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CERTAIN EXTENSIONS AND FACTORIZATIONS OF α-COMPLETE HOMOMORPHISMS IN ARCHIMEDEAN LATTICE-ORDERED GROUPS

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Abstract

As a consequence of general principles, we add to the array of 'hulls' in the category Arch (of archimedean ℓ-groups with ℓ-homomorphisms) and in its non-full subcategory $\boldsymbol{W}$ (whose objects have distinguished weak order unit, whose morphisms preserve the unit). The following discussion refers to either Arch or $\boldsymbol{W}$. Let α be an infinite cardinal number or ∞, let Hom_α denote the class of α-complete homomorphisms, and let $\boldsymbol{R}$ be a full epireflective subcategory with reflections denoted $r_G\colon G \longrightarrow rG$. Then for each G, there is $r_G^\alpha \in \mathrm{Hom}_\alpha(G, \boldsymbol{R})$ such that for each $\varphi \in \mathrm{Hom}_\alpha(G, \boldsymbol{R})$, there is unique $\overline{\varphi}$ with $\overline{\varphi}\, r_G^\alpha = \varphi$. Moreover if every r_G is an essential embedding, then, for every α and every G, $r_G^\alpha = r_G$, and every $\overline{\varphi} \in \mathrm{Hom}_\alpha$. If $\alpha = \omega_1$ and $\boldsymbol{R}$ consists of all *epicomplete* objects, then every $\overline{\varphi} \in \mathrm{Hom}_{\omega_1}$. For $\alpha = \infty$, and for any $\boldsymbol{R}$, every $\overline{\varphi} \in \mathrm{Hom}_\infty$.

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1. General principles

This title alludes to the first phrase of the abstract. We present a categorical theorem, from which the first result in the Abstract follows. We shall refer to [HS] on occasion, but now recall some basics.

PRELIMINARIES 1.1. In a category $\boldsymbol{C}$:

'Subcategories' are always supposed full and isomorphism-closed.

For $\boldsymbol{R}$ a subcategory, and $G \in |\boldsymbol{C}|$, $\mathrm{Hom}(G, \boldsymbol{R}) = \bigcup \{\mathrm{Hom}(G, R) \mid R \in |\boldsymbol{R}|\}$.

The subcategory $\boldsymbol{R}$ is *reflective* if for each $G \in |\boldsymbol{C}|$ there is $r_G \in \mathrm{Hom}(G, \boldsymbol{R})$ such that, for each $\varphi \in \mathrm{Hom}(G, \boldsymbol{R})$, there is unique $\overline{\varphi}$ with $\overline{\varphi}\, r_G = \varphi$. If also, each r_G has a property P, then $\boldsymbol{R}$ is called *P-reflective*. For example, epireflective, monoreflective. (We note that monoreflective implies epireflective [HS].)

For reflective $\boldsymbol{R}$ and $G \in |\boldsymbol{C}|$, the map r_G is usually called the *reflection map* for G, and codomain (r_G) is usually called the *reflection* of G, and denoted rG; thus, $r_G: G \longrightarrow rG$.

The operator $r : \boldsymbol{C} \longrightarrow \boldsymbol{R}$ given by $r(\varphi) = \overline{(r_H\varphi)}$, and $r(G) = rG$, for $\varphi: G \longrightarrow H$, is a functor, called the *reflector*. We shall occasionally use expressions like 'let $(\boldsymbol{R}, r)$ be a reflection'.

$m \in \boldsymbol{C}$ is called *extremal monic* if m is monic, and $m = fe$ with e epic implies e an isomorphism; then, domain(m) is called an *extremal subobject* of codomain(m).

$\boldsymbol{C}$ is said to be an *(epi, extremal mono)*-category if each $f \in \boldsymbol{C}$ has an essentially unique factorization $f = me$ with e epic, m extremal monic.

The meanings of further categorical terms used in the rest of this section are either obvious, or can be extracted from the proof of 1.4 or looked up in [HS]. We set out to generalize the 'if' part of the following major theorem.

THEOREM 1.2 (See [HS; 37.1]). *Suppose that the category $\boldsymbol{C}$ is co-(well-powered), is an (epi, extremal mono)-category, and has products, and let $\boldsymbol{R}$ be a subcategory. Then $\boldsymbol{R}$ is epireflective in $\boldsymbol{C}$ if and only if* (a) *$\boldsymbol{R}$ is closed under formation of products in $\boldsymbol{C}$, and* (b) *$\boldsymbol{R}$ is closed under formation of extremal subobjects in $\boldsymbol{C}$.*

DEFINITIONS 1.3. In the category $\boldsymbol{C}$, let $\boldsymbol{S}$ be a class of morphisms, and let $\boldsymbol{R}$ be a subcategory. For $G, R \in |\boldsymbol{C}|$, let $\boldsymbol{S}(G, R) = \{s: G \longrightarrow R \mid s \in \boldsymbol{S}\}$. Let $\boldsymbol{S}(G, \boldsymbol{R}) = \bigcup\{\boldsymbol{S}(G, R) \mid R \in |\boldsymbol{R}|\}$, and $\boldsymbol{S}(-, \boldsymbol{R}) = \bigcup\{\boldsymbol{S}(G, \boldsymbol{R}) \mid G \in |\boldsymbol{C}|\}$.

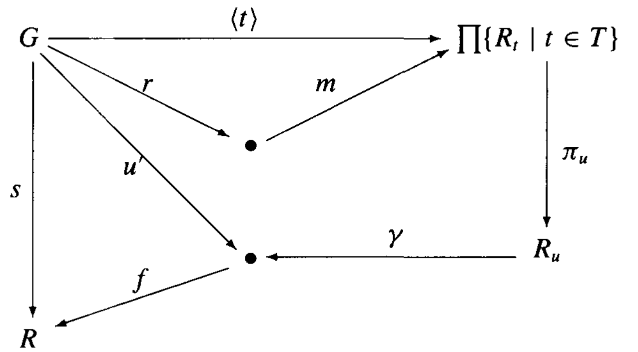
(a) $\boldsymbol{S}(-, \boldsymbol{R})$ is *closed under evaluations* if for each $G \in |\boldsymbol{C}|$ and set I, if $\{s_i \mid i \in I\} \subseteq \boldsymbol{S}(G, \boldsymbol{R})$, then $\langle s_i \rangle \in s(G, \boldsymbol{R})$, where $\langle s_i \rangle: G \longrightarrow \prod_i \text{codomain}(s_i)$ is the evaluation, that is, the unique map for which $\pi_j \circ \langle s_i \rangle = s_j$ for each j. (This is requiring that the products $\prod_i \text{codomain}(s_i)$ exist.)

(b) $\boldsymbol{S}(-, \boldsymbol{R})$ is *monodivisible* (respectively, *epidivisible*) if whenever $s \in \boldsymbol{S}(-, \boldsymbol{R})$ and $s = me$ is its (epi, extremal mono)-factorization, then $e \in \boldsymbol{S}(-, \boldsymbol{R})$ (respectively, $m \in \boldsymbol{S}$).

THEOREM 1.4. *Suppose that $\boldsymbol{C}$ is co-(well-powered) and an (epi, extremal mono)-category. Suppose that $\boldsymbol{S}$ is a class of morphisms, and that $\boldsymbol{R}$ is a subcategory for which* (a) *$\boldsymbol{S}(-, \boldsymbol{R})$ is closed under evaluations, and* (b) *$\boldsymbol{S}(-, \boldsymbol{R})$ is monodivisible. Then for each $G \in |\boldsymbol{C}|$, there is epic $r_G^S \in \boldsymbol{S}(G, \boldsymbol{R})$ such that, for each $s \in \boldsymbol{S}(G, \boldsymbol{R})$ there is (unique) $\overline{s}$ (not asserted to be in $\boldsymbol{S}$) with $\overline{s} r_G^S = s$. If ρ is another map with these properties of r_G^S, $\rho = \overline{\rho} r_G^S$ for an isomorphism $\overline{\rho}$.*

PROOF. This follows the details of [HS; 37.1]: Let $G \in |\boldsymbol{C}|$. Since $\boldsymbol{C}$ is co-(well-powered), the collection of all $s: G \longrightarrow R_S$ with $s \in \boldsymbol{S}$, $R_S \in |\boldsymbol{R}|$, and s epic, has a representative set, say T. Now consider the following diagram.

(1.5)



Here, $\langle t\rangle$ is the evaluation for the set of maps T, and $\langle t\rangle \in \boldsymbol{S}(G, \boldsymbol{R})$ by (a). Then we write the (epi,extremal mono)-factorization $\langle t\rangle = mr_G^S$. We abbreviate r_G^S to just r for the rest of the proof. Note that $r \in \boldsymbol{S}(G, \boldsymbol{R})$ by (b).

