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# THE TRIGONOMETRY OF HYPERBOLIC TESSELLATIONS

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ABSTRACT. For positive integers $p$ and $q$ with $(p-2)(q-2) > 4$ there is, in the hyperbolic plane, a group $[p, q]$ generated by reflections in the three sides of a triangle $ABC$ with angles $\pi/p$, $\pi/q$, $\pi/2$. Hyperbolic trigonometry shows that the side $AC$ has length $\psi$, where $\cosh\psi = c/s$, $c = \cos\pi/q$, $s = \sin\pi/p$. For a conformal drawing inside the unit circle with centre $A$, we may take the sides $AB$ and $AC$ to run straight along radii while $BC$ appears as an arc of a circle orthogonal to the unit circle. The circle containing this arc is found to have radius $1/\sinh\psi = s/z$, where $z = \sqrt{c^2 - s^2}$, while its centre is at distance $1/\tanh\psi = c/z$ from $A$. In the hyperbolic triangle $ABC$, the altitude from $AB$ to the right-angled vertex $C$ is $\zeta$, where $\sinh\zeta = z$.

1. **Non-Euclidean planes.** The real projective plane becomes non-Euclidean when we introduce the concept of orthogonality by specializing one *polarity* so as to be able to declare two lines to be orthogonal when they are conjugate in this 'absolute' polarity. The geometry is elliptic or hyperbolic according to the nature of the polarity.

The points and lines of the *elliptic* plane ([11], §6.9) are conveniently represented, on a sphere of unit radius, by the pairs of antipodal points (or the diameters that join them) and the great circles (or the planes that contain them). The general right-angled triangle $ABC$, like such a triangle on the sphere, has five 'parts': its sides $a, b, c$ and its acute angles A and B. (The right-angled vertex $C$ is opposite to the hypotenuse $c$.) Every three of the five parts are related by a trigonometric identity ([14], p. 34). Five of these ten formulae are:

$$\cos \mathrm{A} = \cos a \sin \mathrm{B}, \tag{1.1}$$

$$\cos \mathrm{B} = \cos b \sin \mathrm{A}, \tag{1.2}$$

$$\cos c = \cos a \cos b \tag{1.3}$$

$$= \cot \mathrm{A} \cot \mathrm{B}, \tag{1.4}$$

$$\sin a = \sin c \sin \mathrm{A}. \tag{1.5}$$

In the 'parabolic' case when the elliptic polarity degenerates to an elliptic involution on a line ('the line at infinity'), the geometry is Euclidean!

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A *hyperbolic* polarity admits self-conjugate points and self-conjugate lines, constituting the points and tangents of a *conic* ([12], p. 72; [5], p. 199) called the *absolute* conic of the hyperbolic plane. The interior points and chords of this conic are the 'ordinary' points and lines of the hyperbolic plane. Two lines are said to be *parallel* if the chords have a common end-point, which is naturally called a 'point at infinity': the 'center' of a 'pencil' of parallel lines. Two lines orthogonal to one line are said to be *ultraparallel*; all the lines orthogonal to one line form a 'pencil' of ultraparallel lines.

These ideas of Cayley and Klein (1871) have been lucidly described by Roberto Bonola ([1], pp. 154–170), who goes on to consider what happens in the Euclidean plane when the points and lines of the hyperbolic plane are mapped on the interior points and chords of a *circle*. This procedure, discovered by Eugenio Beltrami in 1868, is one of two possible ways to map the hyperbolic plane on the inside of a circle. It is called the *projective* model, because straight lines are mapped on straight chords, and the projective concept of *incidence* is preserved, though distances and angles need to be redefined ([5], pp. 206, 207).

The other way to map the hyperbolic plane is Poincaré's *inversive* model, in which lines are represented by circles orthogonal to one chosen circle $\omega$. The three kinds of pencil of lines (intersecting, parallel or ultraparallel) are represented by the three kinds of pencil of coaxal circles (intersecting, tangent or disjoint), all orthogonal to $\omega$. Since orthogonality is preserved and right angles can be repeatedly bisected, an argument using continuity shows that Poincaré's model is *conformal*: angles (and infinitesimal shapes) are represented faithfully although distances need to be redefined.

