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ON A STEIN AND WEISS PROPERTY OF THE CONJUGATE FUNCTION

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1. Introduction.

(1.1) *The conjugate function on locally compact abelian groups.* Let G be a locally compact abelian group with character group $\hat{G}$. Let μ denote a Haar measure on G such that $\mu(G) = 1$ if G is compact. (Unless stated otherwise, all the measures referred to below are Haar measures on the underlying groups.) Suppose that $\hat{G}$ contains a measurable order $P : P + P \subseteq P$; $P \cup (-P) = \hat{G}$; and $P \cap (-P) = \{0\}$. For f in $\mathcal{L}^2(G)$, the *conjugate function of f* (*with respect to the order P*) is the function $\tilde{f}$ whose Fourier transform satisfies the identity

$$\hat{\tilde{f}}(\chi) = -i \operatorname{sgn}_P(\chi)\hat{f}(\chi)$$

for almost all χ in $\hat{G}$, where $\operatorname{sgn}_P(\chi) = 0$, 1, or -1, according as $\chi = 0$, $\chi \in P\backslash\{0\}$, or $\chi \in (-P)\backslash\{0\}$. A generalized version of a theorem of M. Riesz asserts that, for all f in $\mathcal{L}^2 \cap \mathcal{L}^p(G)$, where $1 < p < \infty$, there is a constant A_p such that $\|\tilde{f}\|_p \leqq A_p\|f\|_p$. Moreover, the constant A_p is independent of the order P and the group G. See [**2**], Section 7.(a)); [**3**], Theorem (7.2); and [**4**], Theorem (1.1).

Let E be a measurable subset of G with finite measure. When $G = \mathbf{R}$ or $\mathbf{T}$, and P is the usual ordering on $\mathbf{R}$ or $\mathbf{Z}$, a result of Stein and Weiss [**13**] gives formulas for the distribution functions of the conjugate function of 1_E. These formulas depend only on $|E|$ and not on E itself. (Here, and throughout the paper, we use the symbol 1_E to denote the indicator function of E, and the absolute value to denote Lebesgue measures on $\mathbf{R}$ or $\mathbf{T}$. On $\mathbf{R}$, Lebesgue measure is normalized so that $|[0, 1]| = 1$.)

Using the results of [**13**] on $\mathbf{R}$ only, we will show that, in the general setting of a locally compact abelian group with an arbitrary measurable order on the dual group, the distribution functions of $\tilde{1}_E$ are independent of E, P, and G, and depend only on the measure of E, and whether G is compact or noncompact. Hence the same formulas on $\mathbf{R}$ hold for the conjugate function on an arbitrary locally compact, noncompact, abelian group with ordered dual group; and the formulas on $\mathbf{T}$ hold for the conjugate function on an arbitrary compact abelian group with ordered dual group.

(1.2) *The Stein and Weiss formulas on* $\mathbf{R}$. Let $E \subset \mathbf{R}$ with $|E| < \infty$. The conjugate function (or *Hilbert transform*) of 1_E is given by:

$$H1_E(x) = \lim_{\epsilon\downarrow 0} \frac{1}{\pi} \int_{\epsilon \leqq |t| \leqq \frac{1}{\epsilon}} 1_E(x-t)\frac{1}{t}\, dt. \tag{1}$$

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For $y > 0$, define the distribution functions

$$\begin{aligned}\lambda_{H1_E}(y) &= |\{x \in \mathbf{R} : |H1_E(x)| > y\}|; \\ \lambda^{+}_{H1_E} &= |\{x \in \mathbf{R} : H1_E(x) > y\}|;\end{aligned} \tag{2}$$

and

$$\lambda^{-}_{H1_E}(y) = |\{x \in \mathbf{R} : H1_E(x) < -y\}|.$$

We have

$$\lambda_{H1_E}(y) = \lambda^{+}_{H1_E}(y) + \lambda^{+}_{H1_E}(y), \quad \text{for all } y > 0.$$

The distribution functions of a measurable real-valued function, on an arbitrary measure space, are defined similarly.

From [**13**], Lemmas 3 and 4, we have

$$\lambda^{-}_{H1_E}(y) = \lambda^{+}_{H1_E}(y) = \frac{1}{\sinh(\pi y)}\,|E|; \tag{3}$$

and hence

$$\lambda_{H1_E}(y) = \frac{2}{\sinh(\pi y)}\,|E|. \tag{4}$$

(Note that the definition of the Hilbert transform given in (1) differs from that of [**13**] by a factor of π. For this reason formulas (3) and (4) differ slightly from those in [**13**].)

(1.3) *The ergodic Hilbert transform.* As illustrated in [**1**] and [**3**], some properties of the conjugate function can be derived from those of the ergodic Hilbert transform. To define this transform on the group G, consider a continuous homomorphism ϕ of $\mathbf{R}$ into G, and let $f \in \mathcal{L}^p(G)$, for $1 \leqq p < \infty$. The *ergodic Hilbert transform of f on G* (*with respect to the homomorphism* ϕ) is given by:

$$H_\phi f(x) = \lim_{n\to\infty} \frac{1}{\pi} \int_{\frac{1}{n}\leqq|t|\leqq n} f(x - \phi(t))\frac{1}{t}\,dt. \tag{1}$$

It is shown in [**6**] that the above limit exists μ-a.e. on G; that it is in $\mathcal{L}^p(G)$, whenever f is in $\mathcal{L}^p(G)$ and $1 < p < \infty$; and that it is of weak type (1,1), whenever f is in $\mathcal{L}^1(G)$. (The results of [**6**] are stated for one-parameter groups of measure preserving transformations on a σ-finite measure space. The σ-finiteness condition is not needed in our case where the underlying measure space is G. For details see [**1**], (2.1).) Straightforward computations show that

$$(H_\phi f)\hat{}(\chi) = -i\,\mathrm{sgn}(\psi(\chi))\hat{f}(\chi) \tag{2}$$

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for all f in $\mathcal{L}^p \cap \mathcal{L}^2(G)$, and almost all χ in $\hat{G}$, where ψ denotes the adjoint homomorphism of ϕ. That is, ψ is the continuous homomorphism from $\hat{G}$ into $\mathbf{R}$ such that:

$$\chi \circ \phi(t) = \exp(i\psi(\chi)t) \quad \text{for all } \chi \in \hat{G}, \text{ and all } t \in \mathbf{R}.$$

We will use a separation theorem for measurable orders to approximate $\tilde{f}$ by certain ergodic Hilbert transforms. We will then establish the Stein and Weiss formulas for some particular ergodic Hilbert transforms, and obtain our desired results by approximating the conjugate function by appropriate ergodic Hilbert transforms. This was the course of study in [**1**] where we used the transference methods of Coifman and Weiss [**7**], and some approximation techniques, to derive properties of the conjugate function from those of the ergodic Hilbert transform. However, we show by an example in Section 4 that the Stein and Weiss formulas fail in general for the ergodic Hilbert transform. This will entail a new approach to the problem treated in this essay.

