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Almost all interval exchange transformations with flips are nonergodic†

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Abstract. Here we prove that almost all interval exchange transformations which reverse orientation, in at least one interval, have a periodic point where the derivative is -1. Therefore they are periodic in an open neighborhood of the periodic point.

1. *Introduction*

Let n be in $\mathbb{Z}$, $n \geq 2$, and Λ_n be the positive cone in R^n, that is, the set of all $\lambda = (\lambda_1, \ldots, \lambda_n)$ in R^n where $\lambda_i > 0$ for all i. Let λ be in Λ_n. We set

$$\beta_i = \beta_i(\lambda) = \begin{cases} 0, & \text{if } i = 0 \\ \lambda_1 + \cdots + \lambda_i, & 1 \leq i \leq n. \end{cases} \tag{1}$$

This gives a partition of $(0, |\lambda|)$, $|\lambda| = \lambda_1 + \cdots + \lambda_n$, into n open intervals

$$\Gamma_i = (\beta_{i-1}, \beta_i), \quad \text{for } 1 \leq i \leq n. \tag{2}$$

Let σ be a permutation on $\{1, \ldots, n\}$, and

$$\lambda^\sigma = (\lambda_{\sigma^{-1}(1)}, \ldots, \lambda_{\sigma^{-1}(n)}).$$

We set $\beta_i^\sigma = \beta_i(\lambda^\sigma)$. Let F be a subset of $\{1, \ldots, n\}$. We define $T = T_{(\sigma, F, \lambda)}$, the (σ, F, λ) interval exchange transformation with flip set F, on the interval $(0, |\lambda|)$ by

$$Tx = \begin{cases} x - \beta_{i-1} + \beta^\sigma_{\sigma(i)-1}, & \text{if } i \notin F \text{ and } x \in \Gamma_i, \\ \beta_i - (x - \beta_{i-1}) - \beta_{i-1} + \beta^\sigma_{\sigma(i)-1} = \beta_i - x + \beta^\sigma_{\sigma(i)-1}, & \text{if } i \in F \text{ and } x \in \Gamma_i. \end{cases}$$

T preserves measure and reverses orientation on Γ_i, for i in F, otherwise T preserves orientation. If F is empty, we say that T is a standard interval exchange transformation and we denote it by $T_{(\sigma,\lambda)}$. For convenience, we define T in $D = (0, |\lambda|) \setminus \{\beta_1, \ldots, \beta_{n-1}\}$, in the case where $\lim_{x \uparrow \beta_i} Tx \neq \lim_{x \downarrow \beta_i} Tx$, for every $1 \leq i < n$: otherwise, T exchanges less than n intervals and has less than $n-1$ discontinuities.

In general, standard interval exchange transformations have good ergodic properties. Now we recall some basic results on these maps.

We call T minimal, if for almost all x in D, the orbit of x, $o(x) = \{T^n x : n \in \mathbb{Z}\}$, is dense in the closed interval $[0, |\lambda|]$. Let S'_n be the set of all irreducible permutations σ on $\{1, \ldots, n\}$ such that $\sigma\{1, \ldots, j\} = \{1, \ldots, j\}$ if, and only if $j = n$. We say that λ in Λ_n is irrational, if the only rational relations between $\lambda_1, \ldots, \lambda_n$ are multiples of $|\lambda| = \lambda_1 + \cdots + \lambda_n$. This gives a criterion for the minimality of T.

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THEOREM (Keane [1]). *Let σ in S_n^i and λ in Λ_n be irrational, then the interval exchange $T_{(\sigma,F)}$ is minimal. Therefore, for almost all λ in Λ_n, $T_{(\sigma,\Lambda)}$ is minimal.*

Throughout the paper the 'almost all' concerning interval exchanges is with respect to the Lebesgue measure in the positive cone Λ_n.

For interval exchanges, minimality does not imply unique ergodicity (see Sataev [2]). The latter means that the only invariant Borelian measures for T are the positive multiples of the Lebesgue measure on the interval $(0, |\lambda|)$. Neither does the irrationality condition for λ imply unique ergodicity for the (σ, λ) interval exchange, with σ in S_n^i (see Keane [3]). However we have the following result.

THEOREM (Masur [4] and Veech [5]). *If σ is in S_n^i, then for almost all λ in Λ_n the interval exchange $T_{(\sigma,\lambda)}$ is uniquely ergodic.*

Earlier, Veech [6] had proved this result for standard exchanges in the case $n = 4$. It was noticed by Keane [1] that for $n = 3$ the problem could be reduced to rotations on the circle, the case $n = 2$ intervals, where unique ergodicity is equivalent to the irrationality of $\lambda_1/|\lambda|$.

Here we prove that the opposite to what happens for standard interval exchanges, occurs for interval exchanges with flips: in general they are not minimal.

Let $\mathscr{F}_n$ be the collection of all nonempty subsets of $\{1, \ldots, n\}$.

THEOREM 1. *Let σ be in S_n^i and F in $\mathscr{F}_n$, then for almost all λ in Λ_n there exists a point $x_0 = x_0(\lambda)$ in $(0, |\lambda|)$ and a positive integer $k = k(\lambda)$ such that for the interval exchange $T = T_{(\sigma,F,\lambda)}$*

$$T^k x_0 = x_0 \quad \text{and} \quad \left(\frac{dT^k}{dx}\right) x_0 = -1. \tag{3}$$

Moreover, this is an open property in Λ_n.

A point x_0 which satisfies (3), is called a *flipped periodic point* for T, in this case there is a nonempty interval I such that

$$T^{2k} x = x, \quad \text{for all } x \text{ in } I.$$

Therefore, Theorem 1 implies that, for almost all λ in Λ_n, $T_{(\sigma,F,\lambda)}$ is nonergodic. We call the set $\{x_0, Tx_0, \ldots, T^{k-1}x_0\}$ a *flipped periodic orbit* of T. In particular, if $k = 1$, we say that x_0 is a *flipped fixed point* for T.

It is worthwhile to mention that Keane [1] had noticed that any exchange of two intervals with one flip is periodic.

There are interval exchanges with flips which are minimal. An example is the following: Let $0 < \alpha < \frac{1}{3}$ be irrational and

$$Tx = \begin{cases} \frac{2}{3} - \alpha + x, & \text{if } 0 < x < \alpha \\ \frac{1}{3} + x - \alpha, & \text{if } \alpha < x < \frac{1}{3} \\ \frac{2}{3} + (\frac{2}{3} - x), & \text{if } \frac{1}{3} < x < \frac{2}{3} \\ 1 - x, & \text{if } \frac{2}{3} < x < 1. \end{cases}$$

T is an interval exchange on the interval $(0, 1)$ which reverses orientation in $(\frac{1}{3}, \frac{2}{3}) \cup (\frac{2}{3}, 1)$. T^3 restricted to the interval $(0, \frac{1}{3})$ is a rotation by α, therefore T is uniquely ergodic.

An interesting example is given by Gutierrez [7]. There he proves that on any nonorientable surface of genus ≥ 4 (i.e., a torus with at least two cross-caps) there exists a C^{∞} vector field which has dense nonorientable trajectories on the surface. In order to do this, he constructs a minimal interval exchange with flips, say T, such that any Poincaré's first return map induced on an interval by T has flips. The non-orientability of the trajectories comes from this property. In [**8**], the author introduces a method to construct such minimal interval exchanges and is able to show that inside this class, there are nondenumberable many ones which are uniquely ergodic. According to our present result (Theorem 1), this is in some sense the best possible claim one can make about minimal interval exchanges with flips.

2. *The inducing process*

Before we state some preliminary results, some remarks are in order.

If the permutation is reducible, the interval exchange cannot have a dense orbit. So, as far as Theorem 1 is concerned, we just have to care about irreducible permutations. In order to have an interval exchange with flips which is discontinuous at β_i, for $1 \leq i < n$, that is,

$$\lim_{x\uparrow\beta_i} Tx \neq \lim_{x\downarrow\beta_i} Tx,$$

it is not necessary that

$$\sigma(i+1) \neq \sigma(i)+1, \tag{4}$$

as it is the case for standard ones. The transformations studied in [**7, 8**] do not satisfy (4) for every i.

Theorem 1 is a consequence of a simple fact which is exploited in the proof. Let T be an interval exchange with flips defined in D and $I \subset D$ be an open interval. Assume that for a certain $k \geq 1$

(i) $I \cap T^k I \neq \varnothing$,

(ii) $T^k I$ is a translated flipped image of I.

Therefore T^k has a flipped fixed point in $I \cap T^k I$. We will show that this occurs very often with interval exchanges with flips.

Let σ be in S_n^i and F in $\mathscr{F}_n$. We set

$$\Lambda_{\sigma,F} = \{T_{(\sigma,F,\lambda)}: \lambda \in \Lambda_n\}.$$

We identify $\Lambda_{\sigma,F}$ with Λ_n. Therefore, for almost all $T = T_{(\sigma,F,\lambda)}$ in $\Lambda_{\sigma,F}$, we have either (a) $\lambda_{\sigma^{-1}(n)} < \lambda_n$ or (b) $\lambda_n < \lambda_{\sigma^{-1}(n)}$. Let $I = (0, x_1)$ be an open interval, where

$$x_1 = \begin{cases} |\lambda| - \lambda_{\sigma^{-1}(n)}, & \text{in case } a, \\ |\lambda| - \lambda_n, & \text{in case } b. \end{cases}$$

For almost all y in I, we define

$$Uy = \begin{cases} Ty, & \text{if } Ty \in I, \\ Ty^2, & \text{otherwise.} \end{cases}$$

U is called the Poincaré first return map induced on I by T. This suitable defined map was introduced by Rauzy [**9**]. We define for almost all T in $\Lambda_{\sigma,F}$, the transformation

$$(\sigma, F, \lambda) \in S_n^i \times \mathcal{F}_n \times \Lambda_n \mapsto (\sigma', F', \lambda') \in S_n \times \mathcal{F}_n \times \Lambda_n, \tag{5}$$

where $U = T_{(\sigma', F', \lambda')}$ and this triple is given by:

Case a. (i) If $n \notin F$,

$$\sigma'(i) = \begin{cases} \sigma(i), & \text{if } \sigma(i) \le \sigma(n) \\ \sigma(n)+1, & \text{if } i = \sigma^{-1}(n) \\ \sigma(i)+1, & \text{otherwise.} \end{cases} \quad \text{and } F' = F.$$

(ii) If $n \in F$,

$$\sigma'(i) = \begin{cases} \sigma(i), & \text{if } \sigma(i) < \sigma(n) \\ \sigma(n), & \text{if } i = \sigma^{-1}(n), \\ \sigma(i)+1, & \text{otherwise} \end{cases}$$

if $\sigma^{-1}(n) \notin F$, $F' = F \cup \{\sigma^{-1}(n)\}$; otherwise $F' = F \backslash \{\sigma^{-1}(n)\}$. In any case

$$\lambda_i' = \begin{cases} \lambda_i, & \text{if } i < n \\ \lambda_n - \lambda_{\sigma^{-1}(n)}, & \text{otherwise.} \end{cases}$$

Case b. (i) If $\sigma^{-1}(n) \notin F$,

$$\sigma'(i) = \begin{cases} \sigma(i), & \text{if } i \le \sigma^{-1}(n) \\ \sigma(n), & \text{if } i = \sigma^{-1}(n)+1, \\ \sigma(i-1), & \text{otherwise,} \end{cases}$$

if $n \notin F$, $F' = \{i \in F \colon i < \sigma^{-1}(n)\} \cup \{i+1 \colon i \in F \text{ and } i > \sigma^{-1}(n)\}$; otherwise, $F' = \{i \in F \colon i < \sigma^{-1}(n)\} \cup \{\sigma^{-1}(n)+1\} \cup \{i+1 \colon i \in F \text{ and } i > \sigma^{-1}(n)\}$,

$$\lambda_i' = \begin{cases} \lambda_i, & \text{if } i < \sigma^{-1}(n) \\ \lambda_{\sigma^{-1}(n)} - \lambda_n, & \text{if } i = \sigma^{-1}(n) \\ \lambda_n, & \text{if } i = \sigma^{-1}(n)+1 \\ \lambda_{i-1}, & \text{otherwise.} \end{cases}$$

(ii) If $\sigma^{-1}(n) \in F$,

$$\sigma'(i) = \begin{cases} \sigma(i), & \text{if } i < \sigma^{-1}(n) \\ \sigma(n), & \text{if } i = \sigma^{-1}(n) \\ n, & \text{if } i = \sigma^{-1}(n)+1 \\ \sigma(i-1), & \text{otherwise} \end{cases}$$

if $n \notin F$, $F' = \{i \in F \colon i \le \sigma^{-1}(n)\} \cup \{\sigma^{-1}(n)+1\} \cup \{i+1 \colon i \in F \text{ and } i > \sigma^{-1}(n)\}$, otherwise.

$$F' = \{i \in F \colon i < \sigma^{-1}(n)\} \cup \{\sigma^{-1}(n)+1\} \cup \{i+1 \colon i \in F \text{ and } i > \sigma^{-1}(n)\};$$

$$\lambda_i' = \begin{cases} \lambda_i, & \text{if } i < \sigma^{-1}(n) \\ \lambda_n, & \text{if } i = \sigma^{-1}(n) \\ \lambda_{\sigma^{-1}}(n) - \lambda_n, & \text{if } i = \sigma^{-1}(n)-1 \\ \lambda_{i-1}, & \text{otherwise.} \end{cases}$$

We have neglected the set of all λ such $\lambda_n = \lambda_{\sigma^{-1}(n)}$. As we have noticed above, for each fixed pair (σ, F) in $S_n^i \times \mathscr{F}_n$, there are two choices for (σ', F') in the image by $\mathscr{R}$ (5), namely (σ_1, F_1) in case a, otherwise (σ_2, F_2) (see (6)).

$$\begin{array}{ccc} & & \Lambda_{\sigma_1,F_1} \\ & \overset{a}{\nearrow} & \\ \Lambda_{\sigma,F} & & \\ & \underset{b}{\searrow} & \\ & & \Lambda_{\sigma_2,F_2} \end{array} \tag{6}$$

For almost all λ in $\Lambda_{\sigma,F}$, there exist an $n \times n$ matrix $E = E(\lambda)$, with $\det E = \pm 1$, such that

$$\lambda = E\lambda'.$$

Case a.

$$E_{ij} = \begin{cases} 1, & \text{if } i = j \text{ or } i = n \text{ and } j = \sigma^{-1}(n), \\ 0, & \text{otherwise} \end{cases}$$

Case b.

$$E_{ij} = \begin{cases} 1, & \text{if } i = j \text{ and } i \le \sigma^{-1}(n) \\ & \quad j = i+1 \text{ and } \sigma^{-1}(n) \le i \le n \\ & \quad i = n \text{ and } j = \begin{matrix} \sigma^{-1}(n)+1, \text{if } \sigma^{-1}(n) \notin F \\ \sigma^{-1}(n), \quad \text{otherwise} \end{matrix} \\ 0, & \text{otherwise.} \end{cases}$$

We call E an elementary matrix. If we can carry on this process k times, we get that

$$\lambda = E_1 \cdots E_k \lambda^{(k)}.$$

We call the sequence $A_k(\lambda) = E_1 \cdots E_n$ the expansion of λ. In case this process cannot be carried beyond the k_0th stage, we say that λ has a finite expansion, namely $A_1(\lambda), \ldots, A_{k_0}(\lambda)$. Otherwise, λ has an infinite expansion. In this paper, we prove that the latter form a vanishing set in $\Lambda_{\sigma,F}$. For standard interval exchanges (without flips), it is known that they form a full measure set.

Now we give an example where the expansion is finite, let (σ, F, λ) in $S_n^i \times \mathscr{F}_n \times \Lambda_n$ be such that $\sigma(n) = n-1$, F contains n and $\lambda_{\sigma^{-1}(n)} < \lambda_n$. Let (σ', F', λ') be the image of (σ, F, λ) by $\mathscr{R}$ (5). We have that $\sigma'(n) = n$, therefore σ' is not irreducible. This implies that $\mathscr{R}$ is not defined at (σ', F', λ'), so the process has only one stage.

Let A be a matrix, if all entries of A are positive real numbers, we write $A > 0$.

At the kth stage of the inducing process, we recall that $\lambda = A_k \lambda^{(k)}$. Let T_1 be the image of T by $\mathscr{R}$ and T_k the image of T_{k-1}. Let x_k be the left endpoint of the interval where T_k is defined, therefore x_k is a decreasing sequence $x_0 = |\lambda|$ and $x_k = \lambda_1^{(k)} + \cdots + \lambda_n^{(k)}$.

Let A_{ij} be the entries of A_n and $I_1, \ldots, I_n$ the intervals exchanged by T_k. A_{ij} is the number of times the interval I_j visits the interval Γ_i (see (2)) before returning to $(0, x_k)$.

LEMMA 2.1. *Let $T = T_{(\sigma,F,\lambda)}$ be minimal and λ irrational, then there exists a positive integer m such that at the* mth *stage the matrix $A_m > 0$.*

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Proof. First we prove that there exist positive integers k_1 and k_2 such that $x_{k_1}=\beta_{n-1}$ and $x_{k_2}=\beta_{n-2}$ (see (1)).

For any positive k such that $x_k>\beta_{n-1}$, then $x_{k-1}-x_k=\lambda_i$ for some $1\le i\le n$. Therefore there exists k_1 such that

$$x_{k_1+1}<\beta_{n-1}\le x_{k_1},$$

and this implies that $x_{k_1}=\beta_{n-1}$, otherwise β_{n-1} would be a discontinuity point for T_{k_1} which would be ommited by the sequence x_k.

If $x_k>\beta_{n-2}$, for any positive k, then $x_{k-1}-x_k$ equals λ_i, for some $1\le i\le n$, or $x_{k_1-1}-x_{k_1}$. As above there exists k_2 such that $x_{k_2}=\beta_{n-2}$.

The same reasoning proves that there exist positive integers 1_1, and 1_2 such that $x_{1_1}=\beta^{\sigma}_{n-1}$ and $x_{1_2}=\beta^{\sigma}_{n-2}$ (see § 1).

Let T_{k_2} be equal to $T_{(\sigma',F',\lambda')}$ and $\beta'_i=\lambda'_1+\cdots+\lambda'_i$, for $1\le i\le n$. We note that $\beta_1,\ldots,\beta_{n-3}$ are also discontinuity points for T_{k_2} which is an exchange of n intervals, therefore β_{n-3} equals β'_{n-1} or β'_{n-2}. So there exists k_3 such that $x_{k_3}=\beta_{n-3}$. Besides there exists 1_3 such that $x_{1_3}=\beta^{\sigma}_{n-3}$.

Therefore for any $1\le i<n$, there exist k_i and 1_i such that $x_{k_i}=\beta_{n-i}$ and $x_{1_i}=\beta^{\sigma}_{n-i}$. Let $1=\min\{k_n,1_n\}$, so $x_1<|\lambda|/2$. Therefore $|\lambda^{(k)}|$ goes to zero as $k\to+\infty$.

Since T is minimal, there exist positive integers m and r_{ij}, for any $1\le i,j\le n$, such that $T^{r_{ij}}I_j\subset\Gamma_i$, where $I_1,\ldots,I_n$ are the intervals exchanged by T_m. Moreover, $T^{r_{ij}}$ acts on I_j as a translation, it may be a flipped translation. This means that the entries of the matrix A_m at mth stage are all positive, $A_m>0$. □

3. *Proof of theorem* 1

Let A be a positive $n\times n$ matrix with $\det A=\pm1$. Let $c_j=\sum_{i=1}^n A_{ij}$ be the norm of the jth column of A. We define

$$\rho(A)=\max_{1\le j,l<n}\frac{c_j}{c_l}. \tag{7}$$

We denote by Δ_{n-1} the $n-1$ simplex

$$\Delta_{n-1}=\{\lambda\in\Lambda_n: |\lambda|=1\}$$

and by ν the Lebesgue measure on Δ_{n-1}. We set

$$\mathscr{L}_A: u\in\Delta_{n-1}\to\frac{Au}{|Au|}\in\Delta_{n-1}. \tag{8}$$

Now we consider interval exchanges on the interval $(0,1)$. Let σ in S^i_n and F in $\mathscr{F}_n$ be fixed. We set $\Delta_{\sigma,F}=\{T_{(\sigma,F,\lambda)}: \lambda\in\Delta_{n-1}\}$. We identify $\Delta_{\sigma,F}$ with Δ_{n-1} and consider the Lebesgue measure ν on $\Delta_{\sigma,F}$.

Let T_0 in the $\Delta_{\sigma,F}$ be fixed and assume that at the mth stage of the inducing process we have the matrix A_m. Therefore $\{T_{(\sigma,F,\lambda)}: \lambda\in\mathscr{L}_{A_m}\Delta_{n-1}\}$ is the set of all T in $\Delta_{\sigma,F}$ associated with A_m.

Let m be fixed, then there are only finitely many possible matrices A_m defined by T in $\Delta_{\sigma,F}$. Let $\Delta^{(m)}_{\sigma,F}$ be the disjoint union $U\mathscr{L}_A\Delta_{n-1}$, where A_m runs over all possible matrices A_m. Almost all T in $\Delta_{\sigma,F}$ which is not in $\Delta_{\sigma,F}^{(m)}$, has a flipped periodic orbit. Otherwise, $T = T_{(\sigma,F,\lambda)}$ would have been defined by a λ which is not irrational and they form a vanishing set.