Most simply, the distance between any two ordinary points $A$ and $C$ in the hyperbolic plane can be measured as the distance between two ultraparallel lines (one through $A$ and one through $C$, both orthogonal to the line $AC$). These two lines are represented by disjoint circles, and the distance between them is simply the *inversive* distance between the circles, that is, *the natural logarithm of the ratio* (greater to smaller) *of the radii of any two concentric circles into which the disjoint circles can be inverted* ([15], p. 128; [10], p. 392).

A circle and any interior point can be inverted into a circle and its centre ([15], Figure 5.5B on p. 116); therefore, when discussing interior points of a circle, we lose no generality by assuming one of the points to be the centre.

2. **The angle of parallelism.** Figure 1 shows how the projective and inversive models depict the *angle of parallelism* $\theta = \prod(x)$ between $AC$, the perpendicular from $A$ to the line $CN$, and $AN$, the parallel from $A$ to the ray $CN$. Both models use the same Euclidean circle $\omega$, with centre $A$ and radius $AN = 1$, to represent the set of points at infinity.

In Beltrami's projective mapping, the hyperbolic distance $x = AC$ is evidently represented by the Euclidean distance $\cos\theta$.

In Poincaré's inversive mapping, the ray $CN$ appears as an arc of a circle $\alpha$ with radius $r$, whose centre $O$ lies on the line $AC$ at distance $d$ from $A$. We proceed to determine how the Euclidean distances $d$ and $r$ are related to the hyperbolic distance $x = AC$, which is

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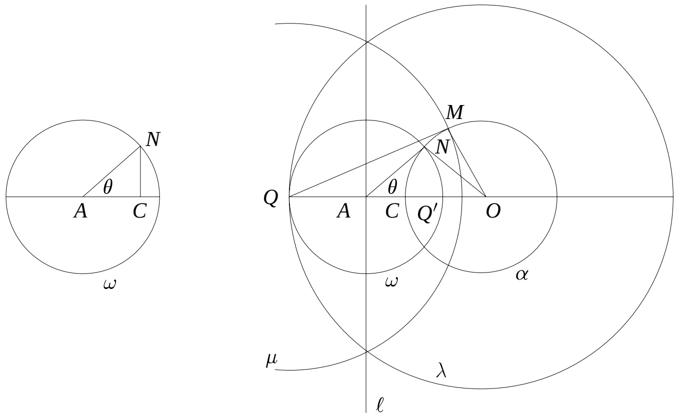


FIGURE 1. Projective and inversive views of the angle of parallelism $\theta = \prod(AC)$.

the inversive distance between two circles, orthogonal to both the circle $\omega$ and the line $AC$. The "circle" through $A$ is simply the *line* $\ell$ through $A$ orthogonal to $AC$, while the circle through $C$ is $\alpha$.

In the right half of Figure 1, $QQ'$ is the diameter through $O$ of $\omega$, with $A$ between $Q$ and $C$; $\ell$ is the perpendicular diameter; $QM$ and $AN$ are the tangents to $\alpha$ at $M$ and $N$; $\lambda$ is the circle with centre $O$ and radius $OQ$; and $\mu$ is the circle with centre $Q$ and radius $QM$. Since the tangent $QM$ to $\alpha$ is perpendicular to the radius $OM$, $\mu$ inverts $\alpha$ into itself. Since $\omega$ is orthogonal to both $\ell$ and $\alpha$, $Q$ and $Q'$ are the limiting points of the coaxal pencil that includes $\alpha$ with radical axis $\ell$. The orthogonal pencil of circles through $Q$ and $Q'$ is inverted by $\mu$ into the pencil of lines through the inverse of $Q'$. (Compare [15], p. 121 and Example 4 on pp. 131, 176; [10], pp. 391-395.)

The right-angled triangle $ANO$ (with $AN = 1$) shows that $r^2 = d^2 - 1$. Since $QQ' = 2$ and $QO = d + 1$, we have

$$QM^2 = QO^2 - MO^2 = (d+1)^2 - r^2 = (d+1)^2 - (d^2-1) = 2(d+1) = QQ' \times QO.$$

Therefore $\mu$ inverts $Q'$ into $O$, and inverts the pencil of circles through $Q$ and $Q'$ into the pencil of lines through $O$, and inverts the orthogonal pencil into the pencil of concentric circles round $O$. In particular, $\mu$ inverts $\ell$ and $\alpha$ into the two concentric circles $\lambda$ and $\alpha$, whose radii are $d + 1$ and $r$.

In terms of the angle $CAN = \theta$, we have

$$d = \sec\theta, \quad r = \tan\theta.$$

Hence the ratio of radii is

$$\frac{d+1}{r} = \operatorname{cosec}\theta + \cot\theta = \cot\frac{1}{2}\theta$$

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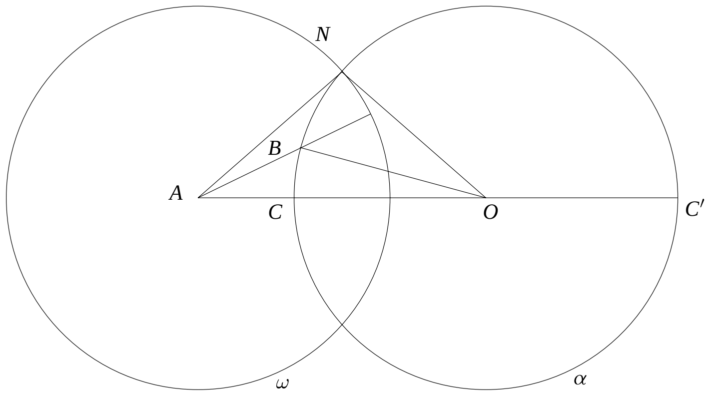


FIGURE 2. The 'triangle' $ABC$ with two straight sides and an arc $BC$.

and the hyperbolic distance $AC$ is

$$x = \log \cot \tfrac{1}{2}\theta.$$

Since $\theta = \prod(x)$, this result agrees with Lobachevsky's famous formula

$$\tan \tfrac{1}{2} \prod(x) = e^{-x}$$

([18], p. 41; [6], p. 82; [11], p. 5A1; [11a], p. 453) and with six simple relations such as

$$\cosh x = \operatorname{cosec} \theta \tag{2.1}$$

which are neatly epitomized by drawing a Euclidean right-angled triangle with sides 1, $\sinh x$, $\cosh x$. In this triangle, the angle opposite to the side of length 1 is $\prod(x)$.

It is of some interest to compare the hyperbolic distance $AC = x$ with the Euclidean distance $AC$, which is

$$\begin{aligned} AO - CO &= \sec\theta - \tan\theta \\ &= \frac{1}{\tanh x} - \frac{1}{\sinh x} \\ &= \frac{e^x + e^{-x} - 2}{e^x - e^{-x}} \\ &= \frac{e^{x/2} - e^{-x/2}}{e^{x/2} + e^{-x/2}} \\ &= \tanh \frac{x}{2}. \end{aligned}$$

Since $\cos\theta = \tanh x$ and $\tanh \infty = 1$, we have obtained the following neat theorem:

THEOREM. *When the hyperbolic plane is mapped, projectively or inversively, on the interior of the Euclidean unit circle, a point at hyperbolic distance x from the centre is at Euclidean distance* $\tanh x$ *or* $\tanh \frac{1}{2}x$*, respectively.*

Since $\tanh \frac{1}{2}x < \tanh x$, the inversive mapping causes less distortion than the projective mapping. Accordingly, we will henceforth abandon the latter in favour of the former.

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3. **The inversive model for a hyperbolic triangle.** If three lines in the hyperbolic plane intersect one another so as to form a triangle $ABC$, they are mapped inversively by three mutually intersecting circles, all orthogonal to the unit circle $\omega$. Two of the three circles may conveniently be replaced by diameters of $\omega$, forming an angle $\mathrm{A} = CAB$, while the third circle intersects them so as to form angles B at $B$, and C at $C$. Figure 2 illustrates the special case when C is a right angle, so that we are considering the general right-angled hyperbolic triangle with acute angles A and B, mapped on the Euclidean plane by a 'triangle' $ABC$ whose sides through $A$ proceed along two radii of $\omega$ while $BC$ is an arc of a circle $\alpha$ (orthogonal to $\omega$) whose centre $O$ lies on the extension of $AC$ outside $\omega$. (Since $BC$ is not straight, $\mathrm{B} < \frac{1}{2}\pi - \mathrm{A}$.) Let $r$ denote the radius of $\alpha$, $C'$ the far end of its diameter through $C$, $N$ one of its intersections with $\omega$, and $d(= AO)$ the distance of its centre from $A$.