2. Preliminaries. We collect in this section various results that will be needed in the sequel. We start by stating a separation theorem for measurable orders which will serve as a link between the conjugate function and the ergodic Hilbert transform.

THEOREM (2.1) ([**3**], Theorem (5.14)). *Let K be a compact non-void subset of $\hat{G}$. There is a real-valued homomorphism ψ on $\hat{G}$ such that ψ is positive on $(K \cap P)\backslash(N \cup \{0\})$, and ψ is negative on $(K \cap (-P))\backslash(N \cup \{0\})$, where N is a subset of $\hat{G}$ with measure zero.*

The proof of Theorem (2.1) is based on the study of Haar-measurable orders of Hewitt and Koshi [**8**]. A particular case of this theorem, dealing with arbitrary orders on $\mathbf{Z}^n$, appears in [**4**], Lemma (2.5). In the proof of our main results, we will need the following corollary of Theorem (2.1).

THEOREM (2.2). *Suppose that $\hat{G}$ is discrete, and let P denote an arbitrary order on $\hat{G}$. Let K be a nonvoid finite subset of $\hat{G}$, and let H be any countable subgroup of $\hat{G}$ that contains K. Then there is a real-valued homomorphism ψ on $\hat{G}$ such that*

(i) $\quad \psi(k \cap (P\backslash\{0\})) \subset]0, \infty[;$

(ii) $\quad \psi(K \cap ((-P)\backslash\{0\})) \subset]-\infty, 0[;$

and

(iii) $\quad \operatorname{Ker}\psi \cap H = \{0\}.$

Proof. First, note that (i) and (ii) are immediate consequences of Theorem (2.1), since in this case N is necessarily empty. To obtain (iii) will require a further construction.

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Let Y denote the group generated by K. Since K is finite and $\hat{G}$ is torsion-free, Y is isomorphic to $\mathbf{Z}^n$, for some n. Order $\mathbf{Z}^n$ by $P \cap \mathbf{Z}^n$, and apply the separation theorem (2.1) to obtain a homomorphism ψ_0 on $\mathbf{Z}^n$ with properties (i)–(ii). We have: $\psi_0(x) = x.\alpha_0 = \psi_{\alpha_0}(x)$ for some α_0 in $\mathbf{R}^n$, and all x in $\mathbf{Z}^n$. (Here $x.\alpha_0$ denotes the usual inner product of vectors in $\mathbf{R}^n$.) The function $(x, \alpha) \longrightarrow x.\alpha$ is continuous on $\mathbf{R}^n \times \mathbf{Z}^n$. Hence there is an open nonvoid neighborhood U of α_0 in $\mathbf{R}^n$ such that (i) and (ii) hold for all the homomorphisms ψ_α with $\alpha \in U$ (for all $x \in \mathbf{Z}^n$, $\psi_\alpha(x) = \alpha . x$). Pick α_1 in U with components independent over $\mathbf{Q}$. The homomorphism ψ_{α_1} defined on $\mathbf{Z}^n$ by $x \longmapsto \alpha_1 . x$ satisfies (i)–(ii), and we have:

$$\operatorname{Ker} \psi_{\alpha_1} \cap \mathbf{Z}^n = \{0\}.$$

If $\mathbf{Z}^n = H$, we are done. If not, let J denote the subgroup of $\hat{G}$ generated by $\mathbf{Z}^n$ and H. Then J is clearly countable. We now proceed as in the proof of [**9**], Theorem (A.7), p. 441, to extend the homomorphism ψ_{α_1} to J in such a way that (iii) remains true. Enumerate $J \backslash \mathbf{Z}^n$ as $\{x_m\}_{m=1}^{\infty}$. Define the extension ψ inductively as follows. If $n_1 x_1 = y_1 \in \mathbf{Z}^n$ for some nonzero integer n_1, then let

$$\psi(x_1) = \frac{1}{n_1} \psi_{\alpha_1}(y_1);$$

otherwise define $\psi(x_1)$ to be any real number y_1 which is independent over $\mathbf{Q}$ from $\psi_{\alpha_1}(\mathbf{Z}^n)$. Having defined $\psi(x_m)$, if $n_{m+1} x_{m+1} = y_{m+1}$ belongs to the linear span of $\mathbf{Z}^n$ and $\{x_1, \ldots, x_m\}$ for some nonzero integer n_{m+1}, let

$$\psi(x_{m+1}) = \frac{1}{n_{m+1}} \psi_{\alpha_1}(y_{m+1});$$

otherwise let $\psi(x_{m+1})$ be any real number independent over $\mathbf{Q}$ of the image under ψ of the linear span of $\mathbf{Z}^n$ and $\{x_1, \ldots, x_m\}$. As defined on J, the homomorphism ψ satisfies (i) and (ii), and is such that $\operatorname{Ker} \psi \cap J = \{0\}$. Once more, use [**9**], Theorem (A.7), to extend the homomorphism in an arbitrary way from J to all $\hat{G}$. Denote the extended homomorphism by ψ. Clearly ψ satisfies (i)–(iii).

Remark (2.3). If $\hat{G}$ is not discrete and P is a measurable order on $\hat{G}$, then $\hat{G}$ is of the form $\mathbf{R}^a \times \Omega$ where a is necessarily positive. (This fact is a consequence of [**8**], Theorem (3.2).) In this case, any real-valued continuous homomorphism ψ on $\hat{G}$ which does not vanish identically on $\mathbf{R}^a \times \{0\}$ has a kernel that intersects every σ-compact subgroup of $\hat{G}$ in a set of measure 0. Hence if K is any compact subset of $\hat{G}$, we enlarge K to contain a compact subset of $\mathbf{R}^a \times \{0\}$ with a nonvoid interior (in the relative topology of $\mathbf{R}^a \times \{0\}$). Then any real-valued homomorphism that is obtained from the separation theorem (2.1) has on locally null kernel.

