac{1}{2}\bar{B}^{ri}\bar{B}^{sj}\sigma_{ijk}\bar{B}^{kt}$, as may be verified using (2.19).

Since the tensor σ_{ijk} of Theorem 1 is determined by choosing V^{rst} arbitrarily, we need only find a tensor T_{ij}, satisfying the conditions of Theorem 1, to complete the specification of M_{ijk}. The non-linearity of equation (4.9) indicates that a simple representation of T_{ij} in terms of arbitrary tensors would be quite difficult to construct. Special solutions are, however, easily derived. The simplest of these is, of course, when T_{ij} is taken to be zero. A second choice would be $T_{ij} = \lambda_i A_{00,j}$, where λ_i is any vector, other than l_i, orthogonal to $A_{00,j}$.

It is worthy of note that, although $l_i V_j$ (arbitrary V_j) is a suitable T_{ij}, it gives the same value to M_{ijk} as the zero solution. Hence, in order to obtain distinct values for M_{ijk}, we may assume (as in § 2) that equation (2.14) holds. In the light of this remark the conditions on T_{ij} become

$$T_{i0} = 0,\ T_{j0} = 0,\ A_{00,r}T^r_k = 0,\ T^i_j T^j_k = 0. \tag{4.10}$$

In summary, then, the work of this and the preceding section allows us to express the most general metric invariant differential in terms of explicitly defined quantities.

5. Uniqueness theorems.

The results of §§ 2, 3, and 4 lead to several theorems concerning the uniqueness of the connection parameters of a metric invariant differential.

THEOREM 1. *There are $\frac{1}{2}n^2(n - 1)$ unspecified scalars in the most general metric connection L_{ijk} of an n-dimensional space. This will be uniquely defined by the further condition of symmetry in i and j if and only if the tensor $K_{hk}{}^{ij}$ of equation* (2.24) *exists.*

Proof. These results are immediate consequences of equation (2.15) and Theorem 2, § 2, as applied to condition (1.27) (a).

Before proceeding further, let us note that, by (1.1) and (1.2), we have

$$\gamma_{ij} = g_{ij} + \dot{x}^r \frac{\partial}{\partial \dot{x}^j} g_{ir} + \dot{x}^r \frac{\partial}{\partial \dot{x}^i} g_{jr} + \frac{1}{2}\dot{x}^r\dot{x}^s \frac{\partial^2 g_{rs}}{\partial \dot{x}^i \partial \dot{x}^j}. \tag{5.1}$$

Hence, using (1.3′) and (1.23) we obtain

$$\gamma_{ij}l^j = F_{\dot{x}^i} = l_i + A_{00,i}. \tag{5.2}$$

It is also readily deduced from (5.1) that a necessary and sufficient condition that the space be a Finsler space ($g_{ij} \equiv \gamma_{ij}$) is

$$A_{0i,j} = 0.^* \tag{5.3}$$

THEOREM 2. *The connection defined by equation* (3.3) *is the only metric connection in a Finsler space which is symmetric in j and k.*

Proof. According to (5.3), the tensor $H_{hk}{}^{ij}$ of § 2 reduces to $\delta_h{}^i\delta_k{}^j + A_{hk,}{}^i l^j$ in a Finsler space. From (2.24), then, the tensor $K_{rs}{}^{hk}$ exists and is given by

$$K^{hk}_{rs} = \delta^h_r\delta^k_s - A_{rs,}{}^h l^k. \tag{5.4}$$

Using this form for $K_{rs}{}^{hk}$ to solve (2.23′) and making the appropriate replacements in the right-hand side of (2.23) (see (3.4)), we find, after some calculation, that the resultant quantity (with middle index raised) coincides with the right-hand side of (3.3).

Uniqueness theorems for M_{ijk} present a great deal more difficulty than the corresponding theorems for L_{ijk} since the auxiliary conditions (1.16) must be taken into account. It is clear, for example, that a symmetry condition together with (1.27) (b) may completely determine an M_{ijk} for which (1.16) is not fulfilled. It is possible, however, to obtain several special results before considering the above problem in full detail.

One such result is embodied in

THEOREM 3. *In a Finsler space, the only solution M_{ijk} of equations* (1.16) *and* (1.27) (b), *which is either* (a) *symmetric in i and k or* (b) *symmetric in j and k, is $A_{ij,k}$.*

Proof. In a Finsler space the tensor $K_{hk}{}^{ij}$ exists and is given by (5.4). Hence, when assumption (a) is applied it follows from Theorem 2 of § 2 that there is an unique solution M_{ijk}. The identity (5.3) shows that $A_{ij,k}$ is this solution.

To prove the second part, we consider the explicit form of M_{ijk} in a Finsler space, which, by (4.1) and (5.3) is

$$M_{ijk} = A_{ij,r}(\delta^r_k - T^r_k) + l_iT_{jk} - l_jT_{ik} + \sigma_{ijk}. \tag{5.5}$$

Condition (b) then yields

$$-A_{ij,r}T^r_k + l_iT_{jk} - l_jT_{ik} + \sigma_{ijk} = -A_{ik,r}T^r_j + l_iT_{kj} - l_kT_{ij} + \sigma_{ijk}. \tag{5.6}$$

Multiplying this by l^i and summing over i, we deduce that $T_{jk} = T_{kj}$, by virtue of (5.3), (4.10), and (2.13). After substituting this in (5.6) we multiply by l^j and sum, thus obtaining $-T_{ik} = \sigma_{ik0} = 0$, by (4.6). We substitute again in (5.6) and find that σ_{ijk} is symmetric in j and k. But this result together with (2.13) implies that σ_{ijk} vanishes. Finally, then, (5.5) reduces to $M_{ijk} = A_{ij,k}$ which completes the proof.

We return now to the question of the uniqueness of M_{ijk} as specified by the original conditions (1.16) and (1.27) (b) (or, equivalently, by (4.1) subject to (2.13), (4.6), and (4.10)). If we expand σ_{ijk} in terms of its components

*For a complete proof of this see **7**, §4.

with respect to the basis introduced in § 2 and apply (2.13) and (4.6) we find that it has the form $c^{\theta}{}_{\mu}\epsilon_{(\theta)ij}l_{(\mu)k}$ $(\mu = 2, \ldots, n, \theta = n+1, \ldots, N = \frac{1}{2}n(n+1)+1)$. Thus the number of undetermined components of σ_{ijk} is

$$\bar{N} = \tfrac{1}{2}(n-1)^2(n-2). \tag{5.7}$$

The corresponding number for T_{ij} may be found with the help of

LEMMA 1. *If A is an $m \times m$ matrix whose square is the zero matrix and whose rank is p, then*

(i) $p \leqslant \frac{1}{2}m$,

and

(ii) *A is determined by $2p(m-p)$ of its entries.*

Proof. Part (i) follows from a theorem of matrix algebra (**1**, p. 87) to the effect that a nilpotent $m \times m$ matrix of index 2, rank p is similar to

$$\left(\begin{pmatrix} O_p & I_p, & O_{p,q} \\ O_p, & O_p, & O_{p,q} \\ O_{q,p}, & O_{q,p}, & O_{q,q} \end{pmatrix}\right) \quad q = m - 2p,$$

where I_p is the p-dimensional unit matrix and the other entries are all zero.

