Glasgow Math. J. **49** (2007) 213–224.
doi:10.1017/S0017089507003679. Printed in the United Kingdom

EXISTENCE OF A SOLUTION FOR A SINGULAR DIFFERENTIAL EQUATION WITH NONLINEAR FUNCTIONAL BOUNDARY CONDITIONS*

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(Received 7 April, 2006; revised 12 December, 2006; accepted 13 February, 2007)

Abstract. In this paper we deal with some boundary value problems related with diffusion processes in the presence of lower and upper solutions. Singularities as well as non local boundary conditions are allowed. We also prove the existence of extremal solutions and the uniqueness of solution for a particular case.

2000 *Mathematics Subject Classification.* 34B15, 34B16.

1. Introduction. Diffusion processes are very important and they appear in many applications ranging from the study of a system of interacting diffusive particles with finite range random interaction [**8**] to the growth on aluminium cluster surfaces [**23**].

Nonlinear diffusion equations arise also in a variety of problems from semiconductor fabrication [**16**] and the determination of the equivalent internal heat source from surface temperature measurements in microwave processing of materials [**11**], to the properties of electromagnetic fields in superconductors with ideal and gradual resistive transitions [**17**].

Equations of the form

$$(k(u)u')'(x) = f(x, u(x), u'(x)),$$

with the initial conditions

$$u(0) = 0, \qquad \lim_{x\to 0^+} k(u)u'(x) = 0,$$

have been recently studied in connection with several diffusion problems such as semiconductor fabrication [**15**], infiltration of water from reservoirs [**20**] and the problem of the diffusion of a dopant through a semiconductor [**2**, **21**, **22**]. Some extensions were given in [**3**, **4**, **5**, **19**] where the authors considered more general problems and weakened considerably the assumptions.

*Partially supported by D.G.I. and F.E.D.E.R. project BFM2001-3884-C02-01, and by Xunta of Galicia and F.E.D.E.R. project PGIDT05PXIC20702PN, Spain.

In this paper we study the equation

$$-(k(t,u(t))u'(t))' = f(t,u(t)) \quad \text{for a. a. } t \in [0,1],$$

subject to different kinds of nonlinear boundary conditions which include, among others, the Dirichlet, periodic or multipoint as particular cases. With this presentation we can consider different boundary value problems under the same formulation. Similar nonlinear boundary conditions for second order ordinary differential equations have been considered in [**1**, **13**], but in that case functional dependence is not allowed. ϕ – laplacian equations with nonlinear functional boundary conditions can be found in [**6**, **7**].

Assuming the existence of a well ordered pair of lower and upper solutions $\alpha \leq \beta$ we prove the existence of at least one solution lying between them. We remark that $k(0,x)$ or $k(1,x)$ may be zero and therefore we are dealing with singular equations.

The paper is organized as follows: in section 2 we present an existence result, in Section 3 we prove the existence of extremal solutions and we give some conditions to ensure the uniqueness of a solution whenever $k(t,x) \equiv k(t)$ and some particular boundary value conditions are considered. Finally, in section 4, we present some examples of the applicability of our results.

2. Existence results. In this section we study the problem

$$\begin{cases} -(k(t,u(t))u'(t))' = f(t,u(t)) & \text{for a.a. } t \in I, \\ L_1(u(0),u(1),u) = 0, \\ L_2(u(0),u(1)) = 0, \end{cases} \tag{2.1}$$

under the following assumptions:

(i) $k : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $k(t,x) > 0$ for all $t \in (0,1)$ and all $x \in \mathbb{R}$. Moreover, for each $r > 0$ there exists $p_r \in L^1(I)$ such that

$$\frac{1}{k(t,x)} \leq p_r(t) \text{ for a.a. } t \in I \text{ and all } x \in [-r,r].$$

(ii) $f : I \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, i.e. for a.a. $t \in I$ the function $f(t,\cdot)$ is continuous, for all $x \in \mathbb{R}$ the function $f(\cdot,x)$ is measurable, and for all $r > 0$ there exists $h_r \in L^1(I)$ such that for a.a. $t \in I$ and all $x \in [-r,r]$ we have that

$$|f(t,x)| \leq h_r(t).$$

(iii) $L_1 : \mathbb{R}^2 \times C(I) \to \mathbb{R}$ is continuous and for all $(x,y) \in \mathbb{R}^2$ the function $L_1(x,y,\cdot)$ is nondecreasing.

(iv) $L_2 : \mathbb{R}^2 \to \mathbb{R}$ is continuous, for all $y \in \mathbb{R}$ the function $L_2(\cdot,y)$ is nonincreasing and for all $x \in \mathbb{R}$ the function $L_2(x,\cdot)$ is injective.

REMARK 2.1. Our boundary conditions include Dirichlet conditions (for the choice $L_1(x,y,u) = x$ and $L_2(x,y) = y$) as well as a great variety of non local boundary conditions such as

$$\max_{t \in I}\{u(t)\} = c, \;\; u(0) = u(1)$$

https://doi.org/10.1017/S0017089507003679 Published online by Cambridge University Press

or

$$\int_0^1 u(s)ds = c, \; u(1) = d.$$

DEFINITION 2.1. We say that $\alpha \in C(I)$ is *a lower solution* of problem (2.1) if for all $t_0 \in (0, 1)$ either

$$D_-\alpha(t_0) < D^+\alpha(t_0),$$

or there exists an open interval $I_0 \subset (0, 1)$ such that $t_0 \in I_0$, $\alpha \in C^1(I_0)$, $k(t, \alpha(t))\alpha'(t) \in AC(I_0)$ and

$$\begin{cases} -(k(t, \alpha(t))\alpha'(t))' \le f(t, \alpha(t)) & \text{for a.a. } t \in I_0, \\ L_1(\alpha(0), \alpha(1), \alpha) \ge 0, \\ L_2(\alpha(0), \alpha(1)) = 0. \end{cases}$$

Analogously we say that $\beta \in C(I)$ is an *upper solution* of problem (2.1) if all the above inequalities are reversed and with the Dini derivatives $D_-\alpha(t_0)$ and $D^+\alpha(t_0)$ changed into $D^-\beta(t_0)$ and $D_+\beta(t_0)$.

We say that $u \in \mathcal{S} := \{u \in AC(I) : k(\cdot, u(\cdot))u'(\cdot) \in AC(I)\}$ is *a solution* of problem (2.1) if it satisfies the equation and the boundary conditions of (2.1).

Whenever $\alpha \le \beta$ we say that a solution x^* of problem (2.1) is the *maximal solution* in the set

$$[\alpha, \beta] := \{u \in \mathcal{C}(I) : \alpha(t) \le u(t) \le \beta(t) \quad \text{for all } t \in I\},$$

if $x^* \in [\alpha, \beta]$ and $x^* \ge x$ for any other solution $x \in [\alpha, \beta]$. The *minimal solution* in $[\alpha, \beta]$, is defined analogously by reversing the inequalities; when both the minimal and the maximal solutions in $[\alpha, \beta]$ exist, we call them the *extremal solutions* in $[\alpha, \beta]$.

REMARK 2.2. The given definitions allow us to consider lower and upper solutions with “corners”. This idea goes back to Nagumo [**18**] and has been used recently by different authors (see [**12**] and references therein).

On the other hand, we point out that the existence of a pair of lower and upper solutions implies the existence of zeros for L_1 and L_2.

The following result asserts the solvability of (2.1) under the presence of a pair of well ordered lower and upper solutions.

THEOREM 2.1. *Let α and β be a lower and an upper solution with $\alpha \le \beta$ and suppose that conditions* (*i*), (*ii*), (*iii*) *and* (*iv*) *hold.*

Then there exists a solution $u \in \mathcal{S}$ of problem (2.1) *with*

$$\alpha(t) \le u(t) \le \beta(t) \textit{ for all } t \in I.$$

Proof. Step 1: The modified problem.