Now let $s \in \boldsymbol{S}(G, \boldsymbol{R})$, and let $s = fu'$ be its (epi,extremal mono)-factorization. By definition of T, there are $u \in T$, and an isomorphism γ with $u' = \gamma u$, and then $\gamma u = \gamma\pi_u\langle t\rangle = \gamma\pi_u mr$. Then $s = fu' = (f\gamma\pi_u m)r$; so $\bar{s} = f\gamma\pi_u m$ is the desired map. It is unique for $\bar{s}r = s$ since r is epic.

If ρ is another such map, then $r = \overline{r}\rho$ and $\rho = \overline{\rho}r$, whence $(\overline{r}\,\overline{\varphi})r = \overline{r}\rho = r = (\mathrm{id})r$, and by uniqueness, $\overline{r}\,\overline{\varphi} = \mathrm{id}$; likewise $\overline{\varphi}\,\overline{r} = \mathrm{id}$; thus $\overline{\rho}$ is an isomorphism.

Theorem 1.4, in the case of $\boldsymbol{S} =$ *all* $\boldsymbol{C}$-morphisms, is exactly the 'if' part of 1.2. In 1.4, a particular r_G^S will be monic if G admits at least one monic $\boldsymbol{S}$-map s to an $\boldsymbol{R}$-object, for then $s = \bar{s}r_G^S$ shows r_G^S is a first factor of a monic, thus monic.

For emphasis, we now de-couple $\boldsymbol{R}$ and $\boldsymbol{S}$ in the hypotheses of 1.4.

COROLLARY 1.6. *Let* **C** *be a category as in* 1.2, *let* **R** *be an epireflective subcategory* (*that is, closed under products and extremal subobjects*), *and let* **S** *be a class of morphisms which is closed under evaluations, and monodivisible. Then the conclusion of* 1.4 *holds.*

In 1.4 (and 1.6) there would seem to be no reason that the extensions $\bar{s}$ should lie in $\boldsymbol{S}$. That this should always be so can be concisely put as $r^S\boldsymbol{S} \subseteq \boldsymbol{S}$, referring to the functor-like operator $r^S\colon (|\boldsymbol{C}|, \boldsymbol{S}) \longrightarrow \boldsymbol{R}$ implicitly defined by 1.4. This issue, in the ℓ-group context described in the Abstract, shall occupy much of the rest of the paper, so we make a formal statement.

THEOREM 1.7. *Assume the hypotheses of* 1.4, *and consider the following further hypotheses.*

(1) $\boldsymbol{S} \circ \boldsymbol{S} \subseteq \boldsymbol{S}$, *and* $\boldsymbol{S}$ *contains all* $\boldsymbol{C}$*-identities.*

(2) $\boldsymbol{S}$ *contains all isomorphisms between* $\boldsymbol{R}$*-objects and all projections from* $\boldsymbol{R}$*-products onto factors.*

(3) $\boldsymbol{S}(-, \boldsymbol{R})$ *is epidivisible.*

If (1) *holds, then* $(|\boldsymbol{C}|, \boldsymbol{S})$ *is a category and* $r^S: (|\boldsymbol{C}|, S) \to \boldsymbol{R}$ *is a functor. If* (1), (2), *and* (3) *hold, then* $r^S\boldsymbol{S} \subseteq \boldsymbol{S}$ *and* $(|\boldsymbol{R}|, S)$ *is an epireflection subcategory of* $(|\boldsymbol{C}|, S)$ *whose reflection maps are the* r^S_G*'s of* 1.4.

PROOF. The first assertion is clear. Concerning $r^S\boldsymbol{S} \subseteq \boldsymbol{S}$, we refer to the equation $\bar{s} = f\gamma\pi m$ in diagram (1.5). By (3), $f, m \in \boldsymbol{S}$, and by (2), $\gamma, \pi_u \in \boldsymbol{S}$; so by (1), $\bar{s} \in \boldsymbol{S}$. Moreover, (1) makes $(|\boldsymbol{C}|, \boldsymbol{S})$ into a category with $(|C|, \boldsymbol{S})$ a subcategory. That $r^S\boldsymbol{S} \subseteq \boldsymbol{S}$ implies the reflectivity statement is now clear.

1.8. A question about Tychonoff spaces. Let Tych be that category (with continuous maps), and let $\boldsymbol{K}$ be the subcategory of compact spaces. As everyone knows, $\boldsymbol{K}$ is epireflective in Tych, *via* the Čech-Stone compactification βX. It has occurred to us to wonder: Is there an $\boldsymbol{S}$ in Tych for which (i) $\boldsymbol{S}(-, \boldsymbol{K})$ is closed under evaluations, and monodivisible, and (ii) for each X, $\boldsymbol{S}(X, \boldsymbol{K})$ contains an embedding, for which (iii) for some X, the r^S_X asserted by 1.4 is *not* the embedding of X in βX ? Such $\boldsymbol{S}$ would create canonical compactifications which are unfamiliar. Li Feng has shown that the answer is yes.

2. Archimedean ℓ-groups and α-complete homomorphisms: Generalities

We shall be concerned now, and for the rest of the paper, with the categories Arch and $\boldsymbol{W}$, according to the discussion in the Abstract. If a discussion, definition, proposition, proof, *et cetera*, fails to specify, it is intended to apply to either. Some of the simpler statements below are valid in all, or all abelian, ℓ-groups, but we shall ignore that. This section simply fits Arch and $\boldsymbol{W}$, and the α-complete homomorphisms, into the context of Section 1. General references for ℓ-groups are [AF] and [BKW]. The most salient reference for $\boldsymbol{W}$ is [BH1].

PROPOSITION 2.1. Arch *and* $\boldsymbol{W}$ (a) *are co-(well-powered),* (b) *are (epi, extremal mono)-categories, and* (c) *have products.*

PROOF. (a) follows from [BH1, 8.3.5 and 8.4.6].
(b) follows from [HS, 34.5], and a little thought.

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(c) In Arch, the categorical product is just the ℓ-group product; that is, the cartesian product with coordinate-wise operations and order; in $\boldsymbol{W}$, the categorical product $\prod_i(G_i, e_i)$, ($e_i$being the distinguished weak unit of G_i) is the Arch-product with $e = (e_i)$ as the weak unit.

In Arch and $\boldsymbol{W}$, the epics are described in [BH1, 8.3.2 and 8.4.4], which underlies (a) and much of the sequel. It is easy to see that monic means one-to-one, but it is not so clear what the extremal monics are.

DEFINITION 2.2. Let α be an infinite cardinal number or the symbol ∞. The ℓ-homomorphism $\varphi: G \longrightarrow H$ is called *α-complete* if, whenever, $\{g_i \mid i \in I\} \subseteq G$, $|I| < \alpha$, $g = \bigvee_i g_i$ in G, then $\varphi(g) = \bigvee_i \varphi(g_i)$ in H. Here, $|I| < \infty$ just means I is a set. The ∞-complete homomorphisms are usually just called complete, or sometimes normal.

Hom_α denotes the class of all α-complete homomorphisms. Note that any ℓ-homomorphism is ω_0-complete: $\mathrm{Hom}_{\omega_0} = \mathrm{Hom}$. A monic in Hom_α will be called an α-embedding. The following is very easy.

LEMMA 2.3. *For the ℓ-homomorphism $\varphi: G \longrightarrow H$, $\varphi \in \mathrm{Hom}_\alpha$ if and only if whenever $\{g_i | i \in I\} \subseteq G$, with $|I| < \alpha$, and $\bigwedge_i g_i = 0$ in G, then $\bigwedge_i \varphi(g_i) = 0$ in H.*

PROPOSITION 2.4. *For $\omega_0 \leq \alpha \leq \infty$, Hom_α is* (a) *closed under evaluations, and* (b) *monodivisible; indeed, whenever $\mu\varphi \in \mathrm{Hom}_\alpha$ with μ monic, then $\varphi \in \mathrm{Hom}_\alpha$.*

PROOF. (a). In a product, any supremum is coordinate-wise.
(b). (See [M, 2.2].) Given $\mu\varphi : G \longrightarrow H$ and $\{g_i\}$ such that $\bigwedge_i \varphi(g_i) \neq 0$, then for some $b > 0$, $\varphi(g_i) \geq b$ for all i. Since μ is one-to-one, $\mu\varphi(g_i) \geq \mu(b) > 0$ for each i, and thus $\bigwedge_i \mu\varphi(g_i) \neq 0$. Since $\mu\varphi \in \mathrm{Hom}_\alpha$, $\bigwedge_i g_i \neq 0$ as desired.

For the same reason as that for (a), Hom_α is closed under products, meaning, if each $f_i : G_i \longrightarrow H_i$ is in Hom_α, then so is $\prod f_i : \prod G_i \longrightarrow \prod H_i$.

COROLLARY 2.5. *Let $\boldsymbol{R}$ be an epireflective subcategory of $\boldsymbol{C} =$ Arch or $\boldsymbol{W}$, and let $\omega_0 \leq \alpha \leq \infty$. Then, for each G, there is an epic $r^\alpha_G \in \mathrm{Hom}_\alpha(G, \boldsymbol{R})$, such that, if $\varphi \in \mathrm{Hom}_\alpha(G, \boldsymbol{R})$, then there is (unique) $\overline{\varphi}$ with $\overline{\varphi} r^\alpha_G = \varphi$. The operator $r^\alpha: (|\boldsymbol{C}|, \mathrm{Hom}_\alpha) \longrightarrow \boldsymbol{R}$ is a functor.*

PROOF. By 1.4 (or 1.6), using 2.4 and 2.1, we have the r^α_G's. One checks 1.7(1), and r^α is a functor.

We note that, in 2.5, the functor r^{ω_0} is just the reflector $r: \boldsymbol{C} \longrightarrow \boldsymbol{R}$.

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COROLLARY 2.6. *With the hypotheses of* 2.5, *if also* $r^{\alpha}\,\mathrm{Hom}_{\alpha} \subseteq \mathrm{Hom}_{\alpha}$, *then* $(|\boldsymbol{R}|, \mathrm{Hom}_{\alpha})$ *is epireflective in* $(|\boldsymbol{C}|, \mathrm{Hom}_{\alpha})$ (*with* $\boldsymbol{C} =$ Arch *or* $\boldsymbol{W}$). *And, if* $\mathrm{Hom}_{\alpha}(-, \boldsymbol{R})$ *is epidivisible, then* $r^{\alpha}\,\mathrm{Hom}_{\alpha} \subseteq \mathrm{Hom}_{\alpha}$ (*for any epireflector* r); *but not conversely.*

PROOF. The first assertion follows from 2.5 and 1.7. The second follows from 1.7 upon checking condition 1.7(2) for $\boldsymbol{S} = \mathrm{Hom}_{\alpha}$. The assertion 'not conversely' follows from Sections 3 and 8 below.

REMARKS 2.7. We need to indicate some specifics about various monoreflective subcategories of Arch and $\boldsymbol{W}$.

(a) We shall be most interested in the subcategory of epicomplete objects. In a category $\boldsymbol{C}$ in which, for simplicity, we suppose monic means one-to-one, an object E is called epicomplete if $E \xrightarrow{\gamma} \bullet$ epic and one-to-one implies γ is an isomorphism. Let $EC(\boldsymbol{C}) = \{E \mid E$ is epicomplete$\}$. It is easy to see (2.11 below) that if $\boldsymbol{R}$ is monoreflective, then $EC(\boldsymbol{C}) \subseteq \boldsymbol{R}$, and so, if $EC(\boldsymbol{C})$ is monoreflective, it is the *smallest* monoreflective subcategory.

(b) In [BH2], the following are shown for $\boldsymbol{C} = \boldsymbol{W}$ or Arch. $E \in EC(\boldsymbol{C})$ if and only if E is divisible, and conditionally and laterally σ-complete, and in $\boldsymbol{W}$ this means that E is $\boldsymbol{W}$-isomorphic to a $D(X)$, X compact and basically disconnected (whose weak unit is the constant function 1). Moreover, $EC(\boldsymbol{C})$ is monoreflective (hence epireflective). Thus there are reflectors $\beta_{\boldsymbol{W}}\colon \boldsymbol{W} \longrightarrow EC(\boldsymbol{W})$ and $\beta_{\mathrm{Arch}} : \mathrm{Arch} \longrightarrow EC(\mathrm{Arch})$. The reflector $\beta_{\boldsymbol{W}}$ is described, quite concretely, in [BH3, 5.1], while not much is known about β_{Arch}. For the sake of the typography, we shall refer to either of these as β unless the context demands otherwise.

(c) By 2.5, for any $\omega_0 \leq \alpha \leq \infty$ we have the functors $\beta^{\alpha} : (|\boldsymbol{C}|\,\mathrm{Hom}_{\alpha}) \longrightarrow EC(\boldsymbol{C})$ ($\boldsymbol{C} =$ Arch or $\boldsymbol{W}$), which have a prominent place in the sequel. Of course, the β^{ω_0}'s are the β's of the previous paragraph. For $\boldsymbol{W}$, the functors β^{ω_1} and β^{∞} are constructed/described in [BH3, 7.2 and 9.6].

(d) In a general category, a monic $\mu : G \to H$ is called *essential* if $\theta\mu$ monic implies θ monic. In Arch or $\boldsymbol{W}$, essentiality of μ is equivalent to each of the following: If I is a non-zero ideal of H, then $\mu^{-1}I \neq (0)$; if $0 < h \in H$, then there are $g \in G$ and $n \in N$, with $0 < \mu(g) \leq nh$.

A monoreflection is called *essential* if each reflection morphism is essential. The β's are not essential [BH3, §9]. However:

(e) [H] characterizes in $\boldsymbol{W}$ those essential monoreflective $\boldsymbol{R}$ for which $|\boldsymbol{R}| \in R \longrightarrow \boldsymbol{Q}$ a surjection implies $\boldsymbol{Q} \in |\boldsymbol{R}|$. [BH4] (respectively [BH5]) describes the least essentially reflective subcategory of $\boldsymbol{W}$ (respectively, Arch), whose objects to some extent deserve the term 'algebraically closed'. We shall have occasion to recall some details of these in Section 8 below.

PROPOSITION 2.8. *In 2.5, if $\boldsymbol{R}$ is monoreflective, then each r_G^α is monic, that is, each $r_G^\alpha : G \longrightarrow r^\alpha G$ is an α-embedding.*

PROOF. According to the comment after 1.4, we see that each G α-embeds into *some* $\boldsymbol{R}$-object. This is an immediate consequence of the following facts, which also will be used later.

LEMMA 2.9. (a) *For each archimedean G, there is an essential embedding $\varepsilon_G : G \longrightarrow \varepsilon G$, with $\varepsilon G = D(X)$ for a certain compact extremally disconnected X. If $G \in |\boldsymbol{W}|$, one may take $\varepsilon_G \in \boldsymbol{W}$.*
(b) *The εG above is epicomplete.*
(c) *An essential embedding is ∞-complete.*
(d) *If E is epicomplete and $\boldsymbol{R}$ is monoreflective, then $E \in |\boldsymbol{R}|$.*

PROOF. (a). See [C, 3.6].
(b) Referring to 2.7(b), an extremally disconnected space is basically disconnected.
(c) See [AF, 8.1.2].
(d) We have $r_E : E \longrightarrow rE$ which is monic and epic (since monoreflective implies epireflective [HS, 36.3]), thus an isomorphism. And $\boldsymbol{R}$ is isomorphism-closed.

3. About the sequel

At this point we can give an overview of the rest of the paper in more detail than the Abstract.

In Section 4, we show that, for $\omega_1 < \alpha < \infty$, $\mathrm{Hom}_\alpha(-, EC)$ is not epidivisible. In spite of this, we cannot produce examples showing $\beta^\alpha \mathrm{Hom}_\alpha \not\subseteq \mathrm{Hom}_\alpha$, though we believe they exist, and perhaps the constructions offer possibilities. The obstacle would seem to be lack of knowledge of what the $\beta^\alpha G$ look like.

In Section 5, we show that $\mathrm{Hom}_{\omega_1}(-, EC)$ *is* epidivisible, for the very special reason that $EC \ni E \xrightarrow{\varphi} \bullet$ in Arch implies $\varphi \in \mathrm{Hom}_{\omega_1}$. This entails $\beta^{\omega_1} \mathrm{Hom}_{\omega_1} \subseteq \mathrm{Hom}_{\omega_1}$, but we have no information about other $(\boldsymbol{R}, r)$'s.

In Section 7, we show that Hom_∞ is epidivisible; thus for any $(\boldsymbol{R}, r)$, $r^\infty \mathrm{Hom}_\infty \subseteq \mathrm{Hom}_\infty$. This involves the surprising theorem: a complete epic embedding is essential.

In Section 8, we show that, for $(\boldsymbol{R}, r)$ *essentially*-reflective, $r^\alpha \mathrm{Hom}_\alpha \subseteq \mathrm{Hom}_\alpha$ for *any* α (in spite of Section 4).

4. Epidivisibility fails

We shall show that, if $\omega_1 < \alpha < \infty$, then $\mathrm{Hom}_\alpha(-, EC)$ is not epidivisible. We shall make an example in $\boldsymbol{W}$ (which turns out to be also an example in Arch), as the

Yosida dual of a topological situation. We describe some preliminaries.

THEOREM 4.1 (The Yosida Representation for $\mathbf{W}$). (a) *For $G \in |\mathbf{W}|$, there is compact Hausdorff YG so that G is $\mathbf{W}$-isomorphic to $\hat{G} \subseteq D(YG)$ and $\hat{G}$ separates the points of YG. If G is also $\mathbf{W}$-isomorphic to $\overline{G} \subseteq D(X)$, with X compact Hausdorff and $\overline{G}$ separating the points, then there is a homeomorphism $\tau : X \longrightarrow YG$ such that $\overline{g} = \hat{g} \circ \tau$ for each $g \in G$.*
(b) *For $\theta : G \longrightarrow H$ in $\mathbf{W}$, there is unique continuous $Y\theta : YH \longrightarrow YG$ for which $\theta(g)^\wedge = \hat{g} \circ (Y\theta)$ for each $g \in G$. The homomorphism θ is one-to-one if and only if $Y\theta$ is onto, and if θ is onto, then $Y\theta$ is one-to-one.*

(These facts are described in [BH1, 8.2.4] and [BH3, 2.2].)

THEOREM 4.2. *The divisible hull of an abelian ℓ-group denoted by $d_G : G \to dG$ is, equivalently,*
(a) *the monoreflection of G into divisible abelian ℓ-groups, or*
(b) *an embedding d_G into a divisible abelian ℓ-group dG such that, for $h \in dG$ there are $g \in G$ and integers p, q with $qh = pg$, or*
(c) *an essential embedding d_G into a divisible abelian ℓ-group dG such that $d_G(G) \leq D \leq dG$, with D divisible implying $D = dG$ (where $\leq$ stands for 'is an ℓ-subgroup of'); or*
(d) *in case $G \in \mathbf{W}$, it is $Q\hat{G} \equiv \{x\hat{g} \mid x \in Q, g \in G\}$ (where Q denotes the rational numbers). Then $YdG = YG$ and for $\theta \in \mathbf{W}$, $Yd\theta = Y\theta$.*

REMARK. The descriptions (a), (b), and (c) are folk theorems. We do not know a reference, but they are not hard to prove. Part (d) follows from these and 4.1.

We now have a topological description of $\theta\,\mathrm{Hom}_\alpha$ (a descendent of a Boolean version in [S]). In it, an α-cozero-set is the union of $< \alpha$ cozero sets, and continuous $\tau : X \longrightarrow Y$ is called α-$SpFi'$ if U being dense α-cozero in Y implies $\tau^{-1}U$ dense in X, where $SpFi$ stands for spaces with filters.

LEMMA 4.3. *For $\theta : G \longrightarrow H$ in $\mathbf{W}$, the following are equivalent:*
(a) $\theta \in \mathrm{Hom}_\alpha$, (b) *$Y\theta$ is α-$SpFi$,* (c) $d\theta \in \mathrm{Hom}_\alpha$.

This is a generalization of [BH3, 4.2 and 9.3] (the cases ω_1 and ∞),and [M, 3.10] (for vector lattices). We can describe the proof of 4.3 by referring to the arguments in [BH3]. The proofs in [BH3] use divisibility, but neglect mention of it. The following serves to correct that and to prove 4.3: In [BH3, 4.2], the proof of (b) implies (a) and the α-generalization is valid, while the proof of (b) implies (a) and the α-generalization need divisibility. But $Y\theta = Yd\theta$, so that the proof works to show (b) implies (c) here.

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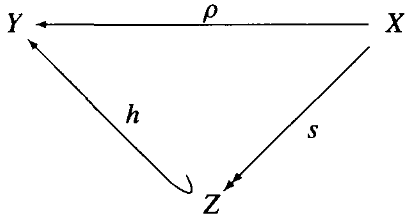
And (c) implies (a): we have $(d\theta)d_G = d_H\theta$, with $d\theta \in \mathrm{Hom}_\alpha$ by hypothesis and $d_G \in \mathrm{Hom}_\alpha$ by 2.11. Thus $d_H\theta \in \mathrm{Hom}_\alpha$. Then $\theta \in \mathrm{Hom}_\alpha$ by 2.4(b).)

Let $\omega_1 < \alpha < \infty$. The topological version of our example is

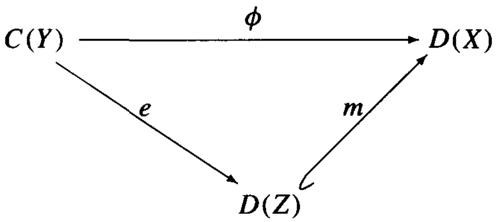
PROPOSITION 4.4. *In compact Hausdorff spaces, there is a commuting triangle*



in which ρ is α-$SpFi$; X and Z are basically disconnected; s is a surjection and h is an injection; s is ω_1-$SpFi$ but not α-$SpFi$.

We construct such a triangle shortly. It implies the algebraic example:

PROPOSITION 4.5. *The triangle in* 4.4 *produces, in* **W** *and in* Arch, *the commuting triangle*



via *the definitions:* $\rho(f) \equiv f \circ \rho$ $(f \in C(Y))$; $e(f) \equiv f \circ h$ $(f \in C(Y))$; $m(f) \equiv f \circ s$ $(f \in D(Z))$; *in which:* $\rho \in \mathrm{Hom}_\alpha$; $D(X)$ *and* $D(Z)$ *are epicomplete;* $\rho = me$ *is the* (*epi, extremal mono*)*-factorization of* ρ; *and* $m \notin \mathrm{Hom}_\alpha$.

PROOF OF 4.5 FROM 4.4. We first concentrate on $\mathbf{W}$. Clearly $\rho, e \in \mathbf{W}$. For $m : D(Z) \longrightarrow D(X)$ to be defined, it is needed that $(f \circ s)^{-1}R = s^{-1}(f^{-1}R)$ be dense; that is so because $f^{-1}R$ is dense ω_1-cozero (= cozero), and s is ω_1-$SpFi$; so $m \in \mathbf{W}$. Clearly $\rho = me$, and this will be the (epi, extremal mono)-factorization if just e is epic and m extremal monic, since those factorizations are (essentially) unique.

Clearly, the ℓ-groups separate the points of their spaces, so that $YC(Y) = Y$, $YD(X) = X$, $YD(Z) = Z$, by the uniqueness statement in 4.1(a). It follows that $Y\varphi =$

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$\rho, Ye = h, Ym = s$, by the uniqueness statement in 4.1(b). Thus, by 4.1(b), m is one-to-one (thus monic) and by 4.3, $\varphi \in \mathrm{Hom}_\alpha$ while $m \notin \mathrm{Hom}_\alpha$.

Since X, Z are basically disconnected, $D(X)$, $D(Z)$ are epicomplete, by 2.7(b). Thus, m is extremal monic.

To see that e is epic: e factors as $C(Y) \xrightarrow{e'} C(Z) \overset{e''}{\hookrightarrow} D(Z)$, where $e'(f) \equiv f \circ h$, and e'' is inclusion. Here e' is epic because it is a surjection (because h is a homeomorphism onto a closed subset of Y, and the Tietze-Urysohn Theorem applies; see [GJ, Chapter 10]). And e'' is epic since Ye'' is the identity on Z; see [BH1, p. 182, remark (c)]. Thus $e = e''e'$ is epic.

We have proved all assertions in $\boldsymbol{W}$.

For Arch, it just remains to note that $D(X)$ and $D(Z)$ are also epicomplete in Arch (2.7(b)), so m is extremal monic in Arch. That e is epic in Arch follows from [BH1, 8.5.2] (which says $\boldsymbol{W}$-epic implies Arch-epic for a map whose codomain is an algebra (like $D(Z)$).

Construction of the triangle in 4.4. Let γ be an infinite cardinal. A space (always completely regular Hausdorff) is said to be a *$P(\gamma)$-space* if the intersection of $< \gamma$ open sets is again open, and an *almost $P(\gamma)$-space* if there are no proper dense α-cozero sets. ($P(\omega_1)$ is what is called P in [GJ].)

The following is routine to verify:

PROPOSITION 4.6. *Let $A(\gamma)$ be the space consisting of a set D of cardinal γ, whose points are isolated, with another point ∞, whose open neighborhoods have the form $\{\infty\} \cup (D - F)$, for $F \subseteq D$, $|F| < \gamma$. Suppose that the cardinal γ is regular.*
(a) *For $E \subseteq D$, $\infty \in \overline{E}$ if and only if $|E| = \gamma$.*
(b) *For $C \subseteq A(\gamma)$, C is cozero if and only if either $C \subseteq D$ and $|C| < \gamma$ or $\infty \in C$ and $|D - C| < \gamma$ if and only if C is γ-cozero.*
(c) *$A(\gamma)$ is a $P(\gamma)$-space, thus an almost $P(\gamma)$-space.*

PROPOSITION 4.7. *For any space Z and regular γ, there is an almost $P(\gamma)$-space W and an embedding $g : Z \to W$ onto a closed subset of W.*

PROOF. (This is similar to [DHH, 5.7], and [BH4, 9.4]). W is the set-theoretic product $Z \times A(\gamma)$, with the product topology refined by decreeing that, for $y \neq \infty$ in $A(\gamma)$, any $\{(x, y)\}$ is open. If B is a dense γ-cozero set, then certainly $y \neq \infty$ implies $(x, y) \in B$, while for any $x \in Z$, $B \cap (\{x\} \times A(\gamma))$ is dense γ-cozero in $\{x\} \times A(\gamma)$, which is a copy of $A(\gamma)$, and thus $(x, \infty) \in B$; so $B = W$. The embedding $g : Z \to W$ is just $g(x) = (x, \infty)$.

LEMMA 4.8. *Let* $\tau : X \to W$ *be continuous, with* W *almost* $P(\gamma)$. *Then* τ *is* γ-$SpFi$, *and so is* $\beta\tau : \beta X \to \beta W$ *(the extension over the Čech-Stone compactifications).*

PROOF. W has no dense γ-cozero sets, so τ is γ-$SpFi$ vacuously, and if B is dense γ-cozero in βW then $B \supseteq W$. Thus $(\beta\tau)^{-1}B \supseteq \tau^{-1}W = X$, which is dense in βX.

We now construct the triangle 4.4: Let $\omega_1 < \alpha < \infty$. Let $Z = \beta A(\omega_1)$; since $A(\omega_1)$ is $(P\omega_1)$, it is basically disconnected, and so is Z. Choose sequentially inaccessible $\gamma > \alpha$ (say $\gamma = \alpha^+$), and then apply 4.7 to produce an almost $P(\gamma)$-space W, and injection $g : Z \to W$. Let $Y = \beta W$, and let $h : Z \to Y$ be $Z \to W \hookrightarrow \beta W$, also an injection.

Let $f : A(\omega_1) + A(\omega_1) \to A(\omega_1)$ be the function which is the identity on the first copy of $A(\omega_1)$, and collapses the second copy to ∞. Let $X = \beta(A(\omega_1) + A(\omega_1))$, which is basically disconnected just as Z was, and let $s = \beta f : X \to Z$. Then, s is a surjection, and is ω_1-$SpFi$ by 4.8, since $A(\omega_1)$ is almost $P(\omega_1)$. But s is not ω_2-$SpFi$: in $Z = \beta A(\omega_1)$, D has cardinal ω_1, and thus is ω_2-cozero, while $s^{-1}D$ is contained in the first copy of $A(\omega_1)$, thus is not dense. Since $\alpha \geq \omega_2$, s is not α-$SpFi$.

Let $\rho \equiv hs : X \to Y$. Notice that ρ is the Čech-Stone extension of a map $A(\omega_1) + A(\omega_1) \to W$. By 4.8, ρ is γ-$SpFi$, hence α-$SpFi$.

This completes 4.4, thus 4.5.

We note that 4.4 is a (rather extensive) modification of [BHM, 2.10] (which shows that the injective $\circ$ surjective factorization of an ω_1-$SpFi$ map need not have the surjective part in ω_1-$SpFi$).

5. Countably complete homomorphisms

We shall prove that $\mathrm{Hom}_{\omega_1}(-, EC)$ is epidivisible. This is for the most special of reasons:

THEOREM 5.1. *If* $\rho : E \to F$ *is an* ℓ*-homomorphism, with* $E, F \in |\,\mathrm{Arch}\,|$, *and if* E *is divisible, conditionally and laterally* σ*-complete (that is,* $E \in EC(\mathrm{Arch})$*), then* $\rho \in \mathrm{Hom}_{\omega_1}$.

A bit more than 5.1 is proved in [F, 1.13]. The result for **W** appears earlier in [Tz] and [V]; this is less general, but (via 6.4 below, say), implies the full 5.1. Related results appear in [Tu]; this seems not to contain 5.1, but does contain theorems not implied by 5.1. [BH3, 4.4] is another contribution.

COROLLARY 5.2. *In* Arch *and* **W**, $\mathrm{Hom}_{\omega_1}(-, EC)$ *is epidivisible.*

PROOF. In C = Arch or W, let $\rho : G \to E$ be ω_1-complete with $E \in EC(C)$. Let $\rho = me$, $e : G \to H$, $m : H \to E$, be the (epi, extremal mono)-factorization. Since $EC(C)$ is epireflective (2.7(b)), $EC(C)$ is closed under forming extremal subobjects (1.2(b)); so $H \in EC(C)$. That means H is divisible and conditionally and laterally σ-complete (2.7(b)), so by 5.1, $m \in \mathrm{Hom}_{\omega_1}$.

COROLLARY 5.3. *For* C = Arch, *or* W, $(|EC(C)|, \mathrm{Hom}_{\omega_1})$ *is monoreflective in* $(|C|, \mathrm{Hom}_{\omega_1})$.

The following will be needed once later. It is a corollary of 5.1, and the historical remarks following 5.1 apply here as well. (One may see [F, 1.14] for the proof.)

THEOREM 5.4. *If* $\varphi : E \to F$ *is a surjective* ℓ*-homomorphism, and if* E *is divisible, conditionally and laterally-*σ*-complete, then so is* F. *(That is,* EC *is closed under formation of quotients.)*

6. Some properties of homomorphisms in Arch, versus W

This section is preliminary to the sections on Hom_∞, and essential reflections. The idea is that the factoring out of 'principal perps' transfers a situation in Arch into situations in W, where the Yosida Representation provides a grip; then one tries to transfer information back. The procedure is employed in [BH1, 2], among other places.

For $G \in$ Arch, and for $S \subseteq G$, $S^\perp$ denotes $\{g \in G \mid |g| \wedge |s| = 0 \text{ for each } s \in S\}$. It is well-known that any $S^\perp$ is a complete ideal in G, $G/S^\perp \in |\,\text{Arch}\,|$, and the quotient $G \to G/S^\perp$ is in Hom_∞. See [AF, p.11] and [BKW, p.227].

In Arch, let $\rho : M \to N$. For each $u \in M^+$, let $\pi_1 : M \to M/u^\perp$ and $\pi_2 : N \to N/\rho(u)^\perp$ be the quotients, and let $\rho_u : M/u^\perp \to N/\rho(u)^\perp$ be defined by $\rho_u(g + u^\perp) = \rho(g) + \rho(u)^\perp$. Then ρ_u is an ℓ-homomorphism with $\pi_2\rho = \rho_u\pi_1$, $u + u^\perp$ and $\rho(u) + \rho(u)^\perp$ are weak units, $\rho_u(u + u^\perp) = \rho(u) + \rho(u^\perp)$, and $\rho_u \in W$.

We record several connections between ρ and the various ρ_u.

PROPOSITION 6.1. ρ *is one-to-one if and only if each* ρ_u *is one-to-one.*

PROOF. If ρ is not one-to-one, we have $u > 0$ with $\rho(u) = 0$. Thus $u + u^\perp \neq 0$, while $N/\rho(u)^\perp = (0)$ whence $\rho_u(u + u^\perp) = 0$. Now suppose ρ is one-to-one, $u \in M^+$, and $\rho_u(g + u^\perp) = \rho(g) + \rho(u)^\perp = 0$. Then $\rho(g) \in \rho(u)^\perp$, that is, $\rho(g) \wedge \rho(u) = 0$, giving $\rho(g \wedge u) = 0$. Thus $g \wedge u = 0$; that is, $g \in u^\perp$, and so $g + u^\perp = 0$.

PROPOSITION 6.2. *$\rho \in \mathrm{Hom}_\alpha$ if and only if each $\rho_u \in \mathrm{Hom}_\alpha$.*

PROOF. We begin with the necessity. A lemma is needed.

LEMMA 6.3. *Let $\psi \in W$, with e the weak unit in* domain(ψ). *Then $\psi \in \mathrm{Hom}_\alpha$ if and only if whenever $\{g_i | i \in I\}$ has $|I| < \alpha$, $g_i \leq e$ for all i, and $\bigwedge_i g_i = 0$, then $\bigwedge_i \psi(g_i) = 0$.*

PROOF. The necessity is clear. Conversely, if $\psi \notin \mathrm{Hom}_\alpha$, then by 2.3 there is $\{g_i | i \in I\}$ with $|I| < \alpha$ and $\bigwedge g_i = 0$, but $\bigwedge \psi(g_i) \neq 0$. So there is h with $0 < h \leq \psi(g_i)$ for all i, and $h \wedge \psi(e) > 0$ since $\psi(e)$ is a weak unit in codomain(ψ). Now $0 = (\bigwedge g_i) \wedge e = \bigwedge(g_i \wedge e)$, and each $g_i \wedge e \leq e$, but $\bigwedge \psi(g_i \wedge e) = \bigwedge(\psi(g_i) \wedge \psi(e)) \geq h \wedge \psi(e) > 0$, so the condition fails.