(2.4) *An approximation scheme for* $\tilde{f}$. We reproduce a construction from [**1**], (3.3) and (3.4), taking into account Theorem (2.2) and Remark (2.3). Suppose that f is a nonzero function in $\mathcal{L}^2(G)$. Its Fourier transform $\hat{f}$ vanishes off of a σ-compact open subgroup $\hat{G}_0$ of $\hat{G}$. Write

$$\hat{G}_0 = \bigcup_{n=1}^{\infty} K_n$$

where each K_n is a compact subset of $\hat{G}_0$ with nonvoid interior, and such that $K_n \subseteq K_m$ whenever $n \leqq m$. Suppose that G is compact. For each n, use Theorem (2.2) to obtain a real-valued homomorphism ψ_n such that

$$(1) \qquad \psi_n(K_n \cap (P \backslash \{0\})) > 0;$$

$$(2) \qquad \psi_n(K_n \cap ((-P) \backslash \{0\})) < 0;$$

and

$$(3) \qquad \operatorname{Ker} \psi_n \cap \hat{G}_0 = \{0\}.$$

Suppose that G is not compact. Use Theorem (2.1) and Remark (2.3) to obtain a sequence of continuous real-valued homomorphisms on $\hat{G}$, and a null subset N of $\hat{G}$, such that

$$(4) \qquad \psi_n(K_n \cap (P \backslash (N \cup \{0\}))) > 0;$$

$$(5) \qquad \psi_n(K_n \cap ((-P) \backslash (N \cup \{0\}))) < 0;$$

and

$$(6) \qquad \operatorname{Ker} \psi_n \text{is locally null.}$$

In either case, properties (1)–(6) imply that

$$(7) \qquad \lim_{j \to \infty} -i \operatorname{sgn}(\psi_j(\chi))\hat{f}(\chi) = -i \operatorname{sgn}_P(\chi)\hat{f}(\chi),$$

for almost all χ in $\hat{G}$. Using (1.3.2), we see that the expression on the left of (7) is the Fourier transform of $H_{\phi_j} f$, where ϕ_j is the adjoint homomorphism of ψ_j. Clearly, the expression on the right side of (7) is the Fourier transform of $\hat{\tilde{f}}$. Using Plancherel's theorem and (7), we see that $H_{\phi_j} f$ converges in the $\mathcal{L}^2$-norm, and hence in measure, to the function $\tilde{f}$. This fact will allow us to reduce the study of the distribution functions of the conjugate function $\tilde{f}$ to those of the ergodic Hilbert transform $H_{\phi_j} f$.

(2.5) *The distribution functions of* $H1_{[0,1]}$. We list a few formulas related to the distribution functions of the Hilbert transform of $1_{[0,1]}$. These formulas will be needed in Section 4.

(i) For all u in $\mathbf{R}$, we have:

$$H1_{[0,1]}(u) = \frac{1}{\pi}\log\left|\frac{1-u}{u}\right|.$$

(ii) For all $y > 0$, we have (from [**13**], or by computing directly):

$$\lambda^{+}_{H1_{[0,1]}}(y) = \frac{1}{\sinh(\pi y)} = \lambda^{-}_{H1_{[0,1]}}(y);$$

and

$$\lambda_{H1_{[0,1]}}(y) = \frac{2}{\sinh(\pi y)}.$$

(iii) For all $y > 0$, using (i) we find

$$|[0,1] \cap \{u \in \mathbf{R} : H1_{[0,1]}(u) > y\}| = \frac{1}{1+\exp(\pi y)};$$

and

$$|[0,1] \cap \{u \in \mathbf{R} : H1_{[0,1]}(u) < -y\}| = \frac{\exp(-\pi y)}{1+\exp(-\pi y)} = \frac{1}{1+\exp(\pi y)}.$$

Using (iii), we find that, for all $y > 0$:

$$\begin{aligned}|[0,1] \cap \{u \in \mathbf{R} : H1_{[0,1]}(u) \leqq y\}| &= 1 - \frac{1}{1+\exp(\pi y)}\\ &= \frac{\exp(\pi y)}{1+\exp(\pi y)}.\end{aligned}$$

Since all the distribution functions that we are dealing with here are continuous on $]0,\infty[$, we obtain from the second identity in (iii)

$$|[0,1] \cap \{u \in \mathbf{R} : H1_{[0,1]}(u) \leqq y\}| = \frac{\exp(-\pi y)}{1+\exp(-\pi y)}.$$

(iv) We combine the last two indentities to obtain the formula

$$|[0,1] \cap \{u \in \mathbf{R} : H1_{[0,1]}(u) \leqq y\}| = \frac{\exp(\pi y)}{1+\exp(\pi y)}$$

which holds for all real numbers y. Hence, for all real numbers y, we have

$$\begin{aligned}|[0,1] \cap \{u \in \mathbf{R} : H1_{[0,1]}(u) > y\}| &= 1 - \frac{\exp(\pi y)}{1+\exp(\pi y)}\\ &= \frac{1}{1+\exp(\pi y)}.\end{aligned}$$

(v) Let $y > 0$. Using the first identity in (ii) and the first identity in (iii), we get

$$|(\mathbf{R}\backslash[0,1] \cap \{u \in \mathbf{R} : H1_{[0,1]}(u) > y\}| = \frac{1}{\sinh(\pi y)} - \frac{1}{1+\exp(\pi y)} = \frac{1}{2\exp(\frac{1}{2}\pi y)\sinh(\frac{1}{2}\pi y)}.$$

(vi) Also, for $y > 0$, using the first identity in (ii) and the second identity in (iii), we find that

$$|(\mathbf{R}\backslash[0,1] \cap \{u \in \mathbf{R} : H1_{[0,1]}(u) < -y\}| = \frac{1}{\sinh(\pi y)} - \frac{1}{1+\exp(\pi y)} = \frac{1}{2\exp(\frac{1}{2}\pi y)\sinh(\frac{1}{2}\pi y)}.$$