Consider the modified boundary value problem

$$\begin{cases} -(k(t, \gamma(t, u(t)))u'(t))' = f(t, \gamma(t, u(t))) & \text{a.a. } t \in I, \\ u(0) = \bar{L}_1(u(0), u(1), u), \\ u(1) = \bar{L}_2(u(0), u(1)), \end{cases} \tag{2.2}$$

https://doi.org/10.1017/S0017089507003679 Published online by Cambridge University Press

where $\gamma : I \times \mathbb{R} \to \mathbb{R}$ is the truncation function defined by

$$\gamma(t,x) = \max\{\alpha(t), \min\{x, \beta(t)\}\}, \tag{2.3}$$
$$\bar{L}_1(x,y,u) = \gamma(0, x + L_1(x,y,u))$$

and

$$\bar{L}_2(x,y) = \gamma(1, y - L_2(x,y)).$$

An easy computation shows that

$$u \in \mathcal{S}_1 := \{u \in AC(I) : k(t, \gamma(\cdot, u(\cdot)))u'(\cdot) \in AC(I)\},$$

is a solution of problem (2.2) if and only if $u \in C(I)$ is a fixed point of the operator $T : C(I) \to C(I)$ defined as

$$\begin{aligned} Tu(t) = &\int_0^1 G_u(t,s) f(s, \gamma(s, u(s)))ds \\ &+ \frac{1}{w_u(1)}[(w_u(1) - w_u(t))\bar{L}_1(u(0), u(1), u) + w_u(t)\bar{L}_2(u(0), u(1))] \end{aligned} \tag{2.4}$$

where

$$w_u(t) = \int_0^t \frac{ds}{k(s, \gamma(s, u(s)))}$$

and

$$G_u(t,s) = \frac{1}{w_u(1)} \begin{cases} w_u(s)(w_u(1) - w_u(t)), & \text{if } s \le t, \\ w_u(t)(w_u(1) - w_u(s)), & \text{if } t \le s, \end{cases} \tag{2.5}$$

is, for fixed $u \in C(I)$, the Green's function associated with the problem

$$\begin{cases} -(k(t, \gamma(t, u(t)))v'(t))' = h(t) & \text{a.a. } t \in I, \\ v(0) = v(1) = 0. \end{cases}$$

We remark that $w_u \in AC(I)$ and $w_u(t) > 0$ for all $t \in (0, 1]$. Moreover, one can verify the following property:

$$\text{For all } u \in C(I) \text{ it holds that } Tu \in C^1(0, 1). \tag{2.6}$$

Step 2: Problem (2.2) has a solution $u \in \mathcal{S}_1$.
It is clear that operator T is bounded in $C(I)$. So, if we show that T is completely continuous, then the Schauder fixed point theorem implies that T has a fixed point which is a solution of (2.2).

2.1.- $T : C(I) \to C(I)$ is a continuous operator.
Let $\{u_n\}_{n\in\mathbb{N}} \subset C(I)$ such that $u_n \to u$ uniformly on I. We shall prove that $Tu_n \to Tu$ uniformly on I.

By using that the functions $f(\cdot, \gamma(\cdot, u(\cdot)))$, $\bar{L}_1$ and $\bar{L}_2$ are continuous and bounded independently of $u \in C(I)$ and from the definition of G_u it suffices to prove that

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$w_{u_n} \to w_u$ uniformly on I. We have for all $t \in I$ that

$$|w_{u_n}(t) - w_u(t)| \leq \int_0^1 \left| \frac{1}{k(s, \gamma(s, u_n(s)))} - \frac{1}{k(s, \gamma(s, u(s)))} \right| ds. \tag{2.7}$$

If we denote

$$g_n(t) := \left| \frac{1}{k(t, \gamma(t, u_n(t)))} - \frac{1}{k(t, \gamma(t, u(t)))} \right|,$$

since γ and k are continuous, and $u_n \to u$ uniformly on I, then we have that for a.a. $t \in I$

$$\lim_{n\to\infty} g_n(t) = 0.$$

Moreover by (i) there exists $r > 0$ such that for a.a. $t \in I$

$$0 < g_n(t) \leq 2p_r(t) \in L^1(I) \quad \text{for all } n \in \mathbb{N}.$$

Thus from the Lebesgue dominated convergence theorem it follows that

$$\lim_{n\to\infty} \int_0^1 g_n(s)ds = 0,$$

which together with (2.7) imply that $w_{u_n} \to w_u$ uniformly on I.