Turning to (ii) we remark that, since A has rank p, (**1**, p. 49) there exist non-singular $m \times m$ matrices P and Q such that

$$PAQ = \left(\begin{pmatrix} I_p, & O \\ O, & O \end{pmatrix}\right) \equiv E_p.$$

It follows that any matrix A of rank p may be obtained from E_p by choosing P and Q to be non-singular and forming PE_pQ. No loss in generality results from the assumption that the first p rows and columns of A form a non-singular matrix. Accordingly, if we let P and Q be given by

$$P = \left(\begin{pmatrix} P_{11}, & P_{12} \\ P_{21}, & P_{22} \end{pmatrix}\right), \quad Q = \left(\begin{pmatrix} Q_{11}, & Q_{12} \\ Q_{21}, & Q_{22} \end{pmatrix}\right),$$

where P_{11} and Q_{11} are $p \times p$ matrices, we find that the general matrix of rank p can be represented in the form

$$A = PE_pQ = \left(\begin{pmatrix} P_{11} \times Q_{11}, & P_{11} \times Q_{12} \\ P_{21} \times Q_{11}, & P_{21} \times Q_{12} \end{pmatrix}\right).^*$$

Hence the matrix A is determined by the three matrices α, β, and γ where $\alpha \equiv P_{11} \times Q_{11}$, $\beta \equiv P_{11} \times Q_{12}$, $\gamma \equiv P_{21} \times Q_{11}$, since the lower right-hand block in A is given by $P_{21} \times Q_{12} = \gamma Q_{11}{}^{-1} P_{11}{}^{-1} \beta = \gamma\alpha^{-1}\beta$, that is,

$$A = \left(\begin{pmatrix} \alpha, & \beta \\ \gamma, & \gamma\alpha^{-1}\beta \end{pmatrix}\right) \tag{5.8}$$

*This follows by the process of block multiplication for matrices, also explained in (**1**). I am indebted to E. Liberman for suggesting this approach to me.

where α, β, and γ are arbitrary apart from the condition that α be non-singular. It then follows that

$$A^2 = \left(\begin{pmatrix} (\alpha^2 + \beta\gamma) \quad , & (\alpha^2 + \beta\gamma)\alpha^{-1}\beta \\ \gamma\alpha^{-1}(\alpha^2 + \beta\gamma), & \gamma\alpha^{-1}(\alpha^2 + \beta\gamma)\alpha^{-1}\beta \end{pmatrix}\right).$$

Thus $A^2 = 0$ if and only if $\alpha^2 + \beta\gamma = 0$. This relation provides p equations for the $p(m - p)$ entries of γ. Accordingly any matrix A satisfying the conditions of the lemma is completely specified by α, β and the remaining entries of γ, that is, by $2p(m - p)$ of its entries. This proves the lemma.

Returning to the tensor T_{ij} occurring in (4.1) ,we expand in terms of the basis of § 2: $T_{ij} = t_{\lambda\mu} l_{(\lambda)\,i} l_{(\mu)\,j}$. In view of the first two parts of (4.10) all the scalars $t_{\lambda\mu}$ whose indices involve a 1 vanish. The last part of (4.10) implies $A^2 = 0$, where A is the matrix of the remaining $t_{\lambda\mu}$. Assuming that the rank of A is p, we apply the above lemma with $m = n - 1$. Consequently the rank p may not exceed $\frac{1}{2}(n - 1)$ and the number of components $t_{\lambda\mu}$ (λ, $\mu = 2, \ldots, n$) necessary to specify A is $2p(n - 1 - p)$. The third part of (4.10) provides 0 or p further conditions on the $t_{\lambda\mu}$ depending on whether or not the vector $A_{00,i}$ vanishes identically.

Combining these results with (5.7) we see that the total number of arbitrary components of M_{ijk} is

$$(5.9)\qquad N_1(p) = \tfrac{1}{2}(n - 1)^2(n - 2) + 2p(n - 1 - p) - p \quad (2p \leqslant n - 1),$$

if $A_{00,i}$ is non-zero, and

$$(5.9')\qquad N_2(p) = \tfrac{1}{2}(n - 1)^2(n - 2) + 2p(n - 1 - p) \qquad (2p \leqslant n - 1),$$

if $A_{00,i}$ vanishes. In either case p is the rank of the matrix A defined above and hence satisfies $0 \leqslant p \leqslant \frac{1}{2}(n - 1)$.

It is easily seen that both N_1 and N_2 are increasing functions for this range of p. Thus we have

THEOREM 4. *The most general M_{ijk} is obtained when the rank of $((T_{ij}))$ assumes its greatest value, namely, $\frac{1}{2}(n - 2)$ when n is even, and $\frac{1}{2}(n - 1)$, when n is odd. In both these cases*

$$(5.10)\qquad (N_1)_{\max} = \tfrac{1}{2}n(n - 1)(n - 2).$$

On the other hand,

$$(5.10')\qquad (N_2)_{\max} = \begin{cases} (k - 1)(4k^2 - 2k + 1) & (n = 2k) \\ 4k^3 & (n = 2k + 1)\,. \end{cases}$$

From this result it is clear that the solution is unique only if $n = 2$. In fact the unique solution is given by (4.2). In general, (5.9) and (5.9′) show that the order of magnitude of N_1 and N_2 is $\frac{1}{2}n^3$. For $(N_1)_{\max}$ we find the values 3, 12, and 30 when n is 3, 4, and 5, respectively, while the corresponding numbers for $(N_2)_{\max}$ are 4, 13, and 32.

6. Autoparallels and extremals. Consider the set of line-elements $(x, \dot{x})$ defined by a curve $C:x^i = x^i(s)$ and a vector field $\dot{x}^i(s)$ given along it. The parameter s of C is the arc-length of C with respect to these line-elements. Thus $ds^2 = g_{ij}(x, \dot{x})dx^i dx^j$ and $dx^i/ds \equiv x'^i$ is the unit tangent to C, while $l^i(x, \dot{x})$ is defined by (1.5).

DEFINITION. *A vector ξ^i is said to be transported by parallel displacement with respect to $\dot{x}^i$ along C if it satisfies*

$$(6.1)\qquad \frac{D\xi^i}{ds} \equiv D_s\xi^i \equiv \xi'^i + \xi^j\left[M_j{}^i{}_k(x,\dot{x})\,\frac{Dl^k}{ds} + L_j{}^i{}_k(x, \dot{x})x'^k\right] = 0,$$

where Dl^k is given by putting $X^i = l^i$ in (1.12) *and solving with the aid of* (1.16) (b):

$$(6.1')\qquad Dl^k = [\delta^k_r + M_0{}^k{}_r(x, \dot{x})][dl^r(x, \dot{x}) + L_0{}^r{}_s(x, \dot{x})dx^s].$$

Since (6.1) is a system of first-order linear differential equations for ξ^i, it is clear that parallel displacement yield a bi-unique map between tangent spaces at points of C.

