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A Note on Integer Symmetric Matrices and Mahler's Measure

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Abstract. We find a lower bound on the absolute value of the discriminant of the minimal polynomial of an integral symmetric matrix and apply this result to find a lower bound on Mahler's measure of related polynomials and to disprove a conjecture of D. Estes and R. Guralnick.

This note was inspired by the work of J. McKee and C. J. Smyth on Mahler's measure of Pisot and Salem numbers defined by graphs [3]. The adjacency matrix of a graph is a symmetric integer matrix, thereby Cauchy's interlacing theorem provides a simple, but effective tool to study its characteristic polynomial. Unfortunately, as we show below, many totally real integral polynomials cannot be represented by symmetric integral matrices. We use the following notation: $S_n(\mathbb{Z})$ denotes the set of all $n \times n$ integer symmetric matrices. For a square matrix M, $p_M(x) = \det(xI - M)$ denotes its characteristic polynomial. Mahler's measure of a polynomial f is defined by

$$M(f) = |a_0| \prod_{i=1}^{n} \max(1, |\alpha_i|),$$

where a_0 is the leading coefficient of f, and the product runs over all its (possibly multiple) zeros. The discriminant of f is denoted $\operatorname{Disc}(f) = a_0^{2n-2} \prod_{i<j}(\alpha_i - \alpha_j)^2$. We say that f is reciprocal if $f(x) = \pm x^{\deg f} f(x^{-1})$. For a monic integral reciprocal polynomial f satisfying an extra condition $f(\pm 1) \neq 0$, we define

$$\tilde{f}(x) = \prod_{i=1}^{n} (x - \alpha_i - \alpha_i^{-1}),$$

where $\alpha_1, \alpha_1^{-1}, \dots, \alpha_n, \alpha_n^{-1}$ is the multiset of zeros of f. Clearly $\tilde{f} \in \mathbb{Z}[x]$. Moreover, the identity $(\alpha_i + \alpha_i^{-1} - \alpha_j - \alpha_j^{-1})^2 = (\alpha_i - \alpha_j)(\alpha_i^{-1} - \alpha_j^{-1})(\alpha_i - \alpha_j^{-1})(\alpha_i^{-1} - \alpha_j)$, implies that $\operatorname{Disc}(\tilde{f})^2$ divides $\operatorname{Disc}(f)$. Further, if all zeros of f are either real or lie on the unit circle then $\tilde{f}$ is totally real. In particular, $\tilde{f}$ is totally real when f is the minimal polynomial of a Salem number.

If g and f are totally real polynomials, $\deg g = n-1$, $\deg f = n$, $\mu_1 \leq \cdots \leq \mu_{n-1}$ are the zeros of g, and $\lambda_1 \leq \cdots \leq \lambda_n$ are the zeros of f, then we write $g \prec f$ if the zeros are interlacing, *i.e.*, $\lambda_1 \leq \mu_1 \leq \cdots \leq \mu_{n-1} \leq \lambda_n$.

Cauchy's interlacing theorem says that $p_{M_i} \prec p_M$, for any submatrix M_i obtained from a square matrix M by deleting its i-th row and column.

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Lemma 1 *Let $M \in S_N(\mathbb{Z})$. If $p_M = f^m$, where f is an irreducible polynomial of degree n, then $|\operatorname{Disc}(f)| \geq n^n$.*

Proof The lemma is trivial for $n = 1$. Suppose that $n \geq 2$. Remove the first row and column from M, and let $\Delta_1(x)$ be the determinant of the resulting matrix. By Cauchy's interlacing theorem, the zeros of Δ_1 interlace with the zeros of f^m. Hence $f^{m-1}|\Delta_1$. Let $f_1 = \Delta_1/f^{m-1}$. Then $f_1 \in \mathbb{Z}[x]$ is a monic polynomial of degree $n-1$, and its zeros still interlace with the zeros of f. Since the zeros of f' also interlace with the zeros of f, we must have $f_1(\lambda)f'(\lambda) > 0$ at every zero λ of f. Denote by $\operatorname{sp}(f)$ the set of all zeros of f. Let t_λ for $\lambda \in \operatorname{sp}(f)$ be positive real numbers such that

$$f_1(\lambda) = t_\lambda f'(\lambda), \text{ for } \lambda \in \operatorname{sp}(f).$$