Applying the rule of sines to the Euclidean triangle $ABO$, whose angle at $B$ is $\mathrm{B} + \frac{1}{2}\pi$, we find

$$\frac{d}{\cos \mathrm{B}} = \frac{r}{\sin \mathrm{A}}.$$

We can express $d$ and $r$ in terms of

$$s = \sin \mathrm{A}, \quad c = \cos \mathrm{B}, \tag{3.1}$$

by observing that, since the sides of the triangle $AON$ are $d, r$ and 1, we have

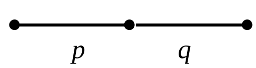
$$d^2 - r^2 = 1,$$
$$\frac{d^2}{c^2} = \frac{r^2}{s^2} = \frac{d^2 - r^2}{c^2 - s^2} = \frac{1}{c^2 - s^2}$$

and

$$d = \frac{c}{\sqrt{c^2 - s^2}} = \frac{1}{\sqrt{1 - (s/c)^2}}, \tag{3.2}$$

$$r = \frac{s}{\sqrt{c^2 - s^2}} = \frac{1}{\sqrt{(c/s)^2 - 1}}. \tag{3.3}$$

4. **The hyperbolic reflection group [$p, q$].** An important special case arises when A and B are submultiples of $\pi$, say $\pi/p$ and $\pi/q$, so that reflections in the lines $CA$, $AB$, $BC$ (or inversions in the representative circles) will generate an infinite group

$[p, q] =$ 


transforming the fundamental region $ABC$ into a tessellation of congruent right-angled triangles filling the whole hyperbolic plane (or the while interior of the circle $\omega$) ([7], p. 201). The group is infinite because we are assuming

$$\mathrm{A} + \mathrm{B} + \mathrm{C} < \pi,$$

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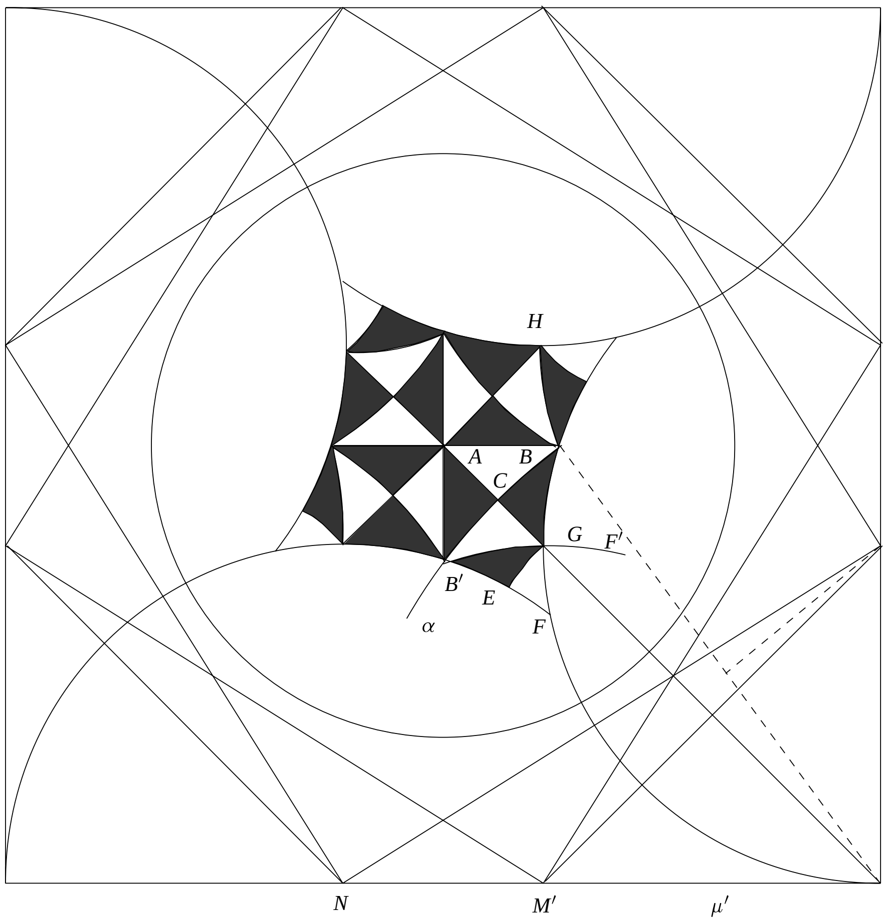