Curves whose tangent vectors are transported by parallelism with respect to themselves are called autoparallels. Their equations therefore follow from (6.1), (6.1′) by identifying the three vector fields ξ^i, $\dot{x}^i$, and x'^i. Thus, denoting the unit vector x'^i by l^i, and using (1.16) (b), we have $D_s l^i = (l'^r + L_{00}{}^r)\,(\delta_r{}^i + M_0{}^i{}_r) = 0$ or, equivalently,

$$(6.2)\qquad l'^r + L_0{}^r{}_0 = x''^r + L_i{}^r{}_j(x, x')x'^i x^{ij} = 0.$$

It is of interest to compare the autoparallels, thus defined, with the geodesics of the space. These curves give extreme values to the integral $\int F(x, dx) = \int ds = \int [g_{ij}(x, dx/dt)dx^i dx^j]^{\frac{1}{2}}$ and hence are defined by the Euler-Lagrange equations:

$$(6.3)\qquad \frac{d}{ds}\,(F_{x'^i}) - \frac{\partial F}{\partial x^i} = 0.$$

It is possible to write these equations in a form suitable for comparison with (6.2).

First we remark that the covariant derivatives (1.18) may be formally applied to tensors which do not belong to H_0. For example,

$$(6.4)\qquad \dot{x}^i{}_{|j} = 0, \qquad \dot{x}^i{}_{;j} = F\delta^i_j,$$

by (1.5) and (1.19). Hence, putting $\dot{x}^i = Fl^i$ and using (1.25) and (1.26), we obtain expressions for $F_{|j}$ and $F_{;j}$:

$$(6.5)\quad (a)\quad F_{|j} \equiv \frac{\partial F}{\partial x^j} - F_{||r}L_0{}^r{}_j = 0,\ (b)\quad F_{;j} \equiv F_{||r}(\delta^r_j - M_0{}^r{}_j) = Fl_j.$$

Also it follows from (5.2) that

$$(6.6)\quad D_s(F_{\dot{x}^i}) \equiv \frac{d}{ds}\,(F_{\dot{x}^i}) - M_i{}^r{}_j F_{\dot{x}^r} D_s l^j - L_i{}^r{}_j F_{\dot{x}^r}\frac{dx^j}{ds} = D_s l_i + D_s A_{00,i}.$$

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In the identities (6.5) (a), (6.6) we replace $\dot{x}^i$ by the unit vector x'^i (henceforth written as l^i) and substitute in (6.3), using the fact that $Dg_{ij} = 0$:

$$(6.7) \qquad (g_{ij} + F_{\dot{x}^r}M_i{}^r{}_j)D_s l^j + D_s A_{00,i} + F_{\dot{x}^r}(L_i{}^r{}_0 - L_0{}^r{}_i) = 0.$$

Comparison of this equation for the geodesics with equation (6.3) for the autoparallels yields

THEOREM 1. (a) *If C_a is an autoparallel along which the conditions*

$$(6.8) \qquad D_s A_{00,i} + F_{\dot{x}^r}(L_i{}^r{}_0 - L_0{}^r{}_i) = 0$$

are satisfied, then C_a is also a geodesic.

(b) *If C_e is a geodesic along which the conditions* (6.8) *and* $|g_{ij} + F_{\dot{x}^r}M_i{}^r{}_j| \neq 0$ *are satisfied, then C_e is also an autoparallel. A theorem of Moór (incorrectly stated) follows from this result when $L_{ijk} = \Gamma^*{}_{ijk}$* **(7**, p. 101**)**.

THEOREM 2. *The class of autoparallels defined by a metric invariant differential coincides with the class of geodesics defined by a metric function $F(x, \dot{x})$ if and only if the tensor T_{ij} occurring in* (3.6) *satisfies*

$$(6.9) \qquad T_{i0} = 0.$$

Proof. Equations (6.3) are equivalent to (3.2) which in turn are equivalent to $x''^i + \Gamma_j{}^i{}_k(x, x')x'^j x'^k = 0$ **(12**, equation (3.1.26)**)**. Thus (6.3) coincides with (6.2), for all (x, x'), if and only if $\Gamma^i{}_{00} = L^i{}_{00}$. Calculating $L^i{}_{00}$ from (3.6′) we find that this condition reads $T^r{}_0(\delta_r{}^i - l^i A_{00,r}) = 0$ or, on inner multiplication with $\delta_i{}^j + l^j A_{00,i}$, $T^j{}_0 = 0$, which is the stated result.

THEOREM 3. *Conditions* (6.8) *and* (6.9) *are equivalent along an autoparallel.*

Proof. That (6.8) implies (6.9) follows from Theorem 1 and the analysis of Theorem 2. The following identities are used to prove the converse.

$$(6.10) \qquad F_{\dot{x}^j|i} = F_{\dot{x}^r}L_k{}^r{}_{i||j}l^k,$$

$$(6.11) \qquad F_{x^r}(L_0{}^r{}_i - \Gamma_0{}^r{}_i) = 0,$$

$$(6.12) \qquad l^i{}_{||j} = \delta^i_j - l^i F_{\dot{x}^j},$$

$$(6.13) \qquad \Gamma_i{}^r{}_{j||k}l^i l^j = \Gamma^r_{00||k} - 2\Gamma^r_{0k} + 2F_{\dot{x}^k}\Gamma_{00}{}^r = 0.$$

The first of these is obtained by differentiating (6.5) with respect to $\dot{x}^i$ and rearranging terms. The second follows from (3.6′), (5.2), and (2.14). The third is a simple consequence of (1.5) and the last is a well-known identity of Finsler geometry **(12**, p. 63**)**.

Now, along an autoparallel, we have $DA_{00,j} = A_{00,j|i}dx^i + A_{00,j;i}Dl^i = A_{00,j|i}dx^i$ by (6.2). In view of (5.2) and (1.29), $A_{00,j|i}$ is the left side of (6.10) and hence, using (6.12) repeatedly, we find $D_s A_{00,j} = F_{\dot{x}^r}[L_0{}^r{}_{0\|j} - L_0{}^r{}_j - L_j{}^r{}_0 + 2F_{\dot{x}^j}L_0{}^r{}_0]$. But (6.9) implies that $L_0{}^r{}_0 = \Gamma_0{}^r{}_0$. Substitution from (6.13) and (6.11) then leads to (6.8) as required.

Theorem 2 shows that the connection parameter L_{ijk} of (3.6) can always be chosen so that geodesics are identical with autoparallels. For some purposes it may not be desirable to do so. For example, the geometry of paths is concerned with curves which may not be expressible as the extremals of a function $F(x, \dot{x})$. It is possible, however, for them to occur as the autoparallels of a connection L_{ijk} of the form (3.6).

