5

From cuts to poles

As we pointed out in the preceding chapter, there are several important differences between the behaviour of the perturbative QCD Pomeron which is the solution of the BFKL equation and that of the 'soft' Pomeron predicted by Regge theory and identified in total hadronic cross-sections and differential cross-sections at small tranverse momenta. Although one might have hoped that a purely perturbative analysis of QCD would yield results which were in qualitative agreement with the behaviour of the 'soft' Pomeron, it is not surprising that the results are in fact very different. Perturbative QCD theory can only be applied reliably to Green functions in which all the momenta and their scalar products are sufficiently large. In the subsequent two chapters we shall be discussing experimental situations in which such criteria are obeyed. However, total hadronic cross-sections or differential cross-sections with low momentum transfer do not obey these criteria and we must therefore expect that non-perturbative features of QCD will play a crucial role in describing such phenomena. Unfortunately a complete analysis of the non-perturbative behaviour of QCD is outside our present grasp. Nevertheless, we can investigate the 'meeting points' of perturbative and non-perturbative QCD in order to obtain some idea of how non-perturbative effects are likely to affect the Pomeron and to what extent we may expect to be able to reproduce the behaviour of hadronic cross-sections in QCD.

One of the most striking differences between the 'soft' Pomeron approach to high energy scattering and the perturbative approach, calculated by summing the leading $\ln s$ terms to all orders, is that the Mellin transform of the scattering amplitude has a cut rather than an isolated pole. Lipatov (1986) pointed out that the origin of the cut is largely due to the fact that, in the leading logarithm derivation, the strong coupling constant, α_s, is kept fixed, whereas

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in QCD we know that it runs. Accounting for the running of the coupling, together with some information about the infra-red behaviour of QCD (provided by the non-perturbative sector) leads to a discrete pole singularity for the Pomeