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THERE ARE NO CHAOTIC MAPPINGS WITH RESIDUAL SCRAMBLED SETS

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A continuous mapping f of a compact real interval I to itself which is chaotic in the sense of Li and Yorke, cannot have a scrambled set residual in an subinterval of I. This shows that chaos in this case cannot be large from a topological point of view.

We shall consider continuous mappings $f: I \to I$, where I is a real compact interval. The notion of chaos for such mappings was introduced by Li and Yorke [4]. However, the following equivalent formulation is more convenient.

DEFINITION ([2], [8]). *A mapping* $f: I \to I$ *is* δ*-chaotic if there is a nonempty perfect set* S *such that for any* $x, y \in S$, $x \neq y$, *and any periodic point* p *of* f,

$$(1) \qquad \limsup_{n \to \infty} |f^n(x) - f^n(y)| > \delta$$

$$(2) \qquad \liminf_{n \to \infty} |f^n(x) - f^n(y)| = 0$$

$$(3) \qquad \limsup_{n \to \infty} |f^n(x) - f^n(p)| > \delta$$

where f^n *denotes the* n*-th iterate of* f, *for* $n = 0, 1, 2, \ldots$. *The*

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set S *is then called* δ*-scrambled (or simply scrambled when* $\delta = 0$*) set for* f .

We recall that f is chaotic when the topological entropy of f is positive (or equivalently, when f has a cycle of order not a power of 2), but there are also chaotic mappings with zero topological entropy (see [8]).

It is well-known that there exist functions with scrambled sets of positive Lebesgue measure (see [3], [5], [6], [7]). By a simple modification of the argument from [7] we can see that (assuming the continuum hypothesis) the standard "tent" mapping defined by $f(x) = 1 - |1 - 2x|$ for $x \in [0, 1]$, has a scrambled set which is of the second Baire category on every subinterval of $[0, 1]$. Another construction can be found in [1].

However, none of the above examples yields a function with a scrambled set residual in a certain interval. The problem is whether such a function does exist. A partial negative solution, for functions satisfying a strong restrictive additional condition, is given by Bruckner and Thakyin Hu [1]. More precisely, they assume that the corresponding function f satisfies on the scrambled set S the condition $\limsup_{n \to \infty} |f^n(x) = f^n(y)| = \mu(I)$ (the measure of I), instead of the much weaker condition (1) with $\delta = 0$.

In the present note we show that the answer is negative in general. Our main result is the following

THEOREM. *Let* $f: I \to I$ *be a continuous function and* S *a scrambled set for* f . *Then* S *is residual in no subinterval of* I .

Our proof of the theorem is based on some lemmas.

LEMMA 1. *Let* $f: I \to I$ *be a continuous function and* S *a scrambled set for* f . *Let* $J_0 \subset I$ *be an interval such that* $S \cap J_0$ *contains at least two different points* u, v . *Then there are positive integers* r, s *such that*

$$\mathrm{int}\,(f^r(J_0)) \cap \mathrm{int}\,(f^{r+s}(J_0)) \neq \emptyset .$$

Proof. By (1) $\limsup_{n \to \infty} |f^n(u) - f^n(v)| = \varepsilon > 0$. Hence

$\limsup_{n\to\infty} (\operatorname{diam} f^n(J_0)) > \varepsilon/2$, that is, $\operatorname{diam} f^m(J_0) > \varepsilon/2$ for infinitely many m's . Since the measure $\mu(I)$ of I is finite, we get immediately the statement. □

LEMMA 2. (see [1]). *Let f be continuous on an interval $[a, b]$ and S be a residual subset of $[a, b]$. If f is one to one on S , then $f(S)$ is residual in $f([a, b])$.*

Since [1] is not generally available, we give here also the proof.

Proof. of Lemma 2. Let H be a dense set of type G_σ contained in S . Then $f(H)$ is a Borel set, being the one to one continuous image of a set of type G_σ .

Thus, if $f(H)$ is not residual in $f([a, b])$, then there exists an open interval J such that $f(H) \cap J$ is of first category. Let I be a component interval of the open set $f^{-1}(J)$ and $H^* = H \cap I$. Since $f(H^*)$ is a first category subset of J, there exist nowhere dense sets $B_1, B_2, \ldots$ such that $f(H^*) = \bigcup_{k=1}^{\infty} B_k \subseteq \bigcup_{k=1}^{\infty} \bar{B}_k \subseteq \bar{J}$. Then $H^* \subseteq \bigcup_{k=1}^{\infty} (f^{-1}(\bar{B}_k) \cap \bar{I})$. Since H^* is residual in I , the same is true for $\bigcup_{k=1}^{\infty} (f^{-1}(\bar{B}_k) \cap \bar{I})$. This implies that there is a k such that the closed set $f^{-1}(\bar{B}_k) \cap \bar{I}$ contains an interval L . But H^* is residual in L and f is one to one on H^* , so the set $f(L)$ is a non-degenerate interval. This is impossible since $f(L) \subseteq \bar{B}_k$, a nowhere dense set. □

LEMMA 3. *Let $f: I \to I$ be a continuous function, let J be a subinterval of I and S be a residual subset of J . If f is one to one on S , then f is monotonic on J .*

Proof. Let f be not monotonic on J . Then there exist points $a < b < c$ such that $f(a) = f(c) \neq f(b)$. Without loss of generality assume $\max\{f(x): x \in [a, b]\} = f(b)$. Since f is one to one on S , by Lemma 2 $f(S \cap [a, b])$ and $f(S \cap [b, c])$ are residual subsets of $f([a, b])$. Thus these two sets have non-empty intersection, which is impossible since f is one to one on S . □

LEMMA 4. *Let f be continuous and S be a scrambled set for f . Then f^n is one to one on $f^m(S)$, for $m, n = 0, 1, 2, \ldots$.*

Proof. Assume that there are $u, v \in f^m(S)$, $u \neq v$, such that $f^n(u) = f^n(v)$. Choose $x, y \in S$ with $f^m(x) = u$, $f^m(y) = v$. Then $x \neq y$ and $f^{m+n}(x) = f^{m+n}(y)$, contrary to (1). □

Now we are able to prove the main result.

Proof of Theorem. Assume contrary to what we wish to show that S is a scrambled set for f, residual in a subinterval L of I. By Lemma 1 there are positive integers r, s such that

(4) $$\operatorname{int}(f^{r+s}(L)) \cap \operatorname{int}(f^r(L)) \neq \emptyset .$$

Let $f^r(L) = J$, $f^r(S) = S_0$. By (4), $\operatorname{int}(f^s(J)) \cap \operatorname{int}(J) \neq \emptyset$ hence $\operatorname{int}(f^{2s}(J)) \cap \operatorname{int}(f^s(J)) \supset \operatorname{int}(f^s(f^s(J) \cap J)) \neq \emptyset$ and by induction we get immediately

(5) $$\operatorname{int}(f^{ks}(J)) \cap \operatorname{int}(f^{(k-1)s}(J)) \neq \emptyset , \; k = 0, 1, 2, \ldots$$

Consequently, $J_0 = \bigcup_{k=0}^{\infty} f^{ks}(J)$ is a connected set, that is an interval. We show that f^s is monotonic on J_0. Since S is residual in L, Lemmas 4 and 2 imply

(6) $$f^{ks}(S_0) \text{ is residual in } f^{ks}(J) , \; k = 1, 2, \ldots$$

and Lemma 4 gives

(7) $$f^s \text{ is one to one on } f^{ks}(S_0), \; k = 1, 2, \ldots .$$

Now by (6), (7) and Lemma 3, f^s is monotonic on each of the intervals $\{f^{ks}(J)\}_{k=1}^{\infty}$ and the intersection property (5) implies that f^s is monotonic on J_0.

Put $g = f^{2s}$. Since f^s is monotonic, g is non-decreasing, and clearly $g^k(x) \in J_0$ for every $x \in S \cap J$ and every k.

Finally, consider the sequence $\{g^k(x)\}_{k=1}^{\infty}$, for $x \in S \cap J$. Since g is non-decreasing, $g(x) < x$ implied $g^k(x) < g^{k-1}(x)$ and $\{g^k(x)\}_{k=0}^{\infty}$ is decreasing. Similarly $\{g^k(x)\}_{k=0}^{\infty}$ is increasing if $g(x) > x$. Thus

$\lim_{k\to\infty} g^k(x) = \alpha$ exists. Since $g(\alpha) = g(\lim_{k\to\infty} g^k(x)) = \lim_{k\to\infty} g^{k+1}(x) = \alpha$, α is a fixed point of g.

For $i = 1, 2, \ldots, 2s$ write $\alpha_i = f^i(\alpha)$. The points α_i form a periodic orbit and by the continuity of f, we have

$$\alpha_i = f^i(\lim_{k\to\infty} g^k(x)) = \lim_{k\to\infty} g^k(f^i(x)) = \lim_{k\to\infty} f^{2ks+i}(x)$$

that is the sequence $\{f^n(x)\}_{n=1}^{\infty}$ is asymptotically periodic, contrary to the fact that x belongs to a scrambled set (see (3)). □

Remarks. 1. In the proof the condition (3) is not necessary, since every set S such that (1) and (2) is true for $x, y \in S$, $x \neq y$, can clearly contain at most one point which has an asymptotically periodic trajectory.

2. A simple modification of the above argument shows that the theorem is also true for continuous mappings of the circle.

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