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ALGEBRAICITY AND TRANSCENDENCY OF BASIC SPECIAL VALUES OF SHINTANI'S DOUBLE SINE FUNCTIONS

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Abstract We study algebraicity and transcendency of certain basic special values of the double sine functions due to Hölder and Shintani by employing the zeta regularized product expressions.

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1. Introduction

The double sine function $F(x,(\omega_1,\omega_2))$ for a 'period' (ω_1,ω_2) was introduced by Shintani [**12**] in 1977 for investigating Kronecker's Jugendtraum for a real quadratic field. It is defined as

$$F(x,(\omega_1,\omega_2)) = \frac{\Gamma_2(\omega_1+\omega_2-x,(\omega_1,\omega_2))}{\Gamma_2(x,(\omega_1,\omega_2))} = \frac{\prod_{n_1,n_2\geqslant 0}(n_1\omega_1+n_2\omega_2+x)}{\prod_{m_1,m_2\geqslant 1}(m_1\omega_1+m_2\omega_2-x)},$$

where

$$\Gamma_2(x,(\omega_1,\omega_2)) = \exp\biggl(\frac{\partial}{\partial s}\zeta_2(s,x,(\omega_1,\omega_2))\biggr|_{s=0}\biggr) = \biggl(\prod_{n_1,n_2\geqslant 0}(n_1\omega_1+n_2\omega_2+x)\biggr)^{-1}$$

is the regularized expression of the double gamma function due to Barnes [**2**]. Here the notation $\prod$ stands for the zeta-regularized product due to Deninger [**3**] and

$$\zeta_2(s,x,(\omega_1,\omega_2)) = \sum_{n_1,n_2\geqslant 0}(n_1\omega_1+n_2\omega_2+x)^{-s}.$$

We remark that $F(x,(1,1))$ is essentially equivalent to the special function

$$F(x) = \mathrm{e}^x \prod_{n=1}^{\infty} \biggl\{ \biggl(\frac{1-x/n}{1+x/n} \biggr)^n \mathrm{e}^{2x} \biggr\}$$

discovered by Hölder [**4**] in 1886. More precisely,

$$F(x) = F(1-x,(1,1)),$$

as proved in [**5**]. We notice here that we are following the original notations $F(x)$ and $F(x,(\omega_1,\omega_2))$ of Hölder [**4**] and Shintani [**12**], respectively. We have generalized $F(x)$ to $\mathcal{S}_r(x)$ from $\mathcal{S}_2(x) = F(x)$ in [**5**, **8**], and we have constructed $S_r(x,(\omega_1,\dots,\omega_r))$ as a generalization of $F(x,(\omega_1,\omega_2)) = S_2(x,(\omega_1,\omega_2))$ in [**5**]. We refer to Manin [**10**] for an excellent survey on multiple sine functions and regularized products.

Shintani expected in [**12**] that the division values $F(\alpha,(1,\tau))$ at $\alpha \in \mathbb{Q}+\mathbb{Q}\tau$ would give abelian extensions of $\mathbb{Q}(\tau)$ for a real quadratic fundamental unit τ. (It may be necessary to make some finite products of these division values.) As an example, Shintani calculated for $\tau = \frac{1}{2}(5+\sqrt{21})$ that

$$F(\tfrac{1}{3},(1,\tau))F(1+\tfrac{1}{3}\tau,(1,\tau))F(\tfrac{1}{3}(2+2\tau),(1,\tau)) = \sqrt{\frac{\frac{1}{2}(1+\sqrt{21}) - \sqrt{\frac{1}{2}(3+\sqrt{21})}}{2}}.$$

Unfortunately we know little, however, about the algebraicity of the special values $F(\alpha,(1,\tau))$ for $\alpha \in \mathbb{Q}+\mathbb{Q}\tau$ except for basic results such as

$$F(\tfrac{1}{2},(1,\tau)) = F(\tfrac{1}{2}\tau,(1,\tau)) = \sqrt{2}$$

(see [**5**]) generalizing Hölder's result $F(\frac{1}{2}) = \sqrt{2}$.