Now let $\rho : M \to N$ be α-complete, and let $u \in M^+$. We use 6.3 to show that $\rho_u \in \mathrm{Hom}_\alpha$, so suppose given $\{g_i + u^\perp\}$, $g_i + u^\perp \leq u + u^\perp$ for each i with $\bigwedge(g_i + u^\perp) = 0$, and $(g_i \wedge u) + u^\perp = g_i + u^\perp$ for each i, while $\bigwedge(g_i \wedge u) = 0$. Recalling $\pi_2\rho = \rho_u\pi_1$, and $\pi_2 \in \mathrm{Hom}_\infty$, so that $\pi_2\rho \in \mathrm{Hom}_\alpha$, we have $0 = \pi_2\rho(\bigwedge(g_i \wedge u)) = \bigwedge \pi_2\rho(g_i \wedge u) = \bigwedge \rho_u\pi_1(g_i \wedge u) = \bigwedge \rho_u(g_i \wedge u + u^\perp) = \bigwedge \rho_u(g_i + u^\perp)$, as desired. Thus $\rho_u \in \mathrm{Hom}_\alpha$.

We turn to the sufficiency in 6.2. Two lemmas are needed.

LEMMA 6.4. *Suppose that $N = \rho(M)^{\perp\perp}$. If each $\rho_u \in \mathrm{Hom}_\alpha$, then $\rho \in \mathrm{Hom}_\alpha$.*

PROOF. Let $\pi\rho_u : \prod M/u^\perp \to \prod N/\rho(u)^\perp$ be the product map (over $u \in M^+$). By the remark after 2.4, $\pi\rho_u \in \mathrm{Hom}_\alpha$. Let $e_1 : M \to \prod M/u^\perp$ be the evaluation for the family $\{M \to M/u^\perp | u \in M^+\}$, and $e_2 : N \to \prod N/\rho(u)^\perp$ the evaluation for $\{N \to N/\rho(u)^\perp | \; u \in M^+\}$. These families are in Hom_∞, hence in Hom_α, so $e_1, e_2 \in \mathrm{Hom}_\alpha$ by 2.4(a). Thus, $(\pi\rho_u)e_1 \in \mathrm{Hom}_\alpha$. Evidently, we have $(\pi\rho_u)e_1 = e_2\rho$, so $e_2\rho \in \mathrm{Hom}_\alpha$. Now, the condition $N = \rho(M)^{\perp\perp}$ is the same as $\rho(M)^\perp = (0)$, or $\bigcap\{\rho(u)^\perp | u \in M^+\} = (0)$; that makes e_2 monic. By 2.4(b), $\rho \in \mathrm{Hom}_\alpha$.

LEMMA 6.5. *If I is an ℓ-ideal in the ℓ-group G, then the embedding $I \hookrightarrow G$ is ∞-complete.*

PROOF. Suppose each $f_i \in I^+$ with $\bigwedge^G f_i \neq 0$. Then there is $g \in G$ with $f_i \geq g > 0$ for each i. Since I is convex, $g \in I$, so $\bigwedge^I f_i \neq 0$.

We conclude the proof of 6.2. Suppose, $\rho : M \to N$ has each $\rho_u \in \mathrm{Hom}_\alpha$. Factor ρ as $\rho = j\rho^0$, where $\rho^0 : M \xrightarrow{\rho^0} \rho(M)^{\perp\perp}$ is the range-restriction, and $j : \rho(M)^{\perp\perp} \hookrightarrow N$ is the inclusion. Then, for each $u \in M^+$, we also have the

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factorization $\rho_u = (j_u)\rho_u^0$ (with obvious notation), and j_u is monic by 6.1. By 2.3(b), $\rho_u^0 \in \mathrm{Hom}_\alpha$. By 6.6, $\rho^0 \in \mathrm{Hom}_\alpha$. By 6.5, $j \in \mathrm{Hom}_\alpha$. Thus $\rho = j\rho^0 \in \mathrm{Hom}_\alpha$, as desired.

An embedding $\rho : M \to N$ in Arch is called *coessential* in Arch if $\alpha\rho = 0$ ($\alpha \in$ Arch) implies $\alpha = 0$. Evidently, an epic embedding is coessential, and it is not hard to see that ρ coessential implies $\rho(M)^\perp = (0)$ [BH1, 8.4.3].

Explicit mention of 6.6 below is barely needed here, but it underlies the crucial 2.7(b) and is in the spirit of this section. We shall need 6.7 later.

PROPOSITION 6.6 (BH1, 8.4.4). *ρ is epic in* Arch *if and only if ρ is coessential and each ρ_u is epic in* **W**.

PROPOSITION 6.7. *ρ is an essential embedding in* Arch *if and only if $\rho(M)^\perp = (0)$ and each ρ_u is an essential embedding in* **W**.

PROOF. Let ρ be an essential embedding. If $h > 0$, and g, n are chosen with $0 < \rho(g) \le nh$, then $h \notin \rho(g)^\perp \supseteq \rho(M)^\perp$. By 6.1, $\rho_u$is an embedding. Suppose $h + \rho(u)^\perp > 0$ with $h > 0$. Then $h \wedge \rho(u) \le h$ and $0 < h \wedge \rho(u) + \rho(u)^\perp$ (since $\rho(u) + \rho(u)^\perp$ is a weak unit). Now choose g, n so that $0 < \rho(g) \le n(h \wedge \rho(u))$. Then $\rho(g) \le n\rho(u)$, and $\rho(g) \notin \rho(u)^\perp$. Thus $0 < \rho(g) + \rho(u)^\perp \le nh + \rho(u)^\perp$, as desired.

Conversely, suppose that $\rho(M)^\perp = (0)$, and that each ρ_u is an essential embedding. Then ρ is an embedding, by 6.1. Let $h > 0$. Then $h \notin \rho(M)^\perp$, so $h \notin \rho(u)^\perp$ for some $u > 0$. Then, $0 < h + \rho(u)^\perp$, and since $\rho(u) + \rho(u)^\perp$ is a weak unit, $0 < h \wedge \rho(u) + \rho(u)^\perp$. Since ρ_u is essential, there are g, n with $0 < \rho(g) + \rho(u)^\perp \le n(h \wedge \rho(u)) + \rho(u)^\perp$, and then $0 < \rho(g) \wedge \rho(u) + \rho(u)^\perp \le n(h \wedge \rho(u)) + \rho(u)^\perp$. We now have $(\rho(g) \wedge \rho(u)) \wedge (n(h \wedge \rho(u))) - \rho(g) \wedge \rho(u) \in \rho(u)^\perp \cap \rho(u)^{\perp\perp} = (0)$, whence $0 < \rho(g \wedge u) \le n(h \wedge \rho(u)) \le nh$, as desired.

7. Complete homomorphisms

We shall prove in 7.4 below that Hom_∞ is epidivisible. Using 2.6 and 2.8, we then have:

THEOREM 7.1. *In $\boldsymbol{C} = \boldsymbol{W}$ or* Arch, *and if $(\boldsymbol{R}, r)$ is any monoreflection, then $r^\infty \mathrm{Hom}_\infty \subseteq \mathrm{Hom}_\infty$, and $(|\boldsymbol{R}|, \mathrm{Hom}_\infty)$ is monoreflective in $(|\boldsymbol{C}|, \mathrm{Hom}_\infty)$.*

The proof of 7.4 has two main constituents: another factorization theorem for Hom_∞ (7.2), and the rather striking fact that complete epic embeddings are essential (7.3).

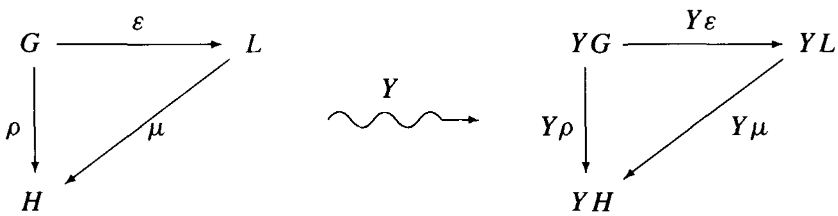
THEOREM 7.2. *In* **W**, *or in* Arch, *suppose that* $\rho \in \mathrm{Hom}_\infty$, *and that* $\rho = \mu\varepsilon$ *with* μ *monic.* (*We recall that* $\varepsilon \in \mathrm{Hom}_\infty$, *by* 2.4(b)).

(a) *If* ε *is surjective, then* $\mu \in \mathrm{Hom}_\infty$. *That is, each* $\rho \in \mathrm{Hom}_\infty$ *has its injective* $\circ$ *surjective factorization in* Hom_∞.

(b) *If range*(ε) *is essentially embedded in* codomain(ε), *then* $\mu \in \mathrm{Hom}_\infty$.

PROOF. (a) for **W**: By 4.3, this has a dual topological statement, which is exactly 2.9 of [BHM].