LEMMA 3.1 (Veech [**10**]). *Let $A > 0$ be an $n \times n$ matrix with* $\det A = \pm 1$. *Then the Jacobian of $\mathscr{L}_A$* (8) *on Δ_{n-1} is given by*

$$J_A(n) = \frac{1}{|A_u|^n}, \quad \text{for all } u \text{ in } \Delta_{n-1}.$$

In particular,

$$\nu(\mathscr{L}_A \Delta_{n-1}) = \frac{1}{n! c_1 \cdots c_n}, \tag{9}$$

where c_j is the norm of the jth column of A.

Let $T = T_{(\sigma,F,\lambda)}$. We note that if i is in F and $\lambda_i > \frac{1}{2}$, then T has a flipped fixed point.

LEMMA 3.2. *Let $A_k > 0$ be a matrix corresponding to the kth stage in the expansion of a certain T in $\Delta_{\sigma,F}$. Then the probability of T in $\mathscr{L}_{A_k}\Delta_{n-1}$ having a flipped periodic point is greater than* $(1 + \rho(A_k))^{-n}$.

Proof. For simplicity, we set $A = A_k$.

Let $\Delta_{n-1}^{1/2} = \{\lambda \in \Delta_{n-1} \colon \lambda_1 > \frac{1}{2}\}$, by Lemma 3.1

$$\nu(\mathscr{L}_A \Delta_{n-1}^{1/2}) \int_{1/2}^{1} du_1 \int_0^{1-u_1} du_2 \cdots \int_0^{1-u_1 \cdots - u_{n-2}} \frac{1}{(c_1 u_1 + \cdots + c_n u_n)^n},$$

where $u_n = 1 - u_1 - \cdots - u_{n-1}$. It follows from a simple change of variables that

$$\nu(\mathscr{L}_A \Delta_{n-1}^{1/2}) = \frac{1}{n! c_1 (c_2 + c_1) \cdots (c_n + c_1)}.$$

Almost all T in $\mathscr{L}_A \Delta_{n-1}$ is an interval exchange with flips. Let T_k be the transformation defined at the kth stage by T, therefore T_k is an interval exchange with flips. If T_k flips the first interval, the probability of T in $\mathscr{L}_A \Delta_{n-1}$ having a flipped periodic point is at least the ratio between the volume of $\mathscr{L}_A \Delta_{n-1}^{1/2}$ and the volume of the whole image of $\mathscr{L}_A$,

$$\frac{\nu(\mathscr{L}_A \Delta_{n-1}^{1/2})}{\nu(\mathscr{L}_A \Delta_{n-1})} = \frac{c_2 \cdots c_n}{(c_2 + c_1) \cdots (c_n + c_1)} > (1 + \rho(A))^{-n},$$

where $\rho(A)$ is defined in (7). Otherwise, T_k flips another interval and the lemma follows. □

We call $\Delta_{\sigma,F}^{b}$ the set of all $T_{(\sigma,F,\lambda)}$ in $\Delta_{\sigma,F}$ such that the expansion of T, namely $A_m = A_m(\lambda)$, is finite, or there exists an increasing sequence m_k such that the subsequence $\rho(A_{m_k})$ is bounded.

COROLLARY 3.3. *Almost all T in $\Delta_{\sigma,F}^{b}$ has a flipped periodic point.*

Proof. Here we consider only irrational λ. In this case, if λ has a finite expansion it means that T has a flipped periodic point. Otherwise by Lemma 3.2 for a fixed sequence m_k at each stage at least a fixed positive fraction of λ in $\mathscr{L}_{A_{m_k}} \Delta_{n-1}$ defines

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interval exchanges in $\Delta^b_{\sigma,F}$ with flipped periodic points. Therefore almost all T in $\Delta^b_{\sigma,F}$ has a flipped periodic point. □

We call $\Delta^\infty_{\sigma,F}$ the complementary of $\Delta^b_{\sigma,F}$ in $\Delta_{\sigma,F}$, that is, the set of all $T_{(\sigma,F,\lambda)}$ in $\Delta_{\sigma,F}$ such that the sequence $\rho(A_m(\lambda))$ goes to $+\infty$. In [**11**]. Kerckhoff proves that for standard interval exchanges, the equivalent set Δ^∞_σ is a vanishing set, recall that in this case since there is no flip we omit F. He obtains from this result an alternative proof of the unique ergodicity property of almost all interval exchanges in Δ_σ. Here we follow [**11**] to obtain a proof for the following lemma.