We notice that the special value $F(r,(1,1))$ for a rational number r is especially interesting from the viewpoint of the Mahler measure of a polynomial with algebraic coefficients. As discovered by Smyth [**13**], Mahler measures are intimately connected with difficult special values of zeta and L-functions. On the other hand, these special values are expressed in terms of multiple sine functions investigated in [**5**, **8**]. Thus, Oyanagi [**11**] obtained the following formula:

$$\begin{aligned} M(x+y+2\sin\pi r) &= F(1-r,(1,1))^2 \\ &= (2\sin\pi r)^2 F(r,(1,1))^{-2} \end{aligned}$$

for $0 < r \leqslant \frac{1}{2}$. We must remark, however, that we do not know the precise nature of the mysterious value $F(r,(1,1))$ except that $F(\frac{1}{2},(1,1)) = \sqrt{2}$.

The purpose of this paper is to present a result on the algebraicity and the transcendency for the basic special value $F(m,(1,\tau))$ at each integer $m \geqslant 1$ when τ is an algebraic number. We obtain the following results.

Theorem 1.1. *Let τ be an algebraic number. Then $F(1,(1,\tau))$ is an algebraic number.*

Theorem 1.2. *Let $m \geqslant 2$ be an integer and let τ be an algebraic number.*

(1) *When τ is rational, $F(m,(1,\tau))$ is algebraic.*

(2) *When τ is irrational, $F(m,(1,\tau))$ is transcendental.*

These results show that the problem concerning the algebraicity and transcendency of division values of double sine functions is quite delicate in general.

2. Regularized products and double sine functions

First we prove necessary properties of the double sine functions and regularized products. One may consult [**5**–**7**] for the general theory of multiple sine functions and regularized products. In what follows, we fix the log-branch by $-\pi \leqslant \arg z < \pi$ for $z \in \mathbb{C}$.

Lemma 2.1. $\prod_{n=1}^{\infty}(na) = \sqrt{2\pi/a}$.

Proof. Since $\sum_{n=1}^{\infty}(na)^{-s} = a^{-s}\zeta(s)$, from the definition we have

$$\begin{aligned}\prod_{n=1}^{\infty}(na) &= \exp\biggl(-\frac{\partial}{\partial s}(a^{-s}\zeta(s))\biggr|_{s=0}\biggr)\\ &= \exp((\log a)\zeta(0) - \zeta'(0))\\ &= \exp(-\tfrac{1}{2}\log a + \tfrac{1}{2}\log(2\pi))\\ &= \sqrt{\frac{2\pi}{a}},\end{aligned}$$

where we have used the well-known facts $\zeta(0) = -\frac{1}{2}$ and $\zeta'(0) = -\frac{1}{2}\log(2\pi)$. □

Lemma 2.2. $F(1,(1,\tau)) = \sqrt{\tau}$.

Proof. Notice that

$$F(1,(1,\tau)) = \frac{\prod_{n_1,n_2\geqslant 0}((n_1+1)+n_2\tau)}{\prod_{m_1,m_2\geqslant 1}((m_1-1)+m_2\tau)} = \frac{\prod_{n_1\geqslant 1,\, n_2\geqslant 0}(n_1+n_2\tau)}{\prod_{m_1\geqslant 0,\, m_2\geqslant 1}(m_1+m_2\tau)}.$$

Since

$$\prod_{n_1\geqslant 1,\, n_2\geqslant 0}(n_1+n_2\tau) = \prod_{n_1,n_2\geqslant 1}(n_1+n_2\tau) \times \prod_{n_1\geqslant 1} n_1$$

and

$$\prod_{m_1\geqslant 0,\, m_2\geqslant 1}(m_1+m_2\tau) = \prod_{m_1,m_2\geqslant 1}(m_1+m_2\tau) \times \prod_{m_2\geqslant 1}(m_2\tau),$$

it follows that

$$F(1,(1,\tau)) = \frac{\prod_{n_1\geqslant 1} n_1}{\prod_{m_2\geqslant 1}(m_2\tau)} = \frac{\sqrt{2\pi}}{\sqrt{2\pi/\tau}} = \sqrt{\tau}$$

by Lemma 2.1. □

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Lemma 2.3. *For each integer $m \geqslant 2$, we have*

$$F(m,(1,\tau)) = \sqrt{\tau}\prod_{k=1}^{m-1}\left(2\sin\left(\frac{k\pi}{\tau}\right)\right)^{-1}.$$

Proof. From Lemma 2.2, it is sufficient to show that

$$\frac{F(x+1,(1,\tau))}{F(x,(1,\tau))} = \left(2\sin\left(\frac{\pi x}{\tau}\right)\right)^{-1}.$$

