Canad. Math. Bull. Vol. **34** (4), 1991 pp. 474–479

ON PRODUCT PARTITIONS OF INTEGERS

V. C. HARRIS AND M. V. SUBBARAO

ABSTRACT. Let $p^*(n)$ denote the number of product partitions, that is, the number of ways of expressing a natural number $n > 1$ as the product of positive integers ≥ 2, the order of the factors in the product being irrelevant, with $p^*(1) = 1$. For any integer $d \geq 1$ let $d_i = d^{1/i}$ if d is an i^{th} power, and $= 1$, otherwise, and let $\bar{d} = \prod_{i=1}^{\infty} d_i$. Using a suitable generating function for $p^*(n)$ we prove that $\prod_{d|n} \bar{d}^{p^*(n/d)} = n^{p^*(n)}$.

1. **Introduction.** The well-known partition function $p(n)$ stands for the number of unrestricted partitions of n, that is, the number of ways of expressing a given positive integer n as the *sum* of one or more positive integers, the order of the parts in the partition being irrelevant.

In contrast to this, we consider here the function $p^*(n)$, which denotes the number of ways of expressing n as the product of positive integers ≥ 2, the order of the factors in the product being irrelevant. For example, $p^*(12) = 4$, since 12 can be expressed in positive integers ≥ 2 as a product in these and only these ways: $12, 6 \cdot 2,\ 4 \cdot 3,\ 3 \cdot 2 \cdot 2$. We may say that $p(n)$ denotes the number of *sum* partitions and $p^*(n)$ the number of *product* partitions of n.

We note that the number of product partitions of $n = {p_1}^{a_1}{p_2}^{a_2} \ldots {p_k}^{a_k}$ in standard form is independent of the particular primes involved; for example, $p^*(12) = p^*(2^2 \cdot 3) = p^*(p_1^2 p_2)$ for every choice of distinct primes p_1 and p_2. For computing it is usually convenient to let p_j be the jth prime; for easily ordering the divisors in increasing order it suffices to take $p_2 > {p_1}^{a_1}, p_3 > {p_1}^{a_1}{p_2}^{a_2}$, etc.

$p^*(n)$ has been studied by Minetola (see Dickson [4]) as the number of repartitions of ℓ distinct objects with α_1 repetitions of the first object, α_2 repetitions of the second object, etc., into n subsets, where $\ell + \alpha_1 + \alpha_2 + \ldots + \alpha_k = m \geq n$.

A generating function in the form of a Dirichlet series was introduced by MacMahon [13], but little use was made of it. He also studied product partitions under the guise of multipartite numbers ([13]).

Also, obviously $p^*(n)$ may be considered as the number of nonnegative solutions of the equation

$$x_2 \ln 2 + x_3 \ln 3 + \ldots + x_n \ln n = \ln n.$$

John F. Hughes and J. O. Shallit [9] proved that $p^*(n) \leq 2n^{\sqrt{2}}$. This has been improved to $p^*(n) \leq n$ by L. E. Mattics and F. W. Dodd ([15], p. 126).

The second author was supported in part by an NSERC research grant.
Received by the editors June 26, 1990.
AMS subject classification: Primary: 11N37; secondary: 11N25.

https://doi.org/10.4153/CMB-1991-076-4 Published online by Cambridge University Press

Canfield, Erdős and Pomerance [2] gave an upper bound for $p^*(n)$, namely,

$$p^*(n) < n\exp\left\{-\frac{\log n}{\log_2 n}\left(\log_3 n + \log_4 n + \frac{\log_4 n - 1}{\log_3 n} + c\frac{\log_4^2 n}{\log_3^2 n}\right)\right\},$$

which is valid for all large n and a suitably chosen constant c.

The authors are not aware of an asymptotic estimate for $p^*(n)$. However, the average value of $p^*(n)$ was considered by Oppenheim [16]. He showed that

$$\frac{1}{x}\sum_{n\le x} p^*(n) \sim \exp(2\sqrt{\log x})/\left(2\sqrt{\pi}(\log x)^{3/4}\right).$$

This result was independently obtained later by Szekeres and Turan [17].

A related problem is to determine the number of ways $F(n)$ of expressing n as a product of integers ≥ 2, the same factors in different order being considered different factorizations. For example, the number of such factorizations of 12 would be 8, namely (12), (6.2), (4.3), (3.4), (2.6), (3.2.2), (2.3.2), (2.2.3). These factorizations could well be called *product compositions,* as Andrews [1] has done.