LEMMA 3.4. *$\Delta^\infty_{\sigma,F}$ is a vanishing set, that is, $\nu(\Delta^\infty_{\sigma,F})=0$.*

The proof of Lemma 3.4 follows from the next three propositions (see Propositions 1.3, 1.4 and 1.5 in [**11**]). Their proofs will be omitted since they are similar to those given in [**11**].

PROPOSITION 3.5. *Let $T_{(\sigma,F,\lambda)}$ have an infinite expansion $A_k=A_k(\lambda)$. Then at any stage k_0 where $A_{k_0}>0$, the probability of a column increasing in norm by a factor of K before being added to another column is less than $n^2/(K-1)$.*

It is clear that as far as the norm of the columns is concerned, we can say that at each stage A_{k+1} has been obtained from A_k having a column being added to another one. By the definition of $\rho(A_k)$, it increases, only if the norm of a certain column of the matrices A_k increases more than the norm of another one. It remains to prove that it is zero the probability that, throughout the process, a subset of columns, other than the entire set, will be added amongst themselves over and over without being added to outside columns. The next result proves that in certain set up this cannot happen.

PROPOSITION 3.6. *Let T be in $\Delta_{\sigma,F}$. Assume that there exist positive integers k and m such that at the* kth *and the* $(k+m)$th *stages the induced interval exchanges, namely T_k and T_{k+m}, have the same permutation and the same flip set. Then between these two stages each column has been added to any other one.*

This is a balancing condition, roughly speaking it implies that ρ is bounded between those extreme stages.

Let A be a positive $n\times n$ matrix, we say that the columns $v_1,\ldots,v_k$ of A are C-distributed, if

$$\max_{1\le i,j\le k}\frac{C_i}{C_j}\le C,$$

where C_1 is the norm of the column v_i. The next proposition assures that for almost all $T_{(\sigma,F,\lambda)}$ in $\Delta_{\sigma,F}$, either λ has a finite expansion, or infinitely often all columns of $A_m(\lambda)$, are C-distributed for a fixed C.

PROPOSITION 3.7. *Let $T_{(\sigma,F,\lambda)}$ be in $\Delta_{\sigma,F}$ and A_m be the* mth *matrix in the expansion of λ. Assume that the collection $v_1,\ldots,v_k$ of columns of A_m is C-distributed. Then with a positive probability, either one of the v_i's will be added to a column outside the collection before the maximum of the normal of the v_i's increases by a factor K, or the* *expansion of λ is finite. Moreover this probability is independent of A_m depending only on C and K.*

Therefore the set of T in $\Delta_{\sigma,F}$ such that the columns of the matrix expansions never become C-distributed is a vanishing set. The set where they become C-distributed finitely many times is the countable union of vanishing sets, hence it is also a vanishing set.

Theorem 1 follows directly from Corollary 3.3 together with Lemma 3.4.

4. *On the number of flipped periodic orbits*

Let σ in S_n^i and F in $\mathcal{F}_n$ be fixed. We take T in $\Delta_{\sigma,F}$ which has a flipped periodic orbit. Let $T_1, \ldots, T_k, \ldots$ be the induced interval exchanges defined by T. Therefore at a certain stage of the inducing process, namely the kth, T_k exchanges n open intervals $I_1, \ldots, I_n$, flips I_n and T_kI_n intersects I_n. Moreover for the permutation η defined by T_k, $\eta(n) = n-1$. This implies that T_k has a flipped fixed point and for the permutation τ defined by T_{k+1}, $\tau(n) = n$, so we can regard T_{k+1} as an exchange of the $n-1$ first intervals. Although, as it stands in (5) $\mathcal{R}$ cannot be applied to T_{k+1}, since τ is reducible. Since an exchange of two intervals with one flipped is always periodic, that is, it has two flipped periodic orbits, by induction on the number n of exchanged intervals, we conclude that an exchange in $\Delta_{\sigma,F}$ has at most n flipped periodic orbits.

LEMMA 4.1. *The set of interval exchanges in $\Delta_{\sigma,F}$ which have n flipped periodic orbits, has positive measure. Moreover, for any $0 \le i < n$ the set of all interval exchanges in $\Delta_{\sigma,F}$ which have more than i flipped periodic orbits is open.*

Proof. The property is open because a flipped periodic orbit is stable.

Let T be in $\Delta_{\sigma,F}$ and assume that T has a flipped periodic orbit. As it was mentioned above at the $(k+1)$st stage, we can define T_{k+1} as an exchange of $n-1$ intervals. If T_{k+1} has flips, the set of interval exchanges in $\Delta_{\sigma,F}$ which have two distinct flipped periodic orbits has positive measure.