3. The Weil formula on G.

(3.1) Throughout this section we suppose that G is a noncompact locally compact abelian group with Haar measure μ. The character group $\hat{G}$ need not be ordered at this point; but, we suppose that we have a continuous homomorphism ϕ from $\mathbf{R}$ into G which is also a topological isomorphism of $\mathbf{R}$ onto $\phi(\mathbf{R})$. As usual, we denote by ψ the adjoint homomorphism of ϕ. Since $\phi(\mathbf{R})$ is closed, we can normalize the Haar measure on $\phi(\mathbf{R})$, then normalize the Haar measure $\mu_{G/\phi(\mathbf{R})}$ on the quotient group $G/\phi(\mathbf{R})$, so that the Weil formula holds as follows:

$$(1) \qquad \int_G f(x)d\mu(x) = \int_{G/\phi(\mathbf{R})}\int_{\phi(\mathbf{R})} f(x+y)dw(y)d\mu_{G/\phi(\mathbf{R})}(x) = \int_{G/\phi(\mathbf{R})}\int_{\mathbf{R}} f(x+\phi(y))dyd\mu_{G/\phi(\mathbf{R})}(x)$$

for all f in $\mathcal{L}^1(G)$. (See **[10]**, Theorem (28.54.iii), p. 91.) To obtain the second equality in (1) we simply define the measure w on $\phi(\mathbf{R})$ by $w(A) = |\phi^{-1}(A)|$, for all Borel subsets A of $\phi(\mathbf{R})$. (See **[11]**, Theorem (12.46), p. 180.)

Fix a measurable set $A \subseteq G$ with $\mu(A) < \infty$. For $y > 0$, let

$$\xi_y(x) = \begin{cases} 1 & \text{if } H_\phi 1_A(x) > y; \\ 0 & \text{otherwise.} \end{cases}$$

We clearly have

$$\lambda^+_{H_\phi 1_A}(y) = \int_G \xi_y(x)d\mu(x).$$

Using the Weil formula, we obtain

$$(2) \qquad \lambda^+_{H_\phi 1_A}(y) = \int_{G/\phi(\mathbf{R})}\int_{\mathbf{R}} \xi_y(x+\phi(u))dud\mu_{G/\phi(\mathbf{R})}(x).$$

Note that, for a fixed x in G, we have

$$\xi_y(x+\phi(u)) = \begin{cases} 1 & \text{if } \dfrac{1}{\pi}\displaystyle\int_{\mathbf{R}} 1_A(x+\varphi(u-t))\frac{1}{t}\,dt > y; \\ 0 & \text{otherwise,} \end{cases}$$

where the last integral is to be computed as in (1.3.1). Using (1.2.2), we get

$$\begin{aligned} \int_{\mathbf{R}} \xi_y(x+\phi(u))du &= |\{u \in \mathbf{R} : H1_{\phi^{-1}(x-A)}(u) > y\}| \\ &= \frac{1}{\sinh(\pi y)} \int_{\mathbf{R}} 1_{\phi^{-1}(x-A)}(u)du \\ &= \frac{1}{\sinh(\pi y)} \int_{\mathbf{R}} 1_A(x+\phi(u))du. \end{aligned}$$

Putting this in (2), and using (1) again, we get

$$\text{(3)} \qquad \begin{aligned} \lambda^{+}_{H_\phi 1_A}(y) &= \int_{G/\phi(\mathbf{R})} \frac{1}{\sinh(\pi y)} \int_{\mathbf{R}} 1_A(x+\phi(u))du\,d\mu_{G/\phi(\mathbf{R})}(x) \\ &= \frac{1}{\sinh(\pi y)}\,\mu(A). \end{aligned}$$

A similar proof applies to $\lambda^{-}_{H_\phi 1_A}$ and yields the same formula as (3). We have thus proved the following theorem.

THEOREM (3.2). *Let G be a locally compact abelian group with Haar measure μ. Let ϕ be a topological isomorphism of $\mathbf{R}$ into G, and let A be a subset of G with finite measure. For every positive real number y, we have*

$$\text{(i)} \qquad \lambda^{+}_{H_\phi 1_A}(y) = \lambda^{-}_{H_\phi 1_A}(y) = \frac{1}{\sinh(\pi y)}\,\mu(A);$$

and so,

$$\text{(ii)} \qquad \lambda_{H_\phi 1_A}(y) = \frac{2}{\sinh(\pi y)}\,\mu(A).$$

4. The Stein and Weiss property of the conjugate function.

(4.1) We continue with the notation of the previous sections. We suppose further that ϕ is an arbitrary continuous homomorphism from $\mathbf{R}$ into G. Since the homomorphism ϕ is not necessarily a topological isomorphism, Theorem (3.2) does not apply directly. To make use of Theorem (3.2), we move the setting to $\mathbf{R} \times G$, and construct a topological isomorphism of $\mathbf{R}$ into $\mathbf{R} \times G$ as follows.

For $\alpha > 0$, define the homomorphism Φ_α, from $\mathbf{R}$ into $\mathbf{R} \times G$, by:

$$\Phi_\alpha(u) = (\alpha u, \phi(u)).$$

We easily verify that Φ_α is a topological isomorphism with adjoint homomorphism Ψ_α, defined on $\mathbf{R} \times \hat{G}$ by:

$$\Psi_\alpha(s, \chi) = \alpha s + \psi(\chi),$$

where ψ is the adjoint homomorphism of ϕ. (An easy way to show that Φ_α is a topological isomorphism follows by using [**9**], Theorem (9.1), p. 84.)

For f in $\mathcal{L}^2(\mathbf{R} \times G)$, and for a fixed $\alpha > 0$, we have from (1.3.2)

$$(1) \qquad (H_{\Phi_\alpha} f)\hat{}(s, \chi) = -i \operatorname{sgn}(\Psi_\alpha(s, \chi))\hat{f}(s, \chi) = -i \operatorname{sgn}(\alpha s + \psi(\chi))\hat{f}(s, \chi)$$

for almost all (s, χ) in $\mathbf{R} \times \hat{G}$. It also follows from Theorem (3.2) that

$$(2) \qquad \lambda^+_{H_\phi 1_A}(y) = \lambda^-_{H_\phi 1_A}(y) = \frac{1}{\sinh(\pi y)} \nu(A),$$

for all $y > 0$, and all measurable subsets A of $\mathbf{R} \times G$ with $\nu(A) < \infty$, where ν denotes the product measure on the group $\mathbf{R} \times G$. Suppose that $A = B \times C$, where $B \subset \mathbf{R}$, $C \subset G$, $|B| < \infty$, and $\mu(C) < \infty$. From (1), we get

$$(3) \qquad (H_{\Phi_\alpha} 1_{B\times C})\hat{}(s, \chi) = -i \operatorname{sgn}(\alpha s + \psi(\chi))\hat{1}_B(s))\hat{1}_C(\chi)$$

for a.a. s in $\mathbf{R}$, and a.a. χ in $\hat{G}$. Letting α decrease to 0, we obtain:

$$(4) \qquad \lim_{\alpha\downarrow 0}(H_{\Phi_\alpha} 1_{B\times C})\hat{}(s, \chi) = \begin{cases} -i \operatorname{sgn}(s)\hat{1}_B(s)\hat{1}_C(\chi) & \text{if } \psi(\chi) = 0, \\ -i \operatorname{sgn}(\psi(\chi))\hat{1}_B(s)\hat{1}_C(\chi) & \text{if } \psi(\chi) \neq 0. \end{cases}$$

To identify the limit in (4), we distinguish two cases.