Kalmar [11] showed that

$$\sum_{n<x} F(n) \sim \frac{-x^\rho}{\rho\zeta'(\rho)}$$

where $\zeta(s)$ is the Riemann zeta function and $\rho > 1$ is given by $\zeta(\rho) = 2$. Other papers on $F(n)$ include those of Erdös [5], Evans [6], Hille [8], Ikehara [10], George Andrews [1] and Long [12]. In [1], Andrews proved a combinatorial formula which was originally conjectured by Long [12].

In this note we obtain a generating function for $p^*(n)$ and utilize it to get a recursion formula for $p^*(n)$. An algorithm for the computation of $p^*(n)$ was given by Canfield, Erdös and Pomerance in ([2], Section 7).

2. **The recursion formula.**

Consider the geometric series

$$\frac{1}{1-x^{\ln 2}} = 1 + x^{\ln 2} + x^{2\ln 2} + \cdots + x^{j\ln 2} + \cdots$$

with ratio $x^{\ln 2}$. Since for $0 < x < 1$ we have $0 < x^{\log 2} < 1$, the series is absolutely convergent for $0 < x < 1$. The same is true for the expansions of

$$\frac{1}{1-x^{\ln 3}}, \frac{1}{1-x^{\ln 4}}, \dots .$$

Consider

$$f(x,N) = \prod_{n=2}^{N} \frac{1}{1-x^{\ln n}} = 1 + \sum_{n=2}^{\infty} p^*(n,N)x^{\ln n} \tag{1}$$

for $N \ge 2$. The function $p^*(n,N)$ counts the number of ways $\ln n$ is expressible as a sum of nonnegative integral multiples of $\ln 2, \ln 3, \dots, \ln N$. This shows $p^*(n,N) =$

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$p^*(n)$ for $n \leq N$. Since N is arbitrary and multiplication by factors $(1 - x^{\ln(N+1)})^{-1}$, $(1 - x^{\ln(N+2)})^{-1}, \ldots$ does not alter the coefficients for $n \leq N$, it results that $p^*(n)$ is the number of product partitions of n.

A generating function for $p^*(n)$ is hence

$$f(x) = \prod_{n=2}^{\infty} \frac{1}{1 - x^{\ln n}} = 1 + \sum_{n=2}^{\infty} p^*(n) x^{\ln n}. \tag{2}$$

Taking logarithms of members of (2) gives

$$-\sum_{n=2}^{\infty} \ln(1 - x^{\ln n}) = \ln\Big(1 + \sum_{n=2}^{\infty} p^*(n) x^{\ln n}\Big)$$

whence by differentiating and multiplying by $x > 0$,

$$\sum_{n=2}^{\infty} \frac{x^{\ln n} \ln n}{1 - x^{\ln n}} = \frac{\sum_{n=2}^{\infty} p^*(n) x^{\ln n} \ln n}{1 + \sum_{n=2}^{\infty} p^*(n) x^{\ln n}}.$$

Therefore, with a change of the index variables,

$$\Big(\sum_{j=2}^{\infty} \ln j \sum_{i=1}^{\infty} x^{i \ln j}\Big)\Big(1 + \sum_{k=2}^{\infty} p^*(k) x^{\ln k}\Big) = \sum_{n=2}^{\infty} p^*(n) x^{\ln n} \ln n. \tag{3}$$

Let d divide n and consider the coefficient of $x^{\ln n}$ of the left member; we have, with $k = n/d$

$$i \ln j + \ln \frac{n}{d} = \ln n$$

so

$$j^i = d \text{ or } j = d^{1/i}, \; j \text{ an integer}$$

For any integer $d \geq 1$ let

$$d_i = \begin{cases} d^{1/i} & \text{if } d \text{ is an } i\text{th power} \\ 1 & \text{otherwise} \end{cases}$$

and let

$$\bar{d} = \prod_{i=1}^{\infty} d_i.$$

Then (3) becomes

$$\sum_{n=2}^{\infty} \sum_{d|n} \ln \bar{d}\, p^*\Big(\frac{n}{d}\Big) x^{\ln n} = \sum_{n=2}^{\infty} p^*(n) \ln n \, x^{\ln n}.$$