7. Commutation formulae. In this section we consider the effect of commuting the various tensorial derivatives introduced in § 1. This leads to the curvature tensors which relate to the integrability of various displacements of the element of support and which are basic to such studies as "geodesic deviation."

For convenience we introduce two preliminary differentiation processes defined for all functions of $(x, \dot{x})$:

$$(7.1) \qquad \text{(a)} \quad \phi_{(i)} \equiv \frac{\partial \phi}{\partial x^i} - \phi_{||r} L_0{}^r{}_i \qquad \text{(b)} \quad \phi_{[i]} \equiv \phi_{||r}(\delta_i^r - M_0{}^r{}_i).$$

For tensorial ϕ, these are the parts of the covariant derivatives (1.18) which do not involve summation over the indices of ϕ. Two useful identities are

$$(7.2) \quad \text{(a)} \quad \phi_{||i} = \phi_{[r]}(\delta_i^r + M_0{}^r{}_i), \quad \text{(b)} \quad \phi_{[0]} \equiv \phi_{[r]} l^r = 0 \qquad (\phi \in H_0).$$

The first of these follows from (7.1) (b) and (1.16) (b) while the second follows from the first by (1.16) (a). It is also noteworthy that $\phi_{[i]} \equiv \phi_{\|i}$ when $n = 2$, $(\phi \in H_0)$ since M_{ijk} is then given uniquely by (4.2).

The commutation formulae for the derivatives (1.19) and (7.1), are found by straightforward calculations using the identities (1.25), (1.26), (4.3), (5.2), (6.5), and (6.12):

$$
\begin{aligned}
(7.3) \quad &\text{(a)} \quad (\phi_{||h})_{(k)} - (\phi_{(k)})_{||h} = -\phi_{||r}[\bar{L}_0{}^r{}_{hk} - L_h{}^r{}_k] \\
&\text{(b)} \quad (\phi_{||h})_{[k]} - (\phi_{[k]})_{||h} = -\phi_{||r}[\bar{M}_0{}^r{}_{hk} - M_{hk}^r + F_{\dot{x}^h}\delta_k^r - l_k\delta_h^k] \\
&\text{(c)} \quad (\phi_{(h)})_{(k)} - (\phi_{(k)})_{(h)} = -\phi_{||r}\bar{R}_0{}^r{}_{hk} \\
&\text{(d)} \quad (\phi_{(h)})_{[k]} - (\phi_{[k]})_{(h)} = -\phi_{||r}[\bar{P}_0{}^r{}_{hk} + L_h{}^s{}_k(\delta_s^r - M_0{}^r{}_s)] \\
&\text{(e)} \quad (\phi_{[h]})_{[k]} - (\phi_{[k]})_{[h]} = -\phi_{||r}[\bar{S}_0{}^r{}_{hk} - (M_h{}^r{}_k - M_k{}^r{}_h) + (l_h\delta_k^r - l_k\delta_h^r)].
\end{aligned}
$$

The barred quantities here (which are easily seen to be tensors) are defined by

$$
\begin{aligned}
(7.4) \quad &\text{(a)} \quad \bar{L}_i{}^r{}_{hk} = -L_i{}^r{}_{k||h} \\
&\text{(b)} \quad \bar{M}_i{}^r{}_{hk} = -M_i{}^r{}_{k||h} \\
&\text{(c)} \quad \bar{R}_i{}^r{}_{hk} = L_i{}^r{}_{h(k)} - L_i{}^r{}_{k(h)} + L_i{}^s{}_h L_s{}^r{}_k - L_i{}^s{}_k L_s{}^r{}_h \\
&\text{(d)} \quad \bar{P}_i{}^r{}_{hk} = L_i{}^r{}_{h[k]} - M_i{}^r{}_{k|h} \\
&\text{(e)} \quad \bar{S}_i{}^r{}_{hk} = M_i{}^r{}_{h[k]} - M_i{}^r{}_{k[h]} + M_i{}^s{}_h M_s{}^r{}_k - M_i{}^s{}_k M_s{}^r{}_k.
\end{aligned}
$$

From (7.3) it is a relatively simple matter to deduce the corresponding formulae for the tensorial derivatives (1.18) and (1.19) of a contravariant vector X^i.

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$$
\begin{aligned}
(7.5)\quad &(a)\quad (X^i{}_{||h})_{|k} - (X^i{}_{|k})_{||h} = -X^i{}_{||r}\bar{L}_0{}^r{}_{hk} + X^r\bar{L}_r{}^i{}_{hk}\\
&(b)\quad (X^i{}_{||h})_{;k} - (X^i{}_{;k})_{||h} = -X^i_{||r}(\bar{M}_0{}^r{}_{hk} + F_{\dot{x}^h}\delta^r_k - l_k\delta^r_h) + X^r\bar{M}_r{}^i{}_{hk}\\
&(c)\quad (X^i{}_{|h})_{|k} - (X^i{}_{|k})_{|h} = -X^i_{||r}\bar{R}_0{}^r{}_{hk} - 2X^i_{|r}\Omega_h{}^r{}_k + X^r\bar{R}_r{}^i{}_{hk}\\
&(d)\quad (X^i{}_{|h})_{;k} - (X^i{}_{;k})_{|h} = -X^i{}_{||r}\bar{P}_0{}^r{}_{hk} - X^i{}_{|r}M_h{}^r{}_k + X^r\bar{P}_r{}^i{}_{hk}\\
&(e)\quad (X^i{}_{;h})_{;k} - (X^i{}_{;k})_{;h} = -X^i{}_{||r}[\bar{S}_0{}^r{}_{hk} - 2\psi_h{}^r{}_k + l_h\delta^r_k - l_k\delta^r_h] -\\
&\qquad\qquad\qquad\qquad 2X^i{}_{;r}\psi_h{}^r{}_k + X^r\bar{S}^i_{rhk}.
\end{aligned}
$$

In these relations we have introduced, for brevity, the *torsion tensors*

$$(7.6)\qquad \Omega_i{}^j{}_k = \tfrac{1}{2}(L_i{}^j{}_k - L_k{}^j{}_i);\ \psi_i{}^j{}_k = \tfrac{1}{2}(M_i{}^j{}_k - M_k{}^j{}_i).$$