(b) for **W**: We first prove this assuming ε is one-to-one. We shall need to know that, in **W**, an injection $G \xrightarrow{\varepsilon} L$ is essential if and only if the surjection $YG \xleftarrow{Y\varepsilon} YL$ is irreducible. That is, U non-void open in YL implies a V non-void open in YG with $(Y\varepsilon)^{-1}V \subseteq U$. This is proved in [HR, 4.1]. Now, since $\rho = \mu\varepsilon$, we have $Y\rho = (Y\varepsilon)(Y\mu)$, as:



Employing 4.3, let U be dense open in YL. Since $Y\varepsilon$ is irreducible, it follows that there is V dense open in YG such that $(Y\varepsilon)^{-1}V$ is dense in U. Since $\rho \in \mathrm{Hom}_\infty$, $(Y\rho)^{-1}V$ is dense in YH (by 4.3). But $(Y\mu)^{-1}V \supseteq (Y\mu)^{-1}(Y\varepsilon)^{-1}V = (Y\rho)^{-1}V$, so $(Y\mu)^{-1}V$ is dense. Thus $\mu \in \mathrm{Hom}_\infty$, by 4.3.

Now, just assuming range(ε) essential in codomain(ε), let $\varepsilon = \varepsilon'\varepsilon^\circ$ be the injective $\circ$ surjective factorization of ε, so $\rho = (\mu\varepsilon')\varepsilon^\circ$. By (a), $\mu\varepsilon' \in \mathrm{Hom}_\infty$. By the previous paragraph, $\mu \in \mathrm{Hom}_\infty$ (since ε' is 1-1).

(b) for Arch. We use Section 6 and its notation. Given $\rho = \mu\varepsilon$ with μ monic and range(ε) essentially embedded, and given $u \in (\mathrm{domain}(\rho))^+$, we have $\rho_u = \mu_u\varepsilon_u$, with $\rho_u \in \mathrm{Hom}_\infty$ (by 6.4), μ_u monic (by 6.3), and range(ε_u) essentially embedded (by 6.9 and 6.3). Then, (b) for **W** says $\mu_u \in \mathrm{Hom}_\infty$. By 6.4, $\mu \in \mathrm{Hom}_\infty$.

(a) for Arch: (b) implies (a).

THEOREM 7.3. *In* **W**, *and in* Arch:

(a) *For each* G, $\beta^\infty G$ *is the only epicompletion of* G *in which* G *is completely embedded, and the only essential epicompletion of* G (*up to isomorphism over* G).

(b) *If* $\varepsilon : G \to H$ *is epic, and* $\varepsilon \in \mathrm{Hom}_\infty$, *then* $\varepsilon(G)$ *is essential in* H.

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PROOF. (a) for $\mathbf{W}$: This is the key to the whole theorem, and is exactly 9.10 of [BH3]. It is not so easy, and depends on an explicit construction of $\beta^\infty G$.

(b) For either $\mathbf{W}$ or Arch, consider the factorization $G \xrightarrow{\varepsilon^\circ} \varepsilon(G) \overset{\varepsilon'}{\hookrightarrow} H$. As the second factor of an epic, ε' is epic. By 7.2(a), $\varepsilon' \in \mathrm{Hom}_\infty$. We are seeking exactly to prove ε' is essential. So for the proof, we might as well, and do, assume ε one-to-one at the outset.

(b) for $\mathbf{W}$: Given our $\varepsilon : G \hookrightarrow H$, we have $\beta_H^\infty \varepsilon : G \hookrightarrow \beta^\infty H$ an epicompletion of G in which G is completely embedded, so $\beta_H^\infty \varepsilon = \beta_G^\infty$ (by (a) for $\mathbf{W}$), and thus $\beta_H^\infty \varepsilon$ is essential. It follows that ε is essential.

(b) for Arch: We use Section 6. Given our $\varepsilon : G \to H$, for each $u \in G^+, \varepsilon_u \in \mathrm{Hom}_\infty$ (by 6.4), and is $\mathbf{W}$-epic (by 6.8). By (b) for $\mathbf{W}, \varepsilon_u$ is essential. But ε is coessential (6.8); thus $\varepsilon(G)^\perp = (0)$ (noted before 6.8), so ε is essential (by 6.9).

(a) for Arch: Given G, $\beta_G : G \hookrightarrow \beta^\infty G$ is essential, by (b). Now, if $\varphi : G \hookrightarrow H$ is another epicompletion, with $\varphi \in \mathrm{Hom}_\infty$, then there is $\Psi : \beta^\infty G \to H$ with $\Psi \beta_G = \varphi$. Since β_G is essential, Ψ is one-to-one. Now, since φ is epic, the embedding $\Psi(\beta^\infty G) \hookrightarrow H$ is epic also. But, by 5.4, $\Psi(\beta^\infty G)$ is epicomplete. Thus Ψ is onto H, and hence an isomorphism.

Next, if $\varphi : G \hookrightarrow H$ is an essential epicompletion, then $\varphi \in \mathrm{Hom}_\infty$ (2.11), and the previous paragraph applies.

The proof of 7.3 is concluded.

COROLLARY 7.4. *In* $\mathbf{W}$ *and in* Arch, Hom_∞ *is epidivisible. Indeed,* any *monic* $\circ$ *epic factorization of a map in* Hom_∞ *lies in* Hom_∞.

PROOF. Let $\varphi \in \mathrm{Hom}_\infty$, and let $\varphi = \mu\varepsilon$ with ε epic and μ monic. By 2.4, $\varepsilon \in \mathrm{Hom}_\infty$. By 7.3(b), range($\varepsilon$) is essentially embedded. Then, by 7.2(b), $\mu \in \mathrm{Hom}_\infty$.

8. Essential reflections

A monoreflective $(\boldsymbol{R}, r)$ is called *essentially reflective* if each reflection map $r_G : G \to rG$ is an essential embedding. Such reflections yield the simplest situations of those we are considering here. The result depends on the descriptions of the maximum essential reflections from [BH4, 9.2] (for $\mathbf{W}$) and [BH5, 11.2] (for Arch).

THEOREM 8.1. *Let* $\omega_0 \leq \alpha \leq \infty$, *and let* $(\boldsymbol{R}, r)$ *be essentially reflective in* $\boldsymbol{C} = \mathbf{W}$ *or* Arch.

(a) *For each* G, r_G^α *is* r_G *in the sense that there is an isomorphism* $\theta : rG \to r^\alpha G$ *with* $r_G^\alpha = \theta r_G$.

(b) $r\,\mathrm{Hom}_\alpha \subseteq \mathrm{Hom}_\alpha$.

(c) $(|\boldsymbol{R}|, \mathrm{Hom}_\alpha)$ *is monoreflective in* $(|\boldsymbol{C}|, \mathrm{Hom}_\alpha)$, *with reflection maps* r_G.

PROOF. (a) Each $\rho \in \mathrm{Hom}(G, R)$ lifts uniquely over rG, *a fortiori* for $\rho \in \mathrm{Hom}_\alpha(G, \boldsymbol{R})$. Moreover, by 2.11, each $r_G \in \mathrm{Hom}_\alpha$. Now (a) follows from the uniqueness statement in 1.4 (using Section 2).

(c) follows from (a), (b), 2.6 and 2.8.

(b) This takes a while. We begin by noting that it is true for the special case of the divisible hull dG (4.2).

LEMMA 8.2. (a) *For each* $G \varepsilon |\mathrm{Arch}|$ *and* $u \varepsilon G^\perp$, $d(G/u^\perp) = dG/u^\perp$.
(b) *For* $\theta \in$ Arch, $\theta \in \mathrm{Hom}_\alpha$ *if and only if* $d\theta \in \mathrm{Hom}_\alpha$.

PROOF. (a) We are viewing $G \leq dG$. By 6.1, $G/u^\perp$ embeds in $dG/u^\perp$. As a quotient of a divisible ℓ-group, $dG/u^\perp$ is divisible. Given $h + u^\perp$ in $dG/u^\perp$, we choose g, m and n, so that $mh = ng$ in dG, and then $m(h + u^\perp) = n(g + u^\perp)$. So $dG/u^\perp$ is $d(G/u^\perp)$ by 4.2(b).