2.2.- $T : C(I) \to C(I)$ maps bounded sets into relatively compact ones.

Let $\{u_n\}_{n\in\mathbb{N}} \subset C(I)$ be a bounded sequence. We shall prove that $\{Tu_n\}_{n\in\mathbb{N}}$ is a relatively compact subset of $C(I)$. It is clear, by the definition of T, that $\{Tu_n\}_{n\in\mathbb{N}}$ is bounded uniformly with respect to n. Then we only have to prove that $\{Tu_n\}_{n\in\mathbb{N}}$ is an equicontinuous family.

For all $t,\ \tilde{t} \in I$ we have

$$\begin{aligned} |Tu_n(t) - Tu_n(\tilde{t})| \leq{}& \int_0^1 |G_{u_n}(t,s) - G_{u_n}(\tilde{t},s)||f(s, \gamma(s, u_n(s)))|ds \\ &+ \frac{|w_{u_n}(t) - w_{u_n}(\tilde{t})|}{w_{u_n}(1)}(|\bar{L}_1(u_n(0), u_n(1), u_n)| + |\bar{L}_2(u_n(0), u_n(1))|). \end{aligned}$$

Thus since $|f(\cdot, \gamma(\cdot, u_n(\cdot)))|$ is bounded independently of $n \in \mathbb{N}$, $\bar{L}_1$ and $\bar{L}_2$ are bounded and the fact that $\{w_{u_n}\}_{n\in\mathbb{N}}$ is a bounded (uniformly with respect to n) and equicontinuous family (as it is not difficult to check) we obtain the desired result.

Step 3: If $u \in \mathcal{S}_1$ is a solution of problem (2.2) then $\alpha(t) \leq u(t) \leq \beta(t)$ for all $t \in I$.

If u is a solution of (2.2) then $\alpha(0) \leq u(0) \leq \beta(0)$ and $\alpha(1) \leq u(1) \leq \beta(1)$. Now we assume that there exists some $t_0 \in (0, 1)$ such that

$$\alpha(t_0) - u(t_0) = \max_{s\in[0,1]} \{\alpha(s) - u(s)\} > 0$$

and $\alpha(t_0) - u(t_0) > \alpha(t) - u(t)$ for all $t_0 < t \leq 1$.

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Note that such a point exists, on the contrary, there exists a sequence $\{t_n\} \to 1$ such that $\alpha(t_n) - u(t_n) = \max_{s\in[0,1]} \{\alpha(s) - u(s)\} > 0$ and thus, by continuity, we arrive at

$$0 < \max_{s\in[0,1]} \{\alpha(s) - u(s)\} = \alpha(1) - u(1) \leq 0.$$

Then we have

$$D_-(\alpha - u)(t_0) \geq D_+(\alpha - u)(t_0).$$

Since the solution u is a fixed point of the operator T we know, from (2.6), that $u \in C^1(0, 1)$ and, in particular, $u'(t_0)$ exists. Therefore

$$D_-\alpha(t_0) - u'(t_0) \geq D_+\alpha(t_0) - u'(t_0).$$

By the definition of a lower solution α there exists an open interval I_0 with $t_0 \in I_0$ such that $\alpha \in C^1(I_0)$ and

$$-(k(t, \alpha(t))\alpha'(t))' \leq f(t, \alpha(t)) \quad \text{for a.a. } t \in I_0. \tag{2.8}$$

Moreover, for some $\delta > 0$ it is verified that

$$u(t) < \alpha(t) \text{ for all } t \in (t_0 - \delta, t_0 + \delta) \subset I_0.$$

Then

$$-(k(t, \alpha(t))u'(t))' = f(t, \alpha(t)) \quad \text{for a.a. } t \in (t_0 - \delta, t_0 + \delta), \tag{2.9}$$

since u is a solution of (2.2).

Now, from (2.8), (2.9) and the fact that $\alpha'(t_0) - u'(t_0) = 0$ it follows that

$$-k(t, \alpha(t))(\alpha'(t) - u'(t)) \leq 0 \quad \text{for a.a. } t \in (t_0, t_0 + \delta),$$

and then $\alpha' - u' \geq 0$ on $(t_0, t_0 + \delta)$ which is a contradiction with the choice of t_0 because $\alpha(t_0) - u(t_0) > \alpha(t) - u(t)$ for all $t_0 < t \leq 1$.