Case I: Ker ψ is locally null. From (4) and (1.3.2), we see that

$$(5) \qquad \lim_{\alpha\downarrow 0}(H_{\Phi_\alpha} 1_{B\times C})\hat{}(s, \chi) = -i \operatorname{sgn}(\psi(\chi))\hat{1}_B(s)\hat{1}_C(\chi) = \hat{1}_B(s)(H_\phi 1_C)\hat{}(\chi),$$

where all the equalities hold almost everywhere on $\mathbf{R} \times \hat{G}$. It follows from (5), Plancherel's theorem, and Lebesgue's dominated convergence theorem, that

$$\lim_{\alpha\downarrow 0} H_{\Phi_\alpha} 1_{B\times C} = 1_B H_\phi 1_C, \quad \text{in the } \mathcal{L}^2(\mathbf{R} \times G)\text{-norm}.$$

Case II: Ker ψ is not locally null. In this case, Ker ψ is necessarily an open subgroup of $\hat{G}$. Let G_0 denote the annihilator in G of Ker ψ. The subgroup

G_0 is compact. Denote by μ_0 the normalized Haar measure on G_0. We have $\hat{\mu}_0 = 1_{\operatorname{Ker}\psi}$. Using (4) and reasoning as in Case I, we see that

$$\lim_{\alpha\downarrow 0} H_{\Phi_\alpha} 1_{B\times C} = 1_B H_\phi 1_C + (1_C * \mu_0) H 1_B, \quad \text{in the } \mathcal{L}^2(\mathbf{R}\times G)\text{-norm.}$$

Summarizing, we have

$$\begin{aligned} (6) \qquad g(.,.) &\equiv \lim_{\alpha\downarrow 0} H_{\Phi_\alpha} 1_{B\times C} \\ &= \begin{cases} 1_B H_\phi 1_C & \text{if } \ker\psi \text{ is locally null;} \\ 1_B H_\phi 1_C + (1_C * \mu_0) H 1_B & \text{if } \ker\psi \text{ is not locally null,} \end{cases} \end{aligned}$$

where the convergence is in the $\mathcal{L}^2(\mathbf{R}\times G)$-norm, and hence in measure. Using (2), and the convergence in measure described by (6), we find, that for all $y > 0$:

$$(7) \qquad \lambda_g^+(y) = \lambda_g^-(y) = \frac{\nu(A)}{\sinh(\pi y)} = \frac{|B|\mu(C)}{\sinh(\pi y)},$$

and

$$\lambda_g(y) = \frac{2\nu(A)}{\sinh(\pi y)} = \frac{2|B|\mu(C)}{\sinh(\pi y)}.$$

We are now ready to state and prove our main theorem. In what follows we will always assume that the Haar measure μ on a compact group G is normalized so that $\mu(G) = 1$.

THEOREM (4.2). *Let ϕ be an arbitrary continuous homomorphism from* $\mathbf{R}$ *into G. Let ψ denote the adjoint homomorphism of ϕ. Let E be an arbitrary measurable subset of G with $\mu(E) < \infty$. Suppose that either*

(i) *G is compact and* $\operatorname{Ker}\psi = \{0\}$,

or

(ii) *G is not compact and* $\operatorname{Ker}\psi$ *is locally null.*

Then, the distribution functions of $H_\phi 1_E$ depend only on $\mu(E)$ and whether G is compact or noncompact.

Proof. If $\mu(E) = 0$, then $H_\phi 1_E = 0$ a.e., and the theorem holds trivially in this case. Suppose throughout the rest of the proof that $\mu(E) \neq 0$. We show that $\lambda_{H1_E}^+$ depends only on $\mu(E)$ and whether G is compact or noncompact. A similar argument applies to $\lambda_{H1_E}^-$ and completes the proof of the theorem. We will start with the easier case.

Suppose that (ii) holds. Take $B = [0, 1]$, and $C = E$, in (4.1.6), and obtain

$$(1) \qquad g(u, x) = 1_{[0,1]}(u) H_\phi 1_E(x) \quad \nu\text{-a.e. on } \mathbf{R}\times G.$$

For all $y > 0$, we have from (4.1.7):

$$\text{(2)} \qquad \begin{aligned} \frac{\mu(E)}{\sinh \pi y} &= \nu(\{(u,x) : 1_{[0,1]}(u) H_\phi 1_E(x) > y\}) \\ &= \mu(\{x : H_\phi 1_E(x) > y\}) = \lambda^+_{H 1_E}(y). \end{aligned}$$

We have thus obtained a formula for $\lambda^+_{H_\phi 1_E}$ which depends only on $\mu(E)$. This proves the theorem in the case (ii).

Suppose that (i) holds. Again, take $B = [0, 1]$, and $E = C$, in (4.1.6); then

$$\text{(3)} \qquad g(u,x) = 1_{[0,1]}(u) H_\phi 1_E(x) + (1_{E^*} \mu_0(x)) H 1_{[0,1]}(u) \quad \nu\text{-a.e. on } \mathbf{R} \times G.$$

Since Ker $\psi = \{0\}$, the annihilator in G of Ker ψ is G, and hence $\mu_0 = \mu$. It follows that

$$1_{E^*} \mu_0(x) = 1_{E^*} \mu(x) = \mu(E),$$

and hence (3) becomes

$$\text{(4)} \qquad g(u,x) = 1_{[0,1]}(u) H_\phi 1_E(x) + \mu(E) H 1_{[0,1]}(u) \quad \nu\text{-a.e. on } \mathbf{R} \times G.$$

Note that (4) is an equation involving $H_\phi 1_E$ and other functions whose distribution functions are either independent of E, or depend only on $\mu(E)$. Therefore it is expected that the distribution functions of $H_\phi 1_E$ have similar properties. The rest of the proof is devoted to justifying this claim.