It is convenient to have the right sides of (7.5) expressed entirely in terms of the covariant derivatives (1.18). To this end we note that $X^i{}_{\|r} = (X_{;s}{}^i - X^kM_k{}^i{}_s)(\delta_r{}^s + M_0{}^s{}_r)$ by (1.18) (b), (7.1) (b), and (7.2) (a). Thus (7.5) becomes

$$
\begin{aligned}
(7.7)\quad &(a)\quad (X^i{}_{||h})_{|k} - (X^i{}_{|k})_{||h} = -X^i{}_{;r}L_0{}^r{}_{hk} + X^rL_r{}^i{}_{hk}\\
&(b)\quad (X^i{}_{||h})_{;k} - (X^i{}_{;k})_{||h} = -X^i{}_{;r}(M_0{}^r{}_{hk} + F_{\dot{x}^h}\delta^r_k - l_k\delta^r_h) + X^rM_r{}^i{}_{hk}\\
&(c)\quad (X^i{}_{|h})_{|k} - (X^i{}_{|k})_{|h} = -2X^i{}_{|r}\Omega_h{}^r{}_k - X^i{}_{;r}R_0{}^r{}_{hk} + X^rR_r{}^i{}_{hk}\\
&(d)\quad (X^i{}_{|h})_{;k} - (X^i{}_{;k})_{|h} = -X^i{}_{|r}M_h{}^r{}_k - X^i{}_{;r}P_0{}^r{}_{hk} + X^rP_r{}^i{}_{hk}\\
&(e)\quad (X^i{}_{;h})_{;k} - (X^i{}_{;k})_{;h} = -X^i{}_{;r}(S_0{}^r{}_{hk} + l_h\delta^r_k - l_k\delta^r_h) \overset{i}{+} X^rS_r{}^i{}_{hk},
\end{aligned}
$$

where

$$
\begin{aligned}
(7.8)\quad &(a)\quad L_r{}^i{}_{hk} = \bar{L}_r{}^i{}_{hk} + M_r{}^i{}_s(\delta^s_t + M_0{}^s{}_t)\bar{L}_0{}^t{}_{hk}\\
&(b)\quad M_r{}^i{}_{hk} = \bar{M}_r{}^i{}_{hk} + M_r{}^i{}_s(\delta^s_t + M_0{}^s{}_t)(\bar{M}_0{}^t{}_{hk} + F_{\dot{x}^h}\delta^t_k - l_k\delta^t_h)\\
&(c)\quad R_r{}^i{}_{hk} = \bar{R}_r{}^i{}_{hk} + M_r{}^i{}_s(\delta^s_t + M_0{}^s{}_t)\bar{R}_0{}^t{}_{hk}\\
&(d)\quad P_r{}^i{}_{hk} = \bar{P}_r{}^i{}_{hk} + M_r{}^i{}_s(\delta^s_t + M_0{}^s{}_t)\bar{P}_0{}^t{}_{hk}\\
&(e)\quad S_r{}^i{}_{hk} = \bar{S}_r{}^i{}_{hk} + M_r{}^i{}_s(\delta^s_t + M_0{}^s{}_t)(\bar{S}_0{}^t{}_{hk} + l_h\delta^t_k - l_k\delta^t_h - 2\psi_h{}^t{}_k).
\end{aligned}
$$

Identities similar to (7.7) are valid for arbitrary tensors. For example, $(X^i{}_{j\|h})_{|k} - (X^i{}_{j|k})_{\|h} = -X^i{}_{j;r}L_0{}^r{}_{hk} + X_j{}^rL_r{}^i{}_{hk} - X^i{}_rL_j{}^r{}_{hk}$. The quantities defined by (7.8) are called the curvature tensors. The barred curvature tensors (7.4) are uniquely determined by the unbarred ones. For example, inner multiplication of (7.8) (c) with l^r yields $R_0{}^i{}_{hk} = \bar{R}_0{}^t{}_{hk}(\delta_t{}^i + M_0{}^i{}_t)$, which, when substituted back into (7.8) (c), leads to

$$(7.9)\qquad \bar{R}_r{}^i{}_{hk} = R_r{}^i{}_{hk} - M_r{}^i{}_sR_0{}^s{}_{hk}.$$

The last three tensors of (7.8) may be shown to be identical to those derived by Moór (7, § 7) in a different manner.

The curvature tensors satisfy several identities. Clearly

$$(7.10)\qquad (a)\ \bar{R}_{ij(hk)} = 0,\ (b)\ R_{ij(hk)} = 0,\ (c)\ \bar{S}_{ij(hk)} = 0,\ (d)\ S_{ij(hk)} = 0.$$

Also, applying the commutation formulae to g_{ij}, it follows that

$$
\begin{aligned}
(7.11)\qquad &(a)\ L_{(ij)hk} = -A_{ij,h|k} \qquad (b)\ M_{(ij)hk} = -A_{ij,h;k}\\
&(c)\ R_{(ij)hk} = 0 \quad (d)\ P_{(ij)hk} = 0 \quad (e)\ S_{(ij)hk} = 0.
\end{aligned}
$$

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In view of (1.25), (7.2) (b), and (1.16) (a), the definitions (9.4) yield

$$\text{(7.12)} \qquad \text{(a) } \bar{L}_i{}^r{}_{0k} = 0 \qquad \text{(b) } \bar{M}_i{}^r{}_{0k} = 0 \qquad \text{(c) } \bar{P}_i{}^r{}_{h0} = 0.$$

Also, since $l^k{}_{[h]} = l^k{}_{;h} - M_0{}^k{}_h$, we have, by (1.26),

$$\text{(7.12}'\text{)} \qquad \text{(d) } \bar{S}_i{}^r{}_{h0} = -M_i{}^r{}_{k[h]}l^k = +M_i{}^r{}_k l^k{}_{[h]} = M_i{}^r{}_k(\delta^k_h - M_0{}^k{}_h).$$

The corresponding equations for the unbarred tensors are

$$\text{(7.13)} \quad \text{(a) } L_i{}^r{}_{0k} = 0, \quad \text{(b) } M_i{}^r{}_{0k} = M_i{}^r{}_s(\delta^s_k + M_0{}^s{}_k) \quad \text{(c) } P_r{}^i{}_{h0} = 0$$
$$\text{(d) } S_r{}^i{}_{h0} = 0.$$

Another set of identities is obtained by straightforward calculation from the definitions:

$$\text{(7.14)} \qquad \text{(a) } [\bar{R}_j{}^r{}_{kh} - 2\Omega_j{}^r{}_{k|h} + 4\Omega_j{}^s{}_k\Omega_s{}^r{}_h] + (\text{cyc.})_{jkh} = 0,$$
$$\text{(b) } [\bar{S}_j{}^r{}_{kh} - 2\psi_j{}^r{}_{k;h} + 4\psi_j{}^s{}_k\psi_s{}^r{}_{h]} + (\text{cyc.})_{jkh} = 0,$$

where $(\text{cyc.})_{jkh}$ denotes the sum of the two terms derived from the quoted one by cyclic permutation of j, k, and h. Similar formulae for the unbarred tensors follow from these by means of relations typified by (7.9).