FIGURE 3. Beginning to draw the scaffolding for [4, 5].

which implies A + B $< \frac{1}{2}\pi$, $\quad p^{-1} + q^{-1} < \frac{1}{2}$,

$$(p-2)(q-2) > 4$$

and

$$\cos^2 \frac{\pi}{p} + \cos^2 \frac{\pi}{q} > 1. \tag{4.1}$$

The 'even' subgroup

$$[p, q]^+ = (p, q, 2),$$

generated by products of pairs of reflections (or inversions), is a Fuchsian group of signature (0; $p, q, 2$) in the notation of Klein and Fricke. This subgroup preserves the 'colouring' when alternate triangles of the tessellation are blackened or shaded. Klein's drawings

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of $[3, 7]^+$ and $[3, 8]^+$ were repeated by Magnus ([19], pp. 183, 187) and the author ([3], pp. 126, 127; [4], pp. 150, 160, 166).

To undertake the precise construction of such a tessellation, one may begin by observing that the circle $\alpha$ of Figure 2 is still determined by (3.2) and (3.3), although now, instead of (3.1), we have

$$s = \sin\frac{\pi}{p}, \quad c = \cos\frac{\pi}{q}. \tag{4.2}$$

For instance, the fundamental region for [6,4] (where $s = \frac{1}{2}$ and $c = \sqrt{\frac{1}{2}}$) is given by

$$d = \sqrt{2}, \quad r = 1,$$

in evident agreement with Figure 15.8b of [11a, p. 284], where the two circles are visibly congruent.

As soon as the first circle $\alpha$ has been drawn, the tessellation can be continued by repeated inversion (or reflection). All the circles through a vertex, such as $B$, have a second common point: the $\omega$-inverse of the first. Being coaxal, these circles have collinear centres, midway between the two common points.

If we regard the lines $AB$ and $AC$ as mirrors of a kaleidoscope, the point $O$ and its images form the vertices of a regular $p$-gon (as in Figure 3, where $p = 4$). They are the centres of $p$ circles, such as $\alpha$, belonging to the tessellation. The line of centres of circles through $B$ (and its $\omega$-inverse) is an edge of this $p$-gon. One of the $q$ centres on this line is at infinity because it is the 'centre' of the 'circle' consisting of the line $AB$. If $q$ is even, another one of the $q$ centres is the midpoint of this edge of the $p$-gon. This edge and other lines of centres form a kind of *scaffolding* for the building of the tessellation. Since the centres lie on pairs of tangents, they are all outside $\omega$, and so too are all the relevant *lines* of centres. (In Figures 6 and 7 of [3], pp. 126–127, and 15.8b of [11], p. 346, those lines which are secants should not have been drawn.)

Usually, each new centre arises as the common point of two such lines; but occasionally a direct appeal to inversion is required. For instance, Figure 3 shows an early stage in the construction of the tessellation for [4,5] ([4], p. 160, Figure 9), where

$$s = \sin\frac{\pi}{4} = 2^{-\frac{1}{2}}, \quad c = \cos\frac{\pi}{5} = \frac{1}{2}\tau$$

so that

$$d = \frac{1}{\sqrt{1 - 2\tau^{-2}}} = \tau^{\frac{3}{2}}, \quad r = \frac{1}{\sqrt{\frac{1}{2}\tau^2 - 1}} = (2\tau)^{\frac{1}{2}}.$$