Using (2.5.i), we rewrite (4) as

$$\text{(5)} \qquad g(u,x) = 1_{[0,1]}(u) H_\phi 1_E(x) + \mu(E) \frac{1}{\pi} \log \left(\left| \frac{1-u}{u} \right| \right).$$

Let (X, ρ) be the measure space $[0, 1] \times G$, where ρ is the product of the Lebesgue measure on $[0, 1]$ and the Haar measure on G. Consider X as a subset of $\mathbf{R} \times G$, and let h denote the restriction of g to X. We have from (5)

$$\text{(6)} \qquad h(u,x) = H_\phi 1_E(x) + \mu(E) \frac{1}{\pi} \log \left(\frac{1-u}{u} \right), \qquad \rho\text{-a.e. on } X.$$

For $y \in \mathbf{R}$, let

$$\text{(7)} \qquad \begin{aligned} &\eta_h(y) = \rho(\{(u,x) \in X : h(u,x) > y\}); \\ &\eta_{H 1_{[0,1]}}(y) = \left| [0,1] \cap \left\{ u \in \mathbf{R} : \frac{\mu(E)}{\pi} \log \left(\frac{1-u}{u} \right) > y \right\} \right|; \end{aligned}$$

and

$$\eta_{H_\phi 1_E}(y) = \mu(\{x \in G : H_\phi 1_E(x) > y\}).$$

(Note that for $y > 0$, $\eta_{H_\phi 1_E}(y) = \lambda^+_{H_\phi 1_E}(y)$; and for $y \leqq 0$, $\eta_{H_\phi 1_E}(y) = 1 - \lambda^-_{H_\phi 1_E}(-y)$.) From (2.5.iv), we have

$$\eta_{H1_{[0,1]}}(y) = \frac{1}{1+\exp\left(\dfrac{\pi}{\mu(E)}y\right)}, \tag{8}$$

for all $y \in \mathbf{R}$.

Suppose for the moment that we have proved that η_h depends only on $\mu(E)$. Let $s \in \mathbf{R}$, then (6) implies that

$$\exp(-ish(u,x)) = \exp(-isH_\phi 1_E(x)) \exp\left(-is\,\frac{\mu(E)}{\pi}\log\left(\frac{u-1}{u}\right)\right). \tag{9}$$

Integrating (9), and rewriting the integrals as Riemann-Stieltjes integrals over $\mathbf{R}$, we get:

$$\begin{aligned}
&\int_X \exp(-ish(u,x))d\rho(u,x) \\
&= \int_G \exp(-isH_\phi 1_E(x))d\mu(x) \int_{[0,1]} \exp\left(-is\,\frac{\mu(E)}{\pi}\log\left(\frac{u-1}{u}\right)\right) du; \\
&- \int_{\mathbf{R}} e^{-isy} d\eta_h(y) = \int_{\mathbf{R}} e^{-isy} d\eta_{H_\phi 1_E}(y) \int_{\mathbf{R}} e^{-isy} d\eta_{H1_{[0,1]}}(y).
\end{aligned} \tag{10}$$

We have from (8):

$$\begin{aligned}
\int_{\mathbf{R}} e^{-isy} d\eta_{H1_{[0,1]}} &= \frac{-\pi}{\mu(E)} \int_{\mathbf{R}} e^{-isy} \frac{\exp\left(\dfrac{\pi}{\mu(E)}y\right)}{\left(1+\exp\left(\dfrac{\pi}{\mu(E)}y\right)\right)^2} dy \\
&= \frac{-\pi}{2\mu(E)} \int_0^\infty \cos sy \operatorname{sech}^2\left(\frac{\pi}{2\mu(E)}y\right) dy \\
&= -\mu(E)s \operatorname{csch}(\mu(E)s),
\end{aligned} \tag{11}$$

where the last integral can be found in a table of Fourier transforms. (See for instance [**12**], 7, p. 34.)

The relations (10) and (11) determine the Fourier-Stieltjes transform of the measure $d\eta_{H_\phi 1_E}$ everywhere on $\mathbf{R}$. They also show that this transform depends only on $\mu(E)$. Hence $\eta_{H_\phi 1_E}$ is determined completely from $\mu(E)$. Thus the proof of the present theorem will be complete once we prove that η_h depends only on $\mu(E)$, and not on E.

Fix $y > 0$. We have

$$\begin{aligned}
&\nu(\{(u,x) \in \mathbf{R} \times G : g(u,x) > y\}) \\
&\quad = \nu(\{(u,x) : u \in [0,1], \text{ and } g(u,x) > y\}) \\
&\quad + \nu(\{(u,x) : u \notin [0,1], \text{ and } g(u,x) > y\}).
\end{aligned} \tag{12}$$

For $u \notin [0, 1]$, we have from (5)

$$(13) \qquad g(u,x) = \mu(E) \frac{1}{\pi} \log \left(\left| \frac{1-u}{u} \right| \right);$$

and so, from (2.5.v), we get

$$(14) \qquad \nu(\{(u,x) : u \notin [0,1], \text{ and } g(u,x) > y\}) = \mu(G) \mid \{u : u \notin [0,1], \quad \text{and} \\ \frac{\mu(E)}{\pi} \log \left(\left| \frac{1-u}{u} \right| \right) > y\}| = \frac{1}{2 \exp\left(\frac{1}{2}\pi \frac{y}{\mu(E)}\right) \sinh\left(\frac{1}{2}\pi \frac{y}{\mu(E)}\right)}.$$

From (4.1.7), we have, for all $y > 0$:

$$(15) \qquad \nu(\{(u,x) \in \mathbf{R} \times G : g(u,x) > y\}) = \frac{\mu(E)}{\sinh(\pi y)}.$$

It follows from (12), (14), and (15) that

$$\nu(\{(u,x) : u \in [0,1], \text{ and } g(u,x) > y\}) \\ = \frac{\mu(E)}{\sinh(\pi y)} - \frac{1}{2 \exp\left(\frac{1}{2}\pi \frac{y}{\mu(E)}\right) \sinh\left(\frac{1}{2}\pi \frac{y}{\mu(E)}\right)}.$$

But since

$$\nu(\{(u,x) : u \in [0,1], \text{ and } g(u,x) > y\}) = \rho(\{(u,x) \in X : h(u,x) > y\})$$

it follows that, for $y > 0$, $\eta_h(y)$ depends only on $\mu(E)$. A similar argument using (2.5.vi) applies to $\eta_h(y)$, for $y < 0$, and completes the proof.