Since

$$\begin{aligned} F(x,(1,\tau)) &= \frac{\prod_{n_1,n_2\geqslant 0}(n_1+n_2\tau+x)}{\prod_{m_1,m_2\geqslant 1}(m_1+m_2\tau-x)} \\ &= \frac{\prod_{n_1\geqslant 1,n_2\geqslant 0}(n_1+n_2\tau+x)\times\prod_{n_2\geqslant 0}(n_2\tau+x)}{\prod_{m_1,m_2\geqslant 1}(m_1+m_2\tau-x)} \end{aligned}$$

and

$$\begin{aligned} F(x+1,(1,\tau)) &= \frac{\prod_{n_1\geqslant 1,n_2\geqslant 0}(n_1+n_2\tau+x)}{\prod_{m_1\geqslant 0,m_2\geqslant 1}(m_1+m_2\tau-x)} \\ &= \frac{\prod_{n_1\geqslant 1,n_2\geqslant 0}(n_1+n_2\tau+x)}{\prod_{m_1,m_2\geqslant 1}(m_1+m_2\tau-x)\times\prod_{m_2\geqslant 1}(m_2\tau-x)}, \end{aligned}$$

we see that

$$\frac{F(x,(1,\tau))}{F(x+1,(1,\tau))} = \prod_{n\geqslant 0}(n\tau+x)\times\prod_{m\geqslant 1}(m\tau-x) = 2\sin\left(\frac{\pi x}{\tau}\right).$$

Here we have used the evaluations

$$\prod_{n\geqslant 0}(n\tau+x) = \sqrt{2\pi}\frac{\tau^{1/2-x/\tau}}{\Gamma(x/\tau)}$$

and

$$\begin{aligned} \prod_{m\geqslant 1}(m\tau-x) &= \sqrt{2\pi}\frac{\tau^{1/2-(\tau-x)/\tau}}{\Gamma((\tau-x)/\tau)} \\ &= \sqrt{2\pi}\frac{\tau^{x/\tau-1/2}}{\Gamma(1-x/\tau)}, \end{aligned}$$

which are obtained by the classical result due to Lerch [**9**] combined with Lemma 2.1. Therefore, the result follows from the reflection formula $\Gamma(x)\Gamma(1-x) = \pi/\sin(\pi x)$. □

Lemma 2.4. *For $m \geqslant 2$ and $\tau \in \bar{\mathbb{Q}} - \mathbb{Q}$, we have*

$$\prod_{k=1}^{m-1}\sin\frac{k\pi}{\tau} \notin \bar{\mathbb{Q}}.$$

Proof. Suppose otherwise. Put $q = \mathrm{e}^{\mathrm{i}\pi/\tau}$. Then we see that

$$\prod_{k=1}^{m-1}(q^k - q^{-k}) \in \bar{\mathbb{Q}}.$$

Hence we get a $\bar{\mathbb{Q}}$-algebraic equation for q, so $q \in \bar{\mathbb{Q}}$. This, however, contradicts the Gelfond–Schneider theorem (see [**1**, Theorem 2.1]) which implies $q = (-1)^{1/\tau} \notin \bar{\mathbb{Q}}$. □

3. Proofs of the theorems

First, we note that Theorem 1.1 follows from the formula $F(1,(1,\tau)) = \sqrt{\tau}$, that is, $[\mathbb{Q}(F(1,(1,\tau))) : \mathbb{Q}(\tau)] = 2$, by Lemma 2.2. Next, Theorem 1.2 (1) is obtained from Lemma 2.3 using the fact that $\sin(k\pi/\tau) \in \bar{\mathbb{Q}}$ for $\tau \in \mathbb{Q}$ and $k = 1,\dots,m-1$. Lastly, Theorem 1.2 (2) follows immediately from Lemmas 2.3 and 2.4. □

Remark 3.1. We obtain similar results for $F(m\tau,(1,\tau))$, because

$$F(m\tau,(1,\tau)) = \frac{1}{\sqrt{\tau}} \times \left(\prod_{k=1}^{m-1} 2\sin \pi k\tau\right)^{-1}.$$

Remark 3.2. Some of the other values at the 'periodic-lattice' are rather complicated, as indicated by the following facts:

$$F(2-\sqrt{-1},(1,\sqrt{-1})) = (-1)^{1/4} \in \bar{\mathbb{Q}}$$

and

$$F(2-\sqrt{2},(1,\sqrt{2})) = -2^{5/4}\cos\left(\frac{\pi}{\sqrt{2}}\right) \notin \bar{\mathbb{Q}}.$$

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