In this case the $p$-gon appears as the peripheral square. Its right and bottom sides, $\mu$ and $\mu'$, are the lines of centres of circles through $B$ and $B'$, respectively. Since the circle $OFGB$ is the inverse, with respect to the circle $\alpha = BC$, of the straight line $AB$, its centre $M$ is located where $\mu$ meets the perpendicular bisector of $OB$. Similarly, $M'$, $N$ and $N'$ are the centres of circles $B'G$, $B'F$ and $BH$. The line $ACG$ reflects $F$ to $F'$, where the circles $BH$ and $B'G$ intersect. Since $MN$ and $M'N'$ are the lines of centres of circles through $F$

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and $F'$, respectively, their common point is the centre of the circle $FF'$. Also the straight line $MM'$ is the line of centres of circles through $G$. One of these circles is $EG$, whose centre is the midpoint of $MM'$.

Understanding that $\omega$ represents the region of infinitely distant points ([16], p. 109), the artist M. C. Escher wrote:

> 'For beyond that there is "absolute nothingness". And yet this round world cannot exist without the emptiness around it, not simply because "within" presupposes "without", but also because it is out there in the "nothingness" that the centre points of the arcs that go to build up the framework are fixed with such geometric exactitude.'

5. **Regular hyperbolic tessellations.** If we regard the lines $BC, CA, AB$ as mirrors in a kaleidoscope, the point $B$ and its images form the vertices of a regular tessellation $\{p,q\}$ ([20], p. 15), consisting of regular $p$-gons, $q$ round each vertex. In other words, the set of vertices of $\{p,q\}$ is the orbit of the point $B$ in the reflection group $[p,q]$. The vertex $A$ of the triangle $ABC$ is the centre of a tile (or 'face') of $\{p,q\}$. This 'face' is a $p$-gon of edge (say) $2\phi$, circumradius $\chi$ and inradius $\psi$. Thus $BC=\phi$, $CA=\chi$, $AB=\psi$.

The group $[p,q]$ is the symmetry group not only of $\{p,q\}$ but also of the dual tessellation $\{q,p\}$, whose vertices are the orbit of $A$.

For fragments of $\{7,3\}$ and $\{3,7\}$, see ([7], pp. 206, 207). Analogous drawings of $\{3,\infty\}$ and $\{\infty,3\}$ were made by L. Fejes Tóth ([17], p. 97).

The general right-angled hyperbolic triangle resembles its spherical counterpart in having five 'parts', every three of which are related by a trigonometric equation. Five of these ten formulae, analogous to (1.1), etc., are:

$$\cos \mathrm{A} = \cosh a \sin \mathrm{B}, \tag{5.1}$$

$$\cos \mathrm{B} = \cosh b \sin \mathrm{A}, \tag{5.2}$$

$$\cosh c = \cot \mathrm{A} \cot \mathrm{B} = \cosh a \cosh b, \tag{5.3}$$

$$\sinh a = \sinh c \sin \mathrm{A}, \tag{5.4}$$

When $\mathrm{B}=0$, so that $\mathrm{A}=\prod(b)$, (5.2) reproduces (2.1)!

Applying (5.1), (5.2), (5.3) with $a, b, c, \mathrm{A}, \mathrm{B}$ replaced by

$$\phi,\ \psi,\ \chi,\ \pi/p,\ \pi/q$$

([7], p. 201), we obtain

$$\cosh\phi = \cos\frac{\pi}{p}\Big/\sin\frac{\pi}{q}, \quad \cosh\psi = \cos\frac{\pi}{q}\Big/\sin\frac{\pi}{p},$$
$$\cosh\chi = \cosh\phi\cosh\psi = \cot\frac{\pi}{p}\cot\frac{\pi}{q}.$$

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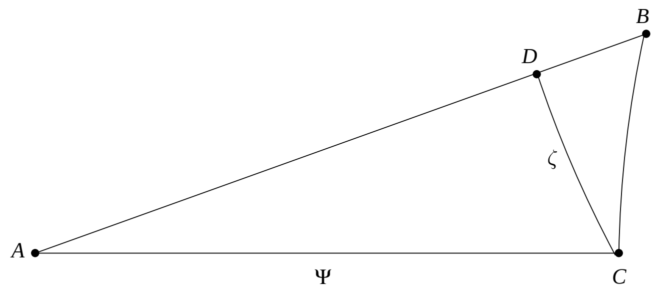


FIGURE 4. The altitude $\zeta = CD$.