(b) Lemma 4.3 asserts this for $\theta \in \boldsymbol{W}$. For $\theta : G \to H$ in Arch, and $d\theta : dG \to dH$ with $(d\theta)d_G = d_H\theta$, we use 6.3. Note that $d_G, d_H \in \mathrm{Hom}_\infty \subseteq \mathrm{Hom}_\alpha$ by 2.9. If $d\theta \in \mathrm{Hom}_\alpha$, then $(d\theta)d_G \in \mathrm{Hom}_\alpha$, and then $\theta \in \mathrm{Hom}_\alpha$ since d_H is monic (2.4). Now suppose $\theta \in \mathrm{Hom}_\alpha$, and let $u \in (dG)^+$. If $v \in G$ has $mu = nv$, then $u^\perp = v^\perp(\perp$ in $dG)$. We have $(d\theta)_v(d_G)_v = (d_H)_v\theta_v$, clearly. By 6.2, $\theta_v \in \mathrm{Hom}_\alpha$, but by 8.2(a), $(d\theta)_v = d(\theta_v)$, and $d(\theta_v)$, and $d(\theta_v) \in \mathrm{Hom}_\alpha$ by 4.3. That is, $(d\theta)_u = (d\theta)_v \in \mathrm{Hom}_\alpha$. By 6.4, $d\theta \in \mathrm{Hom}_\alpha$. (One can, of course, prove 8.2(b) just in abelian ℓ-groups.)

DEFINITION 8.3. Let $\omega_o \leq \beta \leq \infty$, and let $H \leq K$. We say that H is β*-jamd* in K (*join and meet dense*) if whenever $0 \leq k \in K$, there is $\{h_i | i \in I\} \subseteq H$ with $|I| < \beta$, and $k = \bigwedge_i h_i$.

LEMMA 8.4 (cf. [M, 2.9]). *Let* $\omega_0 \leq \beta < \alpha \leq \infty$. *Suppose that* $\rho = \psi\varepsilon$, *with* ε *an embedding with range*(ε) β^+*-jamd in* codomain(ε). *Then* $\rho \varepsilon \mathrm{Hom}_\alpha$ *implies* $\psi \varepsilon \mathrm{Hom}_\alpha$.

PROOF. We write $\varepsilon : G \hookrightarrow G'$ and $\psi : G' \to H$.

Suppose $\{f_i | i \;\varepsilon\; I\} \subseteq G', |I| < \alpha$, and $\bigwedge f_i = 0$. For each i, there is $\{g_i^j \mid j \varepsilon J_i\} \subseteq G$ with $|J_i| \leq \beta$, with $f_i = \bigwedge \varepsilon(g_i^j)$. Thus $0 = \bigwedge_i \bigwedge_j \varepsilon(g_i^j)$, and so $0 = \bigwedge_i \bigwedge_j g_j^i$ since ε is one-to-one. Now the index set for this is $\cup_i J(i)$, whose cardinal is $\sum_i |J(i)| \leq \beta \cdot |I| = \beta \vee |I| < \alpha$. Thus, $0 = \bigwedge_i \bigwedge_j \rho(g_j^i)$, since $\rho \varepsilon \mathrm{Hom}_\alpha$. Since $\rho = \psi\varepsilon$, we have $(*)$ $0 = \bigwedge_i \bigwedge_j \psi\varepsilon(g_j^i)$. If, now, $\bigwedge_i \psi(f_i) \neq 0$, then we have h with $0 < h \leq \psi(f_i)$ for each i, whence $0 < h \leq \psi(f_i) \leq \psi\varepsilon(g_j^i)$ for each i, j, which contradicts $(*)$.

REMARK 8.5. Let $H \leq K$. Then H is *β-jd* in K (*join dense*) if for each $k \in K^+$, there is $\{h_i | i \varepsilon I\} \subseteq H^+$ with $|I| < \beta$ and $k = \bigvee_i h_i$. Then H is β-jamd in K if and only if H is β-jd and *majorizes* K, in the sense that for each $k \in K^+$, there is $h \in H$ with $h \geq k$.

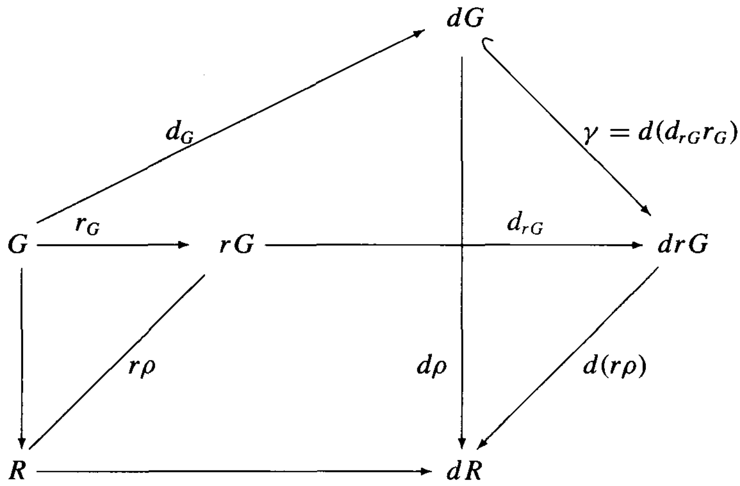
Here is an example showing that 8.4 fails if β^+-jamd is weakened to β^+-jd. Let $G' = C[0, 1]$, $G = M_o = \{g \varepsilon G' | g(0) = 0\}$,which is ω_0^+-jd in G'. Let $\varepsilon : G \to G'$ be the inclusion, $\psi : G' \to R$ the evaluation $\psi(f) = f(0)$, and $\rho : G \to R$ the zero-homomorphism. Then $\rho\varepsilon$ Hom$_\infty$, while $\psi \notin$ Hom$_{\omega_1}$.

In a category, an essentially reflective $(\boldsymbol{M}, m)$ is the *maximum essential reflection* if $(\boldsymbol{R}, r)$ essentially reflective implies, for each G, a map $\theta : rG \to mG$ with $\theta r_G = m_G$; equivalently, $\boldsymbol{M} \subseteq \boldsymbol{R}$. Such an $(\boldsymbol{M}, m)$ exists in most decent categories, though $m = id$ is not uncommon; see [HM].

THEOREM 8.6 ([BH4, 9.2] and [BH5, 11.2 and 12.2]). *The maximum essential reflection* $(\boldsymbol{M}, m)$ *exists in* **W**, *and in* Arch, *and we have* $G \leq dG \leq mG$ *for each* G.
(a) *In* Arch, $dG \leq H \leq mG$ *implies* dG *is* ω_0^+*-jamd in* H, *and in particular,* G *majorizes* mG.
(b) *In* **W**, $dG \leq H \leq mG$ *just implies* dG *is* ω_0^+*-jd in* H (*and for various* G, G *fails to majorize* mG).

We now prove 8.1(b). For $\alpha = \omega_0$, this is part of the hypothesis. We suppose $\alpha > \omega_0$, and for either Arch or **W**, let $\rho \in$ Hom$_\alpha(G, \boldsymbol{R})$, and consider the diagram:

(8.7)



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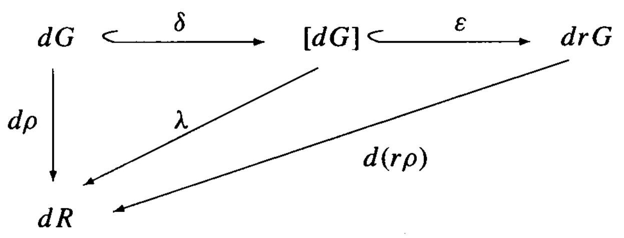
Here γ is monic because $\gamma d_G = d_{rG} r_G$ is monic and d_G is essential. By 8.2, $d\rho \in \mathrm{Hom}_\alpha$. We now must treat Arch and $\boldsymbol{W}$ separately because of the differences noted in 8.6.

Construing (8.7) in Arch, dG is ω_0^+-jamd in drG *via* γ (by 8.6(a)) and 8.4 says $d(r\rho) \in \mathrm{Hom}_\alpha$ (since $\alpha > \omega_0$). By 8.2(b), $r\rho\varepsilon\, \mathrm{Hom}_\alpha$, and we are finished.

Construing (8.7) in $\boldsymbol{W}$, we further resolve the equation $d(r\rho)\gamma = d\rho$ as

(8.8)



where $[dG]$ is the ideal in drG generated by dG, and λ is the restriction $d(r\rho)|[dG]$. By 8.6(b) and 8.5, dG is ω_0^+-jamd in $[dG]$ *via* δ, and 8.4 says that $\lambda \in \mathrm{Hom}_\alpha$ since $d\rho \in \mathrm{Hom}_\alpha$. The next lemma (peculiar to $\boldsymbol{W}$) then says $d(r\rho) \in \mathrm{Hom}_\alpha$, so again by 8.2(b), $r\rho \in \mathrm{Hom}_\alpha$, and we are finished.

LEMMA 8.9. *Suppose* $H \leq K \overset{\psi}{\to} L$ *in* W. *If* $\psi|[H] \in \mathrm{Hom}_\alpha$, *then* $\psi \in \mathrm{Hom}_\alpha$.

PROOF. This is just because $Y[H] = YK$ and $Y(\psi|[H]) = Y\psi$, by the uniqueness statements in 4.1 (a) and (b), and Lemma 4.3.

REMARK. The example in 8.5 shows 8.9 fails in Arch.

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