Finally we mention the *Bianchi identities*:

$$\text{(7.15)} \quad \text{(a) } [\bar{R}_i{}^r{}_{jk|h} + 2\bar{R}_i{}^r{}_{js}\Omega^s_{kh} - \bar{L}_i{}^r{}_{sj}\bar{R}_0{}^s{}_{kh}] + (\text{cyc.})_{jkh} = 0$$
$$\text{(b) } [\bar{S}_i{}^r{}_{jk;h} + 2\bar{S}_i{}^r{}_{js}\psi^s_{kh} - \bar{M}_i{}^r{}_{sj}(\bar{S}_0{}^s{}_{kh} + l_k\delta^s_h - l_h\delta^s_k - 2\psi^s_{kh})] +$$
$$(\text{cyc.})_{jkh} = 0$$
$$\text{(c) } [R_i{}^r{}_{jk|h} + 2R_i{}^r{}_{js}\Omega^s_{kh} + P_i{}^r{}_{js}R_0{}^s{}_{kh}] + (\text{cyc.})_{jkh} = 0$$
$$\text{(d) } [S_i{}^r{}_{jk;h} - 2S_i{}^r{}_{jk}l_h + S_i{}^r{}_{js}S_0{}^s{}_{hk}] + (\text{cyc.})_{jkh} = 0.$$

These formulae may all be established by an adaptation of the method given in (**12**, pp. 109-111). We shall outline this approach in the derivation of (7.15) (d).

The commutation formula (7.7) (e) for a covariant vector X_i reads $X_{i;j;k} - X_{i;k;j} = -X_{i;r}(S_0{}^r{}_{jk} + l_j\delta_k{}^r - l_k\delta_j{}^r) - X_rS_i{}^r{}_{jk}$. If we apply the operation "${}_{;h}$" to this, permute the indices j, k, h and add, the left side becomes $(X_{i;j;k;h} - X_{i;j;h;k}) + (\text{cyc.})_{jkh}$. To this, the commutation formula for the tensor $X_{i;j}$ may be applied. We combine the result with the right side to obtain an equation involving linearly the terms X_r, $X_{i;r}$, and $X_{i;r;j} - X_{i;j;r}$ etc. The latter forms may be simplified again by the commutation formula for X_i. In the resulting relation the left side of (7.15) (d), which we denote by $S_i{}^r{}_{jkh}$, appears as the coefficient of X_r, while the coefficient of $X_{i;r}$ is, by (1.26), (1.27) (b),

$$[-(S_0{}^r{}_{sj} + l_s\delta^r_j - l_j\delta^r_s)(S_0{}^s{}_{kh} + l_k\delta^s_h - l_h\delta^s_k) + S_0{}^r{}_{jk;h} - S_j{}^r{}_{kh} + (g_{jh} - l_jl_h)\delta^r_k -$$
$$-(g_{kh} - l_kl_h)\delta^r_j] + (\text{cyc.})_{jkh}.$$

After some simplification, based on the identities (7.10) (d) and (7.11) (e), this coefficient assumes the form $S_0{}^r{}_{jkh}$ and hence the resulting equation reads $X_{i;r}S_0{}^r{}_{jkh} + X_rS_i{}^r{}_{jkh} = 0$. Since X_i is an arbitrary vector, $S_i{}^r{}_{jkh}$ vanishes and (7.15) (d) is proved. The other formulae of (7.15) follow from the process

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described above applied to the commutation relations (7.7) (c), (7.5) (a), and (7.5) (d).

In a Finsler space it is customary to assume that the connection parameters are symmetric and hence, by Theorems 2 and 3 of § 5, we have $L_i{}^j{}_k = \Gamma_i{}^j{}_k$ and $M_{ijk} = A_{ij,k}$. Since we also have $g_{ij} = \gamma_{ij}$ in this case, $A_{ij,k}$ is completely symmetric and satisfies (5.3). The derivative "$_{[i]}$" then reduces to "$_{\| i}$." In view of these remarks, the results of this section imply corresponding ones for Finsler geometry. For example, (7.15) (a) and (c) reduce to the Bianchi identities of (**12**, pp. 110-111). The Finslerian forms of (7.15) (b) and (d) are of some interest since they have not appeared in the literature. They are

$$(7.16)\qquad \begin{aligned} &\text{(a)}\quad [\bar{S}_i{}^r{}_{jk\|h} + A_i{}^r{}_{,j\|s}(l_k\delta_h^s - l_h\delta_k^s)] + (\text{cyc.})_{jkh} = 0\\ &\text{(b)}\quad [S_i{}^r{}_{jk\|h} - 2S_i{}^r{}_{jk}l_h] + (\text{cyc.})_{jkh} = 0, \end{aligned}$$

where $S_i{}^r{}_{hk} = -(A_i{}^s{}_{,h}A_s{}^r{}_{,k} - A_i{}^s{}_{,k}A_s{}^r{}_{,h})$, $\bar{S}_i{}^r{}_{hk} = S_i{}^r{}_{hk} + A_i{}^r{}_{,h}l_k - A_i{}^r{}_{,k}l_h$.

8. Applications of the curvature tensors. A large number of structural theorems follow from the work of § 7. Some of these will be proved in this section.

THEOREM 1. *If M_{ijk} is independent of $\dot{x}^i$, the space is Riemannian.*

Proof. By hypothesis $\bar{M}_i{}^r{}_{hk} = 0$. It then follows from (7.8) (b), (1.16) (b), and (7.11) (b) that $-A_{ij,h;k} = A_{ij,r}(F_{\dot{x}^h}\delta_k{}^r - l_k\delta_h{}^r)$. Inner multiplication with l^k leads to $-A_{ij,h} = 0$, in view of (1.21), and since this is equivalent to $g_{ij} = g_{ij}(x^k)$, the theorem is proved.

A second result follows immediately from (7.4) (a), (7.8) (a), and (7.11) (a):

THEOREM 2. *If $L_i{}^j{}_k$ is independent of the $\dot{x}^i$, then $A_{ij,k|h}$ vanishes.*

THEOREM 3. *$\Gamma^*{}_j{}^i{}_k$ is independent of the $\dot{x}^i$ if and only if $A_{ij,k|h}$ vanishes.*

Proof. The existence of $\Gamma^*{}_j{}^i{}_k$ implies the existence of $K_{hk}{}^{ij}$ (cf. § 3 and Theorem 2 of § 2). Also $\Gamma^*{}_{ijk}$ is a member of the family (3.6) and hence may replace L_{ijk} in the formulae of § 7. The necessity of $A_{ij,k|h} = 0$ then follows from Theorem 2. To prove the sufficiency we note that $\bar{L}_{(ij)hk} = -A_{ij,h|k} - A_{ij,r}\bar{L}_0{}^r{}_{hk}$, by (7.8) (a), (7.11) (a) and (1.27) (b). Denoting $\bar{L}_{ijhk}$, when $L_{ijk} = \Gamma^*{}_{ijk}$, by $\bar{L}^*{}_{ijhk}$ we solve the above relations together with

$$\bar{L}^*{}_{ijhk} = \bar{L}^*{}_{kjhi}:$$

$$(8.1)\quad \bar{L}^*_{ijhk} = -(A_{ij,h|k} + A_{jk,h|i} - A_{ik,h|j}) - A_{ij,r}\bar{L}^{*r}_{0hk} - A_{jk,r}\bar{L}^{*r}_{0hi} + \\ + A_{ik,r}\bar{L}^{*r}_{0hj}.$$

On inner multiplication by l^i, it follows that

$$(8.2)\qquad \bar{L}^*_{0rhs}H^{rs}_{jk} = -(A_{0j,h|k} + A_{jk,h|0} - A_{0k,h|j}).$$

Since $K_{hk}{}^{ij}$ exists, we can find $\bar{L}^*{}_{0rhs}$ from (8.2) (cf. (2.24)) and, substituting in (8.1), we will have $\bar{L}^*{}_{ijhk}$ expressed as a linear combination of the $A_{ij,k|h}$. The second part of the theorem follows at once. The special case of this expression for $\bar{L}^*{}_{ijhk}$ when the space is Finslerian is well known (**12**, p. 81, equation (3.13)).