In the notation of (4.2), $\cosh\psi = c/s$. Thus the distance and radius of (3.2) and (3.3) are functions of $\psi$ alone, namely

$$d = 1/\sqrt{1 - \operatorname{sech}^2\psi} = 1/\tanh\psi, \tag{5.5}$$

$$r = 1/\sqrt{\cosh^2\psi - 1} = 1/\sinh\psi. \tag{5.6}$$

In the triangle $ABC$ (see Figure 4), the altitude $\zeta = CD$, from the hypotenuse $AB$ to the right-angled vertex $C$, is obtained by applying (5.4) to the smaller triangle $ACD$:

$$\sinh\zeta = \sinh\psi \sin\frac{\pi}{p}.$$

Since $\cosh\psi = c/s$, we have $\sinh\psi = \sqrt{c^2 - s^2}/s$, and

$$\sinh\zeta = \sqrt{c^2 - s^2}. \tag{5.7}$$

The altitude $\zeta$ provides simple expressions for $\sinh\phi$ and $\sinh\psi$:

$$\sinh\phi = \sinh\zeta/\sin\frac{\pi}{q}, \quad \sinh\psi = \sinh\zeta/\sin\frac{\pi}{p}.$$

It is thus the hyperbolic counterpart of the spherical arc $\pi/h$ ([8], p. 9; [9], p. 21) where $h$ is the 'Coxeter number' for a finite reflection group $[p, q]$ ([2], p. 117; [21]). In other words, $\{p, q\}$ has for its Petrie polygon a zigzag, and the length $\zeta$ measures the translation component of the glide-reflection that takes the zigzag one step (one 'zig' or 'zag') along itself.

6. **Quasi-regular tessellations.** We have seen that, in the reflection group $[p, q]$, with fundamental region $ABC$, the orbits of $A$ and $B$ yield the regular tessellations $\{q, p\}$ and $\{p, q\}$. Analogously, the orbit of $C$ yields the *quasi-regular* tessellation $\left\{\begin{matrix} p \\ q \end{matrix}\right\}$ ([9], pp. 18-19, 101) which is analogous to two of the Archimedean solids: the cuboctahedron

$\begin{Bmatrix} 3 \\ 4 \end{Bmatrix}$ and the icosidodecahedron $\begin{Bmatrix} 3 \\ 5 \end{Bmatrix}$. Of course, $\begin{Bmatrix} p \\ q \end{Bmatrix}$ is the same as $\begin{Bmatrix} q \\ p \end{Bmatrix}$. Exceptionally, when $p = q$ the tessellation is regular:

$$\begin{Bmatrix} p \\ p \end{Bmatrix} = \{p, 4\}$$

([9], p. 60).

A small part of Figure 3 shows four right-angled triangles, two white and two black, surrounding the point $C$ which is a typical vertex of $\begin{Bmatrix} p \\ q \end{Bmatrix}$. The white and black pairs of triangles are crossed by pairs of edges of $\begin{Bmatrix} p \\ q \end{Bmatrix}$ (such as $CE$, of length $2\zeta$): one pair crossing the two white triangles and then crossing two 'new' black triangles, the other pair crossing the two black triangles and then crossing two 'new' white triangles. Thus the complete set of edges forms a network of infinitely long lines spreading out over the whole plane. The Euclidean case when $p = q = 4$ is, of course, very familiar. (For fragments of other cases, namely

$$\begin{Bmatrix} 5 \\ 4 \end{Bmatrix}, \begin{Bmatrix} 6 \\ 4 \end{Bmatrix}, \begin{Bmatrix} 7 \\ 3 \end{Bmatrix},$$

see [3], pp. 136, 137, 139.)

Such 'infinitely long lines' are the 'axes' of the above-mentioned Petrie polygons of $\{p, q\}$: they join the midpoints of the edges in the zigzag.

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