We have $f'(\lambda) = g_\lambda(\lambda)$, where $g_\lambda(x) = \frac{f(x)}{x-\lambda}$. Consider the polynomial $G(x) = f_1(x) - \sum_{\lambda\in\operatorname{sp}(f)} t_\lambda g_\lambda(x)$. Since $G(\lambda) = 0$ for every $\lambda \in \operatorname{sp}(f)$, and $\deg G \leq n-1$, we conclude that G is identically 0. Hence

$$f_1(x) = \sum_{\lambda\in\operatorname{sp}(f)} t_\lambda g_\lambda(x). \tag{1}$$

Moreover,

$$\sum_{\lambda\in\operatorname{sp}(f)} t_\lambda = 1, \tag{2}$$

since all polynomials involved are monic. By using the fact that $g_\lambda(\mu) = 0$ for $\mu \in \operatorname{sp}(f)$ other than λ, and the arithmetic-geometric means inequality, we get

$$1 \leq \left|\prod f_1(\lambda)\right| = \left|\prod t_\lambda g_\lambda(\lambda)\right| = \left|\prod t_\lambda f'(\lambda)\right| \leq \left(\frac{\sum t_\lambda}{n}\right)^n |\operatorname{Disc}(f)|,$$

where the products and the sum run over $\lambda \in \operatorname{sp}(f)$. Thus, by (2), $|\operatorname{Disc}(f)| \geq n^n$. ■

D. Estes and R. Guralnick [2, p.84] conjectured that every totally real separable monic polynomial can occur as the minimal polynomial of a symmetric integral matrix. Lemma 1 shows that without additional conditions this is not the case. We have the following.

Corollary 1 *There are infinitely many totally real monic polynomials that cannot occur as minimal polynomials of a symmetric integral matrix.*

This conclusion is based on the fact that there are infinitely many totally real polynomials of degree n with root discriminant smaller than n. For example, D. Simon [4, Proposition IV.3.5] showed that if m is a product of consecutive primes, $m = \prod_{p<x} p$, then

$$|\operatorname{Disc}\Phi_m|^{\frac{1}{\phi(m)}} \sim e^{2\gamma}\phi(m)\frac{\log\log\phi(m)}{\log\phi(m)},$$

where Φ_m denotes the m-th cyclotomic polynomial. To deal with totally real polynomials it suffices to notice that $|\operatorname{Disc}\tilde{\Phi}_m| \leq |\operatorname{Disc}\Phi_m|^{1/2}$.

Another application of the Lemma is given by the following.

Theorem 1 *Suppose that $f \in \mathbb{Z}[x]$ is a monic, irreducible, and reciprocal polynomial that is not cyclotomic and its zeros $\alpha_1, \alpha_1^{-1}, \dots, \alpha_n, \alpha_n^{-1}$ are either real or lie on a unit circle. If $\tilde{f} = \prod_{i=1}^{n}(x - \alpha_i - \alpha_i^{-1})$ is a characteristic polynomial of an integer symmetric matrix, then*

$$M(f) > 1.043.$$

Note: For the sake of simplicity no attempt was made to optimize the bound in this theorem. In fact, a stronger result by the graph theory approach is expected.

Proof By Lemma 1, $|\operatorname{Disc}(\tilde{f})| \geq n^n$. Hence, $|\operatorname{Disc}(f)| \geq \operatorname{Disc}(\tilde{f})^2 \geq n^{2n}$. Let A be the $4n \times 4n$ Vandermonde matrix with rows $[1, \alpha_i, \dots, \alpha_i^{4n-1}]$, $i = 1, \dots, 2n$, and $[1, \alpha_i^p, \dots, \alpha_i^{(4n-1)p}]$, $i = 1, \dots, 2n$, where p is a prime that will be determined later. Consider the determinant of A. By Hadamard's inequality on the left, and [1, Lemma 2] on the right-hand side, we get

$$(4n)^{2n} M(f)^{(4n-1)(p+1)} \geq |\det(A)| \geq |\operatorname{Disc}(f)| p^{2n} \geq \operatorname{Disc}(\tilde{f})^2 p^{2n}.$$

Hence, by Lemma 1,

$$M(f) \geq \left(\frac{p}{4}\right)^{\frac{1}{2(p+1)}}.$$

With $p = 11$, this gives $M(f) > 1.043$. ■

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