According to Theorem (4.2), to compute the distribution function of the ergodic Hilbert transform on a group G, when ϕ and ψ are as in Theorem (4.2), it is enough to consider the cases $G = \mathbf{R}$, for noncompact groups, and $G = \mathbf{T}$ for compact groups.

Consider the case $G = \mathbf{R}$, with $\hat{G} = \mathbf{R}$. Let ψ be the identity homomorphism: $x \longmapsto x$. The ergodic Hilbert transform is in this case the usual Hilbert transform. So the distribution functions of the ergodic Hilbert transform, when Ker ψ is locally null, are given by the Stein and Weiss formulas for the Hilbert transform on $\mathbf{R}$. This is also given by Theorem (3.2).

Now consider the case $G = \mathbf{T}$, and let ψ be the identity homomorphism from $\mathbf{Z}$ into $\mathbf{R}$. Then Ker $\psi = \{0\}$; and the adjoint homomorphism ϕ is given by $\phi(s) = s(\bmod 2\pi)$, for all $s \in \mathbf{T}$. In this case, the ergodic Hilbert transform becomes

$$(1) \qquad H_\phi f(x) = \lim_{n \to \infty} \frac{1}{\pi} \int_{\frac{1}{n} \leqq |t| \leqq n} f(x-t) \frac{1}{t}\, dt,$$

for all $f \in \mathcal{L}^1(\mathbf{T})$, where in the above integral the functions on $\mathbf{T}$ are considered as 2π-periodic functions on $\mathbf{R}$. From the uniqueness of the Fourier transform, it follows from (1.3.2) that the function $H_\phi f$ is *the* usual conjugate function of f on $\mathbf{T}$. (See Remark (4.4.a) *infra*.) Thus, according to Theorem (4.2), the distribution functions of the ergodic Hilbert transform, when G is compact and $\operatorname{Ker}\psi = \{0\}$, are precisely those of the conjugate function on $\mathbf{T}$. Thus, using the Stein and Weiss formulas on $\mathbf{T}$ and $\mathbf{R}$, we obtain the following useful corollary of Theorem (4.2).

COROLLARY (4.3). *Let ϕ, G, and E be as in Theorem* (4.2). *Let $y > 0$. Then,*
(i) *if G is not compact, we have*

$$\lambda_{H_\phi 1_E}(y) = \frac{2}{\sinh(\pi y)}\,\mu(E);$$

(ii) *and if G is compact, we have*

$$\exp\left(\frac{i}{2}\,\pi\lambda_{H_\phi 1_E}(y)\right) = \frac{\sinh(\pi y) + i\sin\left(\frac{1}{2}\,\mu(E)\right)}{\sinh(\pi y) - i\sin\left(\frac{1}{2}\,\mu(E)\right)}.$$

Remarks (4.4). (a) The relationship between the Hilbert transform on $\mathbf{R}$ and the conjugate function on $\mathbf{T}$ expressed by the limit (1) preceding Corollary (4.3) is well-known. See [**15**], (7.6), pp. 56–57; or, [**7**], pp. 6–7. In the cited references, the limit of the integral is identified using a series expansion of the cotangent function.

(b) The proof of Theorem (4.2) uses the Stein and Weiss results on $\mathbf{R}$ only. For this reason, one can derive (4.3.ii) without appealing to the Stein and Weiss results on $\mathbf{T}$. This can be done as follows. Given a measurable subset E of G, where G is compact. Let $\alpha = \mu(E)$, and suppose that ϕ is a continuous homomorphism of $\mathbf{R}$ into G as in Theorem (4.2). Then, according to (4.2.i), $H_\phi 1_E$ and $\tilde{1}_{[0,\alpha]}$ have the same distribution functions, where $\tilde{1}_{[0,\alpha]}$ denotes the conjugate function on $\mathbf{T}$ of $1_{[0,\alpha]}$. Hence, to derive (4.3.ii), it is enough to consider the distribution function of $\tilde{1}_{[0,\alpha]}$. One shows, with some effort, that the latter satisfies (4.3.ii).

We now establish the Stein and Weiss formulas for the conjugate function on locally compact, noncompact, abelian groups.

THEOREM (4.5). *The notation is borrowed from* (1.1) *and* (1.2). *Let G be a locally compact, noncompact, abelian group with Haar measure μ. Let P denote an arbitrary Haar measurable order on $\hat{G}$. Let E be a measurable subset of G with $\mu(E) < \infty$. For $y > 0$, we have*

(i) $$\lambda_{\tilde{1}_E}(y) = \frac{2\mu(E)}{\sinh \pi y}.$$

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Proof. From (2.4) (the noncompact case), $\tilde{1}_E$ can be approximated in the $\mathcal{L}^2$-norm by ergodic Hilbert transforms that are defined with respect to homomorphisms with the properties enunciated in Theorem (4.2.ii). By Corollary (4.3.i), each one of these transforms has distribution functions that satisfy identity (i) of the present theorem. The theorem follows now, since these transforms converge in measure to $\tilde{1}_E$.

Next we establish the Stein and Weiss formulas for the conjugate function on compact abelian groups.