Let λ in Λ_n and α in Λ_{n-1} be the lengths of the intervals exchanged, respectively, by T_k and T_{k+1}. The map

$$\lambda \in \Lambda_n \to \alpha \in \Lambda_{n-1}$$

is just a projection, that is, $\alpha = (\lambda_1, \ldots, \lambda_{n-1})$. Therefore its image is a full measure set in Λ_{n-1} and Theorem 1 can be applied to it.

Using that any exchange of two intervals with one flipped has two distinct flipped periodic orbits, by induction on n the claim of the lemma follows. So we only have to prove that there exist such exchanges T_{k+1} with flips.

First we consider a case where there is only one flipped periodic orbit. Let τ in S_n^i and $G = \{\tau^{-1}(n), n\}$. Let λ in Λ_n be such that $\lambda_{\sigma^{-1}(n)} < \lambda_n$. Therefore $T' = T_{(\tau,G,\lambda)}$ has only one flipped periodic orbit, defined by the point $x_0 = |\lambda| - \frac{1}{2}(\lambda_{\sigma^{-1}(n)} + \lambda_n)$. The first return map induced on the interval $(0, |\lambda| - \lambda_n)$ by T' exchanges $n-1$ intervals and preserves orientation. This would not be the case if G had another element.

Now we assume that G contains $\sigma^{-1}(n)$ and n. Let λ in R_+^n be such that $\lambda_{\sigma^{-1}(n)+1}+\cdots+\lambda_n<\lambda_{\sigma^{-1}(n)}$. Therefore the first return map U induced on the interval $(0, \lambda_1+\cdots+\lambda_{\sigma^{-1}(n)})$ by $T_{(\tau,G,\lambda)}$ has a flipped fixed point. Moreover U flips the last maximal interval

$$I=(|\lambda|-\lambda_{\sigma^{-1}(n)}, \lambda_1+\cdots+\lambda_{\sigma^{-1}(n)})$$

and $UI=I$. U restricted to the interval $(0, |\lambda|-\lambda_{\sigma^{-1}(n)})$ is an exchange of $n-1$ intervals with flips.

It follows that for any G containing $\tau^{-1}(n)$ and n the set of T in $\Delta_{\tau,G}$ which has two distinct flipped periodic orbits has positive measure. Therefore this is true for any G which contains n.

Let T be in $\Delta_{\sigma,F}$ and assume that T has a flipped periodic orbit. Therefore at a certain stage, say the kth, T_k has a flip set which contains n. Let A_k be the matrix associated with T at this stage. According to what we have proved above, there exists a nonvanishing subset of $\mathscr{L}_{A_k}\Delta_{n-1}$ such that if λ is in there, $T_{(\sigma,F,\lambda)}$ has two flipped periodic orbits.

This proves Lemma 4.1. □

5. *Billiards with flips*

Motivated by interval exchanges with flips, we can define billiards with flips in polygons. In a polygon P, we fix the set of the so-called flip sides. When the billiard ball reaches a point in one of the flip sides, it leaves from the symmetric point of this side with respect to its middle point with the reflected angle. When the ball hits any other side of P, it behaves as a standard billiard ball.

Let us consider a billiard in a convex polygon P such that the flip sides of P are chosen in such manner that the perpendicular to each flip side at its middle point splits P into two symmetrical parts. It is interesting to note that in this case all well defined trajectories of a billiard ball in P are periodic. This is the case for the billiard in a rectangle with only one flip side (see figure 1).

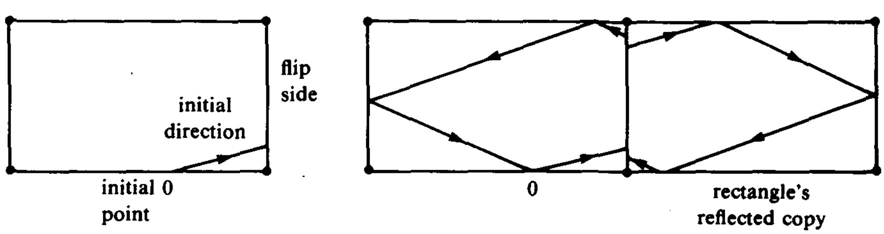

FIGURE 1

A billiard with flips in a rational polygon may induce on a given side an interval exchange with flips. If this is the case and we make a small perturbation in the initial angle of the trajectory, we still get a trajectory which induces an interval exchange with flips. So this is an open property.

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