A vector field $\xi^i(x)$, defined throughout a region, whose elements are obtained from each other by parallel displacement with respect to themselves (see § 6) must satisfy the system of partial differential equations $\xi^i{}_{|j} = 0$, $(L_j{}^i{}_h = L_j{}^i{}_k\,(x, \xi))$ whose integrability conditions are easily seen to be $\xi^r \bar{R}_r{}^i{}_{jk}(x, \xi) = 0$. Similarly, the integrability conditions for n linearly independent vector fields $\eta_{(\mu)}{}^i$ $(\mu = 1, \ldots, n)$, parallel with respect to ξ^i, are $\eta_{(\mu)}{}^r \bar{R}_r{}^i{}_{jk}(x, \xi) = 0$, that is, $\bar{R}_r{}^i{}_{jk}(x, \xi) = 0$. But if this condition is satisfied, there exists a co-ordinate system (dependent on ξ^i) in which $\Gamma^*{}_j{}^i{}_k$ vanishes (see (**12**, pp. 135-136)). In order that this special co-ordinate system be independent of ξ^i it is sufficient that $\Gamma^*{}_j{}^i{}_k$ be independent of direction. These remarks lead to

THEOREM 4. *If $\Gamma^*{}_j{}^i{}_k$ exists, and if both $\bar{R}_r{}^i{}_{jk}$ and $A_{ij,h|k}$ vanish identically, the space is Minkowskian, that is, $g_{ij} = g_{ij}\,(\dot{x}^k)$.*

Proof. The proof, based on Theorem 3, is formally identical to that given in (**12**, p. 136).

The general theory of curve deviation is another example of the applicability of the work of § 7. We begin by considering a subspace L_2 defined by $x^i = x^i(u, v)$, rank $((\partial x^i/\partial u, \partial x^i/\partial v)) = 2$, $(i = 1, 2, \ldots, n)$.* The tangent vectors to the parameter curves of L_2 satisfy

$$(8.3) \qquad \text{(a)}\quad \xi^i = \frac{\partial x^i}{\partial u} \equiv x^i_u,\ \eta^i = \frac{\partial x^i}{\partial v} \equiv x^i_v; \qquad \text{(b)}\quad \xi^i_v = \eta^i_u.$$

Having chosen a vector field $\dot{x}^i = \dot{x}^i(x)$ on L_2 we introduce the derivatives

$$(8.4) \qquad \text{(a)}\ \nabla_u X^i = X^i{}_{|h}\xi^h \qquad \text{(b)}\ \nabla_v X^i = X^i{}_{|h}\eta^h,$$

defined for all vectors X^i attached to the line-element $(x, \dot{x})$. For example, we have $\nabla_v \xi^i = \xi^i{}_v + L_r{}^i{}_h(x, \dot{x})\xi^r\eta^h$ and a similar equation for $\nabla_u \eta^i$. Combining these and using (8.3) (b) and (7.6), we find

$$(8.5) \qquad \nabla_v \xi^i - \nabla_u \eta^i = 2\Omega^i_{hk}(x, \dot{x})\xi^h\eta^k.$$

It is also easily seen from (7.5) (c) and (8.5) that

$$(8.6) \quad \nabla^2_{vu}\xi^i - \nabla^2_{uv}\xi^i = (\xi^i_{|k|h} - \xi^i_{|h|k})\xi^h\eta^k + \xi^i_{|h}(\nabla_v\xi^h - \nabla_u\eta^h) = \bar{R}_r{}^i{}_{hk}(x, \dot{x})\xi^r\xi^h\eta^k.$$

Now let c, ϵ (small) be constants and consider the neighbouring curves $C: v = c$ and $C': v = c + \epsilon$. A natural correspondence between points A of C and B of C' is given by $A(u, c) \leftrightarrow B(u, c + \epsilon)$. We adopt, however, a general correspondence $A(u, c) \leftrightarrow B(u', c + \epsilon)$, where $u' = u + f(u)$, $f(u)$ being of the same order as ϵ. If $z^i(x)$ represents the displacement vector from $A(x^i) = A(u, c)$ to $B(x^i + dx^i)\ C\ B(u', c + \epsilon)$ we therefore have

$$(8.7) \qquad z^i(x) = dx^i = \xi^i du + \eta^i dv = f(u)\xi^i + \epsilon\eta^i,$$

*The method of approach here is essentially that of Rund. See (**12**, pp. 111–118).

neglecting quantitites of the order ϵ^2. The differential equation satisfied by z^i is known as the deviation equation. To find it, we apply the operator ∇_u to (8.7), noting (8.5) and the skew-symmetry of $\Omega_h{}^i{}_k$: $\nabla_u z^i = \nabla_u(f\xi^i) - 2\Omega_h{}^i{}_k\xi^h z^k + \epsilon\nabla_v\xi^i$. Thence, a second differentiation, together with (8.6) and (7.10) (b), yields

$$(8.8)\quad \nabla_u^2 z^i + 2\Omega_{hk}^i\xi^h\nabla_u z^i + [\bar{R}_j{}^i{}_{hk}\xi^j\xi^h + 2\nabla_u(\Omega_{hk}^i\xi^h)]z^k = \epsilon\nabla_{vu}^2\xi^i + \nabla_u^2(f\xi^i),$$

or, since $\epsilon\nabla_{vu}{}^2\xi^i = (\nabla_u\xi^i)_{|h}(z^h - f\xi^h)$,

$$(8.9)\quad \nabla_u^2 z^i + 2\Omega_{hk}^i\xi^h\nabla_u z^k + [\bar{R}_j{}^i{}_{hk}\xi^j\xi^h + 2\nabla_u(\Omega_{hk}^i\xi^h) - (\nabla_u\xi^i)_{|k}]z^k = f''\xi^i + \\ + 2f'\nabla_u\xi^i.$$

This second-order differential equation involving arbitrary $f(u)$ and $\dot{x}^i$ is the most general deviation equation for the family of curves $v =$ constant.

