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A PARTITION THEOREM OF SUBBARAO

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In a recent paper, Subbarao [1] has used generating functions to prove the

THEOREM. *Let $C(n)$ be the number of partitions of n such that all even multiplicities of the parts are less than $2m$, $m>1$; and all odd multiplicities are at least $(2r+1)$ and at most $2(m+r)-1$, $r\geq 0$. Let $D(n)$ be the number of partitions of n into parts which are either odd multiples of $(2r+1)$ or are even and not divisible by $2m$. Then*

$$C(n) = D(n).$$

We give here a straight-forward

Proof. Let

$$a_1^{m_1} \quad a_2^{m_2} \quad a_3^{m_3} \quad \cdots \quad a_k^{m_k} \tag{1}$$

be a C-type partition of n with a's distinct positive integers and

$$m_j = \begin{cases} 2x_j, & 0 < x_j < m; \\ 2(y_j+r)+1, & 0 \leq y_j < m; \end{cases} \tag{2}$$

according as m_j the multiplicity of a_j is even or odd.

Without loss of generality, we can assume that in (1) the first k_0 of the a's have an even multiplicity, the next k_1 an odd multiplicity exceeding $(2r+1)$ and the rest an odd multiplicity equal to $(2r+1)$.

Then, we have

$$\begin{aligned} n &= 2\sum_{j=1}^{k_0} a_j x_j + \sum_{j=k_0+1}^{k_0+k_1} \{2(y_j+r)+1\}a_j + (2r+1)\sum_{j=k_0+k_1+1}^{k} a_j \\ &= 2\left\{\sum_{j=1}^{k_0} a_j x_j + \sum_{j=k_0+1}^{k_0+k_1} a_j y_j\right\} + (2r+1)\sum_{j=k_0+1}^{k} a_j \\ &= 2M+(2r+1)N. \end{aligned}$$

Notice that

$$N = \sum_{j=k_0+1}^{k} a_j \tag{3}$$

provides a partition of N into distinct summands, to which corresponds in the manner of Sylvester [2] a unique partition of N into odd parts repetitions allowed.

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Let this be given by

(4) $$b_1^{u_1}\ \ b_2^{u_2}\ \cdots\ \ b_h^{u_h}.$$

Again

$$M=\sum_{j=1}^{k_0}a_jx_j+\sum_{j=k_0+1}^{k_0+k_1}a_jy_j$$
$$=\sum_{j=1}^{k_0+k_1}a_jz_j,$$

where $z_j=x_j$ or y_j according as j is or is not

(5) $$\leqslant k_0;\ z_j>0;$$

provides a partition of M in which the multiplicity of a_j is at most $(m-1)$. Corresponding to this we have the unique partition

(6) $$c_1^{d_1}\ \ c_2^{d_2}\ \ c_3^{d_3}\ \cdots\ \ c_v^{d_v},\qquad v=k_0+k_1$$

of M, with

(7) $$c_j=a_j/m^t,\qquad d_j=m^tz_j$$

t being the index of the highest power of m which divides a_j, $t\geq 0$. Evidently the c's in (6) are not necessarily all distinct.

We thus obtain the D-type partition

(8) $$(2c_1)^{d_1}(2c_2)^{d_2}\cdots(2c_v)^{d_v}\{(2r+1)b_1\}^{u_1}\{(2r+1)b_2\}^{u_2}\cdots\{(2r+1)b_h\}^{u_h}$$

of n from the C-type partition (1).

In the reverse process, (7) and the fact that $z_j<m$, play an important role but the reconstruction of (1) from (8) offers no special difficulty. The correspondence between the C-type and D-type partitions of n being one-one and onto, the theorem follows.

EXAMPLE. Let us find the C-type partition corresponding to the D-type partition

$$2^3\ \ 4^{37}\ \ 8^{11}\ \ 14^2\ \ 9^2\ \ 15^1$$

of 303, when $r=1$, $m=5$.

Taking the even parts first, since

$$37=25+2.5+2;\qquad\text{and}\qquad 11=2.5+1;$$

we have

$$d_1=3,\ \ d_2=25,\ \ d_3=10,\ \ d_4=2,\ \ d_5=10,\ \ d_6=1,\ \ d_7=2;$$
$$c_1=1,\ \ c_2=2,\ \ c_3=2,\ \ c_4=2,\ \ c_5=4,\ \ c_6=4,\ \ c_7=7.$$

Thus

$$z_1=3,\ \ z_2=1,\ \ z_3=2,\ \ z_4=2,\ \ z_5=2,\ \ z_6=1,\ \ z_7=2;$$
$$a_1=1,\ \ a_2=50,\ \ a_3=10,\ \ a_4=2,\ \ a_5=20,\ \ a_6=4,\ \ a_7=7.$$

Taking the odd parts now, we get

$$b_1 = 3, \quad b_2 = 5; \quad u_1 = 2, \quad u_2 = 1.$$

The Sylvester partition into distinct parts corresponding to the partition $3^2\ 5$ into odd parts, is readily found to be $5^1\ 4^1\ 2^1$. We use the graph:

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for the purpose. The required C-type partition, therefore, is

$$1^6 \quad 50^2 \quad 10^4 \quad 2^4 \quad 20^4 \quad 4^2 \quad 7^4 \quad 5^3 \quad 4^3 \quad 2^3$$

i.e. $1^6 \quad 50^2 \quad 10^4 \quad 20^4 \quad 7^4 \quad 5^3 \quad 4^5 \quad 2^7$.

References

1. M. V. Subbarao, *On a partition theorem of MacMahon-Andrews*, Proc. Amer. Math. Soc., **27** (1971), 449–450.

2. H. Gupta, *On Sylvester's theorem in partitions*, Indian Jour. Pure and Applied Maths., **2** (1971), 740–748.

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