In a similar way we prove that $u \leq \beta$ on I.

Step 4: If $u \in \mathcal{S}_1$ is a solution of problem (2.2) then $u \in \mathcal{S}$ and it is a solution of problem (2.1) .

By using Step 3, we know that $\alpha(t) \leq u(t) \leq \beta(t)$ for all $t \in I$ and, as a consequence, $u \in \mathcal{S}$.

Obviously it suffices to prove that in this case u satisfies the nonlinear boundary conditions of problem (2.1).

If $u(1) - L_2(u(0), u(1)) < \alpha(1)$ then $u(1) = \alpha(1)$ and by (iv)

$$0 < L_2(u(0), \alpha(1)) \leq L_2(\alpha(0), \alpha(1)) = 0,$$

which is a contradiction. Therefore $u(1) - L_2(u(0), u(1)) \geq \alpha(1)$. In a similar way we prove that $u(1) - L_2(u(0), u(1)) \leq \beta(1)$ and thus $L_2(u(0), u(1)) = 0$.

On the other hand if $u(0) + L_1(u(0), u(1), u) < \alpha(0)$ then $u(0) = \alpha(0)$ and

$$L_2(\alpha(0), u(1)) = L_2(u(0), u(1)) = 0 = L_2(\alpha(0), \alpha(1)).$$

Since $L_2(x, \cdot)$ is injective we have that $\alpha(1) = u(1)$. But in this case, by using (*iii*), we deduce that

$$0 > L_1(\alpha(0), \alpha(1), u) \geq L_1(\alpha(0), \alpha(1), \alpha) \geq 0,$$

which is a contradiction. Then $u(0) + L_1(u(0), u(1), u) \geq \alpha(0)$.

The fact that $u(0) + L_1(u(0), u(1), u) \leq \beta(0)$ is obtained in a similar way.

These two properties imply that $L_1(u(0), u(1), u) = 0$. □

REMARK 2.3. If, instead of problem (2.1), we consider the problem

$$\begin{cases} -(k(t, u(t))u'(t))' = f(t, u(t)) & \text{for a.a. } t \in I, \\ L_1(u(0), u) = 0, \\ L_2(u(1), u) = 0. \end{cases} \tag{2.10}$$

We can deduce similar existence results by redefining, in this case, the lower solution α as in definition 2.1 but assuming

$$L_1(\alpha(0), \alpha) \geq 0 \geq L_2(\alpha(1), \alpha),$$

and the reversed conditions in β.

We note that these conditions include the Dirichlet ones as a particular case. In this case the definition of α and β allow them to be different from 0 at the endpoints of the interval. So we improve the previous definition of lower and upper solutions for Dirichlet problems given in the framework of problem (2.1).

It is important to note that the multipoint boundary value conditions

$$u(0) = \sum_{i=0}^{k} a_i\, u(\tau_i), \qquad u(1) = \sum_{j=0}^{l} b_j\, u(\xi_j),$$

with $a_i \geq 0,\ i = 0, \ldots, k,\ b_j \geq 0,\ j = 0, \ldots, l,\ 0 < \tau_0 < \cdots < \tau_k \leq 1,\ 0 \leq \xi_0 < \cdots < \xi_l < 1$, are also covered.

The corresponding existence result is the following:

THEOREM 2.2. *Let α and β be a lower and an upper solution of problem (2.10) with $\alpha \leq \beta$, and suppose that conditions* (*i*), (*ii*) *are satisfied together with*

- $(iii)'$ *$L_1 : \mathbb{R} \times C(I) \to \mathbb{R}$ is continuous and the function $L_1(x, \cdot)$ is nondecreasing for all $x \in \mathbb{R}$.*
- $(iv)'$ *$L_2 : \mathbb{R} \times C(I) \to \mathbb{R}$ is continuous and the function $L_2(x, \cdot)$ is nonincreasing for all $x \in \mathbb{R}$.*

Then problem (2.10) *has at least one solution in the sector* $[\alpha, \beta]$.