This gives the recursion formula:

$$\sum_{d|n} \ln \bar{d}\, p^*\Big(\frac{n}{d}\Big) = p^*(n) \ln n \tag{4}$$

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or, alternatively,

$$\prod_{d|n} \bar{d}^{p^*(\frac{n}{d})} = n^{p^*(n)}. \tag{5}$$

REMARK 1. We justify the above procedures as follows: The products

$$\prod_{n=2}^{\infty} \frac{1}{1-x^{\ln n}}, \quad \prod_{n=2}^{\infty} \left(1 + \frac{x^{\ln n}}{1-x^{\ln n}}\right)$$

and the series

$$\sum_{n=2}^{\infty} \frac{x^{\ln n}}{1-x^{\ln n}}, \quad \sum_{n=2}^{\infty} x^{\ln n}$$

all converge or diverge together. But since $x^{\ln n} = n^{\ell n\, x}$ for x real and positive,

$$\sum_{n=2}^{\infty} x^{\ln n} = \sum_{n=2}^{\infty} n^{\ln x}$$

which converges absolutely and uniformly for any x in any closed interval contained in $-\infty < \ln x < -1$, i.e., in $0 < x < 1/e$. Thus arranging the series according to powers of x, taking logarithms, differentiating termwise and equating coefficients can be done (G. H. Hardy and E. M. Wright, [7], pp. 244, 245).

REMARK 2. Since $p^*(n)$ depends only on the vector $\vec{\alpha}(n) = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ in the representation of n as $\prod_{j=1}^{k} q_j^{\alpha_j}$, the recursion formula may also be written

$$\sum_{\vec{0} \leq \vec{\beta} \leq \vec{\alpha}} p^*(\vec{\alpha} - \vec{\beta})\lambda(\vec{\beta})\|\vec{\beta}\| = p^*(\alpha)\|\vec{\alpha}\|, \tag{4*}$$

where $p^*(\vec{\alpha}) = p^*(q_1^{\alpha_1} q_2^{\alpha_2} \ldots q_k^{\alpha_k})$, $\|\vec{\alpha}\| = \prod_{j=1}^{k} \alpha_j$, $\lambda(\vec{\alpha}) = \Sigma_{i:i|\alpha_j \text{ for } 1 \leq j \leq k\}} 1/i$ and where $\vec{\beta} < \vec{\alpha}$ means that $\beta_j \leq \alpha_j$ for $1 \leq j \leq k$.

3. **Two special cases.** In cases $n = q^{\alpha}$, where q is a prime, the function reduces to the sum-partition function. Thus $p^*(q^{\alpha}) = p(\alpha)$ and

$$\alpha\, p(\alpha) = \sum_{j=1}^{\alpha} \sigma(j) p(\alpha - j), \tag{6}$$

a result due to Euler. For if $d = q^j$, $j = 1, 2, \ldots, \alpha$, and q is a prime, then $\bar{d} = q^{\sigma(j)}$ and (4) becomes

$$p(\alpha) \ln q^{\alpha} = \sum_{j=1}^{\alpha} \ln q^{\sigma(j)} p(\alpha - j)$$

which is equivalent to (6).

In case $n = p_1 p_2 \ldots p_k$, where the p's are distinct primes, the function reduces to the Bell number called A_{k-1}, by Carlitz [3]:

$$p^*(n) = \sum_{j=1}^{k} \binom{k-1}{j-1} p^*\left(\frac{n}{p_1 p_2 \ldots p_j}\right) = A_{k-1}. \tag{7}$$

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Consider the coefficient of $p^*\left(\frac{n}{p_1p_2\cdots p_j}\right)$ in $p^*(n)\ln n = \sum_{d|n}\ln \bar{d}p^*\left(\frac{n}{\alpha}\right)$. Here $\bar{d} = p_1p_2\cdots p_j$. There are $\binom{k-1}{j-1}$ ways of choosing primes from $p_2, p_3, \ldots, p_k$ by which to multiply p_1 to get products of j primes which have p_1 as a factor. Thus $\ln p_1$ will occur $\binom{k-1}{j-1}$ times, each time multiplied by p^* (a product of the remaining $k-j$ primes) and these p^* (product of the remaining $k-j$ primes) are all equal. By symmetry there will be the same number of occurrences of $\ln p_2, \ln p_3, \ldots, \ln p_k$. The coefficient of $p^*\left(\frac{n}{p_1\cdots p_j}\right)$ will be $\binom{k-1}{j-1}\ln n$. Thus (4) becomes

$$\ln n\, p^*(n) = \sum_{j=1}^{k} \ln n \binom{k-1}{j-1} p^*\left(\left(\frac{n}{p_1\cdots p_j}\right)\right)$$

which is equivalent to (7).

The authors wish to acknowledge their indebtedness to the referees for Equation (4*) and for valuable suggestions.

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Department of Mathematics
San Diego State University
San Diego, California 92182
USA

Department of Mathematics
University of Alberta
Edmonton, Alberta T6G 2G1