THEOREM 4.6. *Let G be a compact abelian group with normalized Haar measure μ. Let P be an arbitrary order on $\hat{G}$, and let E be a measurable subset of G. For $y > 0$, we have*

$$\text{(i)} \qquad \exp\left(\frac{i}{2}\pi\lambda_{\tilde{1}_E}(y)\right) = \frac{\sinh(\pi y) + i\sin(\frac{1}{2}\mu(E))}{\sinh(\pi y) - i\sin(\frac{1}{2}\mu(E))}.$$

Proof. We treat a particular case first: $\hat{G}$ is countable. Take $\hat{G}_0 = \hat{G}$ and $f = 1_E$ in (2.4) (the compact case). To prove (ii), argue as in Theorem (4.5), using Theorem (4.2.i) and Corollary (4.3.ii). Now suppose that $\hat{G}$ is arbitrary. Let $\hat{G}_0$ be the subgroup of $\hat{G}$ generated by supp $\hat{1}_E$. Then $\hat{G}_0$ is a countable subgroup of $\hat{G}$. Order $\hat{G}_0$ by $P \cap \hat{G}_0$. Let H denote the annihilator in G of $\hat{G}_0$, and let μ_H denote the normalized Haar measure on H. We have

$$\hat{1}_E = \hat{1}_E \hat{\mu}_H;$$

and hence from the uniqueness of the Fourier transform

$$1_E = 1_E * \mu_H \ \mu\text{-a.e. on } G.$$

Let $\pi : G \longrightarrow G/H$ denote the natural homomorphism of G onto G/H. We have

$$\text{(1)} \qquad 1_E = 1_{\pi(E)} \circ \pi \ \mu\text{-a.e. on } G.$$

For an arbitrary f in $\mathcal{L}^p(G)$, where $1 \leqq p < \infty$, and such that f is constant on the cosets on H, we have

$$f = f * \mu_H = F \circ \pi \ \mu\text{-a.e. on } G,$$

where $F \in \mathcal{L}^p(G/H)$. (See [**10**], Theorem (28.55), p. 95.) By comparing Fourier transforms, it is easy to verify that

$$\text{(2)} \qquad \tilde{f} = \tilde{F} \circ \pi \ \mu\text{-a.e. on } G,$$

where as usual $\tilde{f}$ denotes the conjugate function of f on G (with respect to the order P), and $\tilde{F}$ denotes the conjugate function of F on G/H (with respect to the order $P \cap \hat{G}_0$). (Here we have identified the character group of G/H with $\tilde{G}_0$. See [9], Theorem (24.11).) From (1) and (2), we get

$$(3) \qquad \tilde{1}_E = \tilde{1}_{\pi(E)} \circ \pi \ \mu\text{-a.e. on } G.$$

Since the character group of G/H is countable, we appeal to the particular case treated earlier to infer that the distribution function of $\tilde{1}_{\pi(E)}$ satisfies the identity (i) on the group G/H, with the normalized Haar measure $\mu_{G/H}$. Using [10], Theorem (28.54.v), we see that: for $y > 0$,

$$\begin{aligned}\mu(\{x \in G : |\tilde{1}_E(x)| > y\}) &= \mu(\{x \in G : |\tilde{1}_{\pi(E)} \circ \pi(x)| > y\}) \\ &= \mu_{G/H}(\{x \in G/H : |\tilde{1}_{\pi(E)}(x)| > y\});\end{aligned}$$

and

$$\mu(E) = \mu_{G/H}(\pi(E)).$$

The theorem follows now from the last identities, and the fact that (i) holds on G/H.

Different proofs of the Stein and Weiss formulas on **R** or **T** can be found in [**5**], and [**14**], p. 238–240.

We end the essay with an example that shows that the Stein and Weiss formulas fail in general for the ergodic Hilbert transform.

Example (4.7). Suppose that G is compact, so that $\hat{G}$ is discrete. Let $\psi : \hat{G} \longmapsto \mathbf{R}$ be an arbitrary homomorphism with a kernel $K \neq \{0\}$. Let G_0 be the annihilator in G of K, and let A be an arbitrary subset of G, with positive Haar measure $\mu(A)$, and such that 1_A is constant on the cosets of G_0. It follows that $\hat{1}_A$ is supported in K. Using (1.3.2), we see that $H_\phi 1_A \equiv 0$ on G. Clearly the Stein-Weiss formulas do not hold in this case.

Added in proof. After this paper was to appear, Professor Guido Weiss kindly brought to our attention the paper *Conjugate function theory in weak* Dirichlet algebras*, by I. I. Hirschmann, Jr. and R. Rochberg, in J. Functional Analysis *16* (1974) 359–371. Our last two theorems can be derived from this paper.

References

1. N. Asmar, *Exponential estimates for the conjugate function on locally compact abelian groups*, Can. Math. Bull., (to appear).
2. N. Asmar, Earl Berkson, and T. A. Gillespie, *Representation of groups with ordered duals and generalized analyticity*, J. of Funct. Anal. (to appear).
3. N. Asmar and Edwin Hewitt, *Marcel Riesz's theorem on conjugate Fourier series and its descendants*, Proceedings of the Analysis Conference (Singapore, 1986). North-Holland, Mathematics Series *150* (Elsevier Publishers, New York, 1988), 1–56.
4. E. Berkson, and T. A. Gillespie, *The generalized M. Riesz theorem and transference*, Pacific J. Math. *120* (1985), 279–288.

5. A. Calderón, *Singular integrals*, Bull. A.M.S. *72* (1966), 427–465.
6. ——— *Ergodic theory and translation-invariant operators*, Proc. Nat. Acad. Sc., U.S.A. *59* (1968), 349–353.
7. R. R. Coifman and G. Weiss, *Transference methods in analysis*, Regional Conference Series in Math. *31* (Amer. Math. Soc., Providence, R.I., 1977).
8. Edwin Hewitt and S. Koshi, *Orderings in locally compact abelian groups and the theorem of F. and M. Riesz*, Mat. Proc. of Camb. Phil. Soc. *93* (1983), 441–457.
9. Edwin Hewitt and K. A. Ross, *Abstract harmonic analysis*, Vol. I, Grundlehren der mathematischen Wissenschaften *115*, Second Edition (Springer-Verlag, 1970).
10. ——— *Abstract harmonic analysis*, Vol. II, Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen *152* (Berlin, Heidelberg, New York, 1970).
11. Edwin Hewitt and K. Stromberg, *Real and abstract analysis*, Graduate Texts in Mathematics *25*, (Berlin, Heidelberg, New York: Springer, 1969).
12. F. Oberhettinger, *Tabellen zur Fourier Transformation*, Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen *90* (Berlin, Göttingen, Heidelberg, Springer-Verlag, 1957).
13. E. M. Stein and G. Weiss, *An extension of a theorem of Marcinkiewicz and some applications*, Journal of Math. and Mechanics *8* (1959), 263–284.
14. ——— *Introduction to Fourier analysis on Euclidean spaces*, 2nd Printing (Princeton Mathematical Series, Princeton Univ. Press, Princeton, New Jersey, 1975).
15. A. Zygmund, *Trigonometric series*, Vol. I and II (Cambridge University Press, Second Edition, 1959).

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