Proof. The proof follows the lines of the proof of Theorem 2.1. However we have some differences in the proof of Step 4. *If $u \in \mathcal{S}_1$ is a solution of problem (2.2) (with obvious notation) then $u \in \mathcal{S}$ and it is a solution of problem (2.1).*

As in the Step 3 of Theorem 2.1, we know that $\alpha(t) \leq u(t) \leq \beta(t)$ for all $t \in I$ and, as a consequence, $u \in \mathcal{S}$.

Now, to prove that u satisfies the nonlinear boundary conditions of problem (2.10), we argue by contradiction:

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If $u(0) + L_1(u(0), u) < \alpha(0)$ then $u(0) = \alpha(0)$ and, by using $(iii)'$,

$$0 > L_1(u(0), u) = L_1(\alpha(0), u) \geq L_1(\alpha(0), \alpha) \geq 0.$$

If $u(1) - L_2(u(1), u) < \alpha(1)$ then $u(1) = \alpha(1)$ and, by $(iv)'$, we arrive at

$$0 < L_2(u(1), u) = L_2(\alpha(1), u) \leq L_2(\alpha(1), \alpha) \leq 0.$$

It is clear that, from these properties and the analogous ones for β, the solution of the truncated problem is also a solution of (2.10). □

3. Existence of extremal solutions and uniqueness. In this section we deal with the existence of extremal solutions and with the uniqueness of solutions for the problem

$$\begin{cases} -(k\,u')'(t) = f(t, u(t)) & \text{a.a. } t \in I, \\ L_1(u(0), u(1), u) = 0, \\ L_2(u(0), u(1)) = 0. \end{cases} \tag{3.1}$$

In this case the function k only depends on t. Clearly problem (3.1) is a particular case of problem (2.1), but we were not able to prove the existence of extremal solutions for the general case. However we remark that even for problem (3.1) this result seems to be new.

Before proving our main results we need the following technical result which is inspired by [**12**, Theorem 1.2].

PROPOSITION 3.1. *Let α_i $(i = 1, 2)$ be lower solutions and β_i $(i = 1, 2)$ be upper solutions of* (3.1) *and $\alpha := \max\{\alpha_1, \alpha_2\}$ and $\beta := \min\{\beta_1, \beta_2\}$ be such that $\alpha \leq \beta$. Then, if conditions* (i)*,* (ii)*,* (iii) *and* (iv) *are satisfied, then problem* (3.1) *has a solution $u \in \mathcal{S}$ such that $\alpha(t) \leq u(t) \leq \beta(t)$ for all $t \in I$.*

Proof. Step 1: The modified problem.
Consider the modified problem

$$\begin{cases} -(k\,u')'(t) = \bar{f}(t, u(t)) & \text{a.a. } t \in I, \\ u(0) = \bar{L}_1(u(0), u(1), u), \\ u(1) = \bar{L}_2(u(0), u(1)), \end{cases} \tag{3.2}$$

where

$$\begin{aligned} \bar{L}_1(x, y, u) &= \gamma(0, x + L_1(x, y, u)), \\ \bar{L}_2(x, y) &= \gamma(1, y - L_2(x, y)), \end{aligned}$$

with γ defined in (2.3), and $\bar{f} : I \times \mathbb{R} \to \mathbb{R}$ given by

$$\bar{f}(t, u) = \begin{cases} \max\{f(t, \max\{\alpha_1(t), u\}), f(t, \max\{\alpha_2(t), u\})\}, & \text{if } \quad u < \alpha(t), \\ f(t, u), & \text{if } \quad \alpha(t) \leq u \leq \beta(t), \\ \min\{f(t, \min\{\beta_1(t), u\}), f(t, \min\{\beta_2(t), u\})\}, & \text{if } \quad \beta(t) < u. \end{cases}$$

Step 2: Problem (3.2) has a solution $u \in \mathcal{S}$.

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It can be proven as in Step 2 of Theorem 2.1.

Step 3: If $u \in \mathcal{S}$ is a solution of problem (3.2) then $\alpha(t) \leq u(t) \leq \beta(t)$.
If we suppose that

$$\min_{t \in I}\{u(t) - \alpha(t)\} < 0,$$

it follows that there exists $i_0 \in \{1, 2\}$ such that

$$\min_{t \in I}\{u(t) - \alpha(t)\} = \min_{t \in I}\{u(t) - \alpha_{i_0}(t)\} < 0.$$

Since $\alpha(0) \leq u(0) \leq \beta(0)$ and $\alpha(1) \leq u(1) \leq \beta(1)$ we can choose $t_0 \in (0, 1)$ such that

$$\min_{t \in I}\{u(t) - \alpha(t)\} = u(t_0) - \alpha_{i_0}(t_0) < u(t) - \alpha_{i_0}(t) \quad \text{for all } t_0 < t \leq 1.$$

Reasoning as in Step 3 of Theorem 2.1, we know that there exists an open interval I_0 with $t_0 \in I_0$ such that $\alpha_{i_0} \in C^1(I_0)$ and

$$-(k\,\alpha_{i_0}')'(t) \leq f(t, \alpha_{i_0}(t)) \leq \bar{f}(t, u(t)) = -(k\,u')'(t) \quad \text{for a.a. } t \in I_0.$$

The proof follows now as in Step 3 of Theorem 2.1.

Step 4: If $u \in \mathcal{S}$ is a solution of problem (3.2) then it is a solution of problem (3.1).
We only must verify that u satisfies the nonlinear boundary conditions of problem (3.1).

The fact that $L_2(u(0), u(1)) = 0$ is analogous to Step 4 in Theorem 2.1.

On the other hand, if $u(0) + L_1(u(0), u(1), u) < \alpha(0)$ then $u(0) = \alpha(0) = \alpha_{i_0}(0)$ ($i_0 \in \{1, 2\}$) and, since $L_2(x, \cdot)$ is injective we have that $\alpha_{i_0}(1) = u(1)$.

Now, by (*iii*) we arrive at the following contradiction

$$0 > L_1(u(0), u(1), u) \geq L_1(\alpha_{i_0}(0), \alpha_{i_0}(1), \alpha_{i_0}) \geq 0,$$

and the result is proved. □

Now we are in a position to prove the existence of extremal solutions for problem (3.1).

THEOREM 3.1. *Let α and β be a lower and an upper solutions of* (2.1) *with $\alpha \leq \beta$ and suppose that conditions* (*i*), (*ii*), (*iii*) *and* (*iv*) *hold.*

Then there exists the minimal solution $u_{min} \in \mathcal{S}$ and the maximal solution $u_{max} \in \mathcal{S}$ of problem (3.1) *with*

$$\alpha(t) \leq u_{min}(t) \leq u_{max}(t) \leq \beta(t) \textit{ for all } t \in I.$$

Proof. The proof of Theorem 2.1 shows that the set of solutions between α and β is equal to the set of fixed points of the completely continuous operator T defined by (2.4). Moreover, by Proposition 3.1, given two solutions (which in particular are lower solutions) there exists another solution greater than both of them, that is, the set of fixed points of T is upward directed. Then by [**9**, Theorem 2.1] there exists u_{max} the maximal fixed point of T which is also the maximal solution of (3.1) between α and β. By a similar argument we prove the existence of u_{min} the minimal solution of (3.1) in $[\alpha, \beta]$. □

https://doi.org/10.1017/S0017089507003679 Published online by Cambridge University Press

REMARK 3.1. If we consider conditions

$$L_1(u(0), u) = L_2(u(1), u) = 0$$

as in Theorem 2.2, the Theorem 3.1 is valid too.

Next we deal with the uniqueness of solutions for problem (3.1).

THEOREM 3.2. *Let α and β be a lower and an upper solution with $\alpha \leq \beta$ and assume conditions (i), (ii) and moreover*

(U1) *for a. a. $t \in I$ the function $f(t, \cdot)$ is nonincreasing in $[\alpha(t), \beta(t)]$,*
(U2) *$L_1(x, y) \equiv L_1(x, y, u)$ is continuous, nonincreasing in y and $x + L_1(x, y)$ is nonincreasing in x,*
(U3) *$L_2(y) \equiv L_1(x, y)$ is continuous, injective and $y - L_2(y)$ is nonincreasing.*

Then problem (3.1) has a unique solution in $[\alpha, \beta]$.

Proof. From Theorem 2.1 we see that the set of solutions between α and β is equal to the set of fixed points of the operator T defined by (2.4). Moreover from (2.4), (2.5) and our hypotheses it follows that $T : C(I) \to C(I)$ is a nondecreasing operator (considering in $C(I)$ the pointwise partial ordering). On the other hand, Proposition 3.1 implies that the set of fixed points of T is directed. Then [**10**, Theorem 2.1] ensure us that T has at most one fixed point and therefore problem (3.1) has at most one solution in $[\alpha, \beta]$. Since Theorem 2.1 asserts the solvability of (3.1) we deduce the existence of a unique solution of problem (3.1) between α and β. $\square$

REMARK 3.2. The conditions $(U2)$ and $(U3)$ are stronger than (iii) and (iv). In particular they include the Dirichlet boundary conditions $L_1(x, y) = -x$ and $L_2(y) = y$.

4. Examples.

In this section we present two different boundary value problems in which we apply the existence results given in sections 2 and 3.

EXAMPLE 4.1. Consider the problem

$$\begin{cases} -\left(\dfrac{\sqrt[4]{t(1-t)}}{u^2(t)+1}u'(t)\right)' = t - u^3(t) \quad \text{for a.a. } t \in I = [0, 1], \\ u(0) = u\left(\frac{1}{2}\right) = u(1), \end{cases} \tag{4.1}$$

which is of the form (2.1) with

$$k(t, x) = \frac{\sqrt[4]{t(1-t)}}{x^2+1},$$
$$f(t, x) = t - x^3,$$
$$L_1(x, y, u) = u\left(\frac{1}{2}\right) - x$$

and

$$L_2(x, y) = y - x.$$

https://doi.org/10.1017/S0017089507003679 Published online by Cambridge University Press

It is an easy matter to check that assumptions *(i)*, *(ii)*, *(iii)* and (*iv*) are satisfied and moreover that $\alpha(t) = -1$ and $\beta(t) = 1$ for all $t \in I$ are lower and upper solutions, respectively. Then Theorem 2.1 ensures us the existence of a solution of problem (4.1) between -1 and 1.

EXAMPLE 4.2. Let the problem

$$\begin{cases} -\left(\sqrt{t}u'(t)\right)' = \dfrac{e^{-(t^2+u^2)}}{2} \quad \text{for a.a. } t \in I = [0, 1], \\ u(0) = u(1) = 0. \end{cases} \tag{4.2}$$

It is not difficult to verify that it is of the form

$$\begin{cases} -(k\,u')'(t) = f(t, u(t)) \quad \text{a.a. } t \in I, \\ L_1(u(0), u(1)) = 0, \\ L_2(u(1)) = 0. \end{cases}$$

with

$$k(t) = \sqrt{t},$$
$$f(t, x) = \frac{e^{-(t^2+x^2)}}{2},$$
$$L_1(x, y, u) = -x$$

and

$$L_2(x, y, u) = y.$$

One can verify that $\alpha \equiv 0$ is a lower solution of this problem and numerical experiments show that $\beta_c(t) = c\,(1 - t)$ is an upper solution for every $c \geq 0.5191$.

Since the assumptions of theorem 3.2 hold we obtain that for all $c \geq 0.5191$ this problem has a unique solution satisfying

$$0 \leq u(t) \leq c\,(1 - t).$$

On the other hand, since $k(1) > 0$, by using the expression of the operator T defined in the proof of Theorem 2.1, we conclude that every solution of problem (4.2) belongs to the following set

$$S^* = \{u \in C^1(0, 1]; \quad u(0) = u(1) = 0\}.$$

It is clear that every function in this space satisfies that there exists $c > 0$ such that $u(t) \leq c\,(1 - t)$. So, we conclude that problem (4.2) has a unique nonnegative solution.

Note that we have additional information about the unique nonnegative solution: it is less than or equal to 0.5191 and $-0.5191 \leq u'(1) \leq 0$.

ACKNOWLEDGEMENTS. The authors thank the anonymous referee for his/her detailed reading of the manuscript and valuable comments.

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