If we assume that u represents arc-length s and that the members of the family $v =$ constant have tangent vectors which satisfy $\nabla_s\xi^i = 0$, we have, by (7.1) (a),

$$0 = (\nabla_s\xi^i)_v = \left[\frac{\partial}{\partial x^h}(\nabla_s\xi^i) + F^{-1}(x, \dot{x})(\nabla_s\xi^i)_{||r}\frac{\partial \dot{x}^r}{\partial x^h}\right]\eta^h = (\nabla_s\xi^i)_{(h)}\eta^h + \\ F^{-1}(\nabla_s\xi^i)_{||r}\nabla_v\dot{x}^r.$$

For such curves $v =$ constant we also have $\nabla_{vs}{}^2\xi^i \equiv [(\nabla_s\xi^i)_{(h)} + (\nabla_s\xi^j)L_j{}^i{}_h]\eta^h = (\nabla_s\xi^i)_{(h)}\eta^h$ and $(\nabla_s\xi^i)_{\|r} = -\bar{L}_h{}^i{}_{rk}\xi^h\xi^k$, by (7.4) (a). Finally, if we assume that the fields $\dot{x}^i$ and ξ^i coincide, then $\dot{x}^i$ is a unit vector and the curves $v =$ constant become autoparallels (cf. (6.2)). When the line elements are (x, ξ), with $\nabla_s\xi^i = 0$, we may replace the symbol ∇_s by D_s. In view of the above remarks, the right-hand side of (8.8) becomes $\epsilon(D_s\xi^i)_{(h)}\eta^h + f''(s)\xi^i = \epsilon\bar{L}_h{}^i{}_{rk}(x, \xi)\xi^h\xi^k\nabla_v\xi^r + f''\xi^i = \bar{L}_h{}^i{}_{rk}\xi^h\xi^k[D_s(z^r - f\xi^r) + 2\Omega_p{}^r{}_q\xi^p z^q] + f''\xi^i$, where (8.5) and (8.7) have been used. *Thus the equation of autoparallel deviation is*

$$(8.10)\quad D_s^2 z^i + (2\Omega_{0k}^i - \bar{L}_0{}^i{}_{0k})D_s z^k + (\bar{R}_0{}^i{}_{0k} + 2D_s\Omega_0{}^i{}_k - 2\bar{L}_0{}^i{}_{0r}\Omega_0{}^r{}_k)z^k = f''(s)\xi^i,$$

in which we have put $\Omega_j{}^i{}_k\xi^j = \Omega_0{}^i{}_k$ etc. This special case of (8.9) has been given in (**8**, equation (3.7)).

The following lemma will be useful in the sequel.

LEMMA. *Let L_2 be defined by $x^i = x^i(s, v)$, where s is the arc-length of the curves $v =$* constant. *Then ,with $\dot{x}^i \equiv \xi^i$, η^i, and ∇_s defined as above,*

$$(8.11)\qquad (a)\quad F_{\dot{x}^i}(x, \xi)\nabla_s\xi^i = 0 \qquad (b)\quad F_{\dot{x}^i}(\nabla_s\eta^i + 2\Omega_{0s}^i\eta^s) = 0.$$

Proof. Since $ds^2 = F(x, dx)$, we have $F(x, \xi) \equiv 1$ and hence

$$0 = dF = F_{x^i}(\xi^i ds + \eta^i dv) + F_{\dot{x}^i}(\xi_s^i ds + \xi_v^i dv).$$

Substituting for F_{x^i} from (6.5) and using the definitions of ∇_s and $\Omega_j{}^i{}_k$ (cf. (7.6)) we obtain the parts of (8.11) as coefficients of the arbitrary du, dv.

It follows from this lemma, (8.7), and the fact that $F \in H_1$ that

$$(8.12)\qquad F_{\dot{x}^i}(D_s z^i + 2\Omega_{0k}^i z^k) = f'(s)$$

is a *first integral* of the equation of autoparallel deviation (8.10). It is sometimes convenient to assume that the variation vector z^i is normal, that is, $g_{ij}(x, \xi)\xi^i z^j = 0$. If this condition is imposed on z^i, then (8.12) must be considered as a definition of $f(s)$.

Equation (8.9) may also be used to obtain the equations of geodesic deviation. This is, however, essentially a problem of Finsler space (cf. § 6) and the results of **(13)** give a simpler description.

The Bianchi identities of § 9 may be used to obtain results analogous to the *Schur Theorem* of Riemannian geometry. However the close relationship between isotropic spaces and spaces for which the Schur theorems hold, which extends to Finsler geometry, seems to break down here.

We hope to consider the extensions of these and related problems at a later date. For the present we remark that the preceding work provides a generalization of the requisite analytical tools.

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References

1. A. Albert, *Modern higher algebra* (Chicago, 1937).

2. L. Berwald, *Untersuchung der Krümmung allgemeiner metrischer Räume auf Grund des in ihnen herrschenden Parallelismus*, Math. Z., *25* (1926), 40–73.

3. ——— *Ueber Finslersche und Cartansche Geometrie III. Two-dimensional Finsler spaces with rectilinear extremals*, Ann. Math. (2), *42* (1941), 84–112.

4. C. Caratheodory, *Ueber die starken Maxima und Minima bei einfachen Integralen*, Math. Ann., *62* (1906), 449–503.

5. E. Cartan, *Les espaces de Finsler*, Actualités, *79* (Paris 1934).

6. P. Finsler, *Ueber Kurven und Flächen in allgemeinen Räumen* (Dissertation, Göttingen, 1918). Unveränderter Nachdruck (Basel: Birkhäuser, 1951).

7. A. Moór, *Entwicklung einer Geometrie der allgemeinen metrischen Linienelementräume*, Acta Sci. Math. Szeged, *17* (1956), 85–120.

8. ——— *Ueber die Autoparallele Abweichung in Allgemeinen metrischen Linienelementräumen*, Publicationes Mathematicae, Debrecen, *5* (1957), 103–118.

9. B. Riemann, *Ueber die Hypothesen, welche der Geometrie zugrunde liegen*, Habilitationsvortrag 1854, Ges. math. Werke., 272–287.

10. M. Riesz, *L'intégrale de Riemann-Liouville et le problème de Cauchy*, Acta Math., *81* (1949), 1–223.

11. H. Rund, *Die Hamiltonsche Funktion bei allgemeinen dynamischen* Systemen, Arch. Math., *3* (1952), 207–215.

12. ——— *Differential Geometry of Finsler Spaces* (Berlin: Springer-Verlag, 1959).

13. ——— *The scalar form of Jacobi's equations in the calculus of variations*, Ann. Math. pura appl. (4), *35* (1953), 183–202.

14. J. Synge, *A generalization of the Riemannian line-element*, Trans. Amer. Math. Soc., *27* (1925), 61–67.

15. ———*The Geometry of Dynamics*, Phil. Trans. Roy. Soc. London (A), *226* (1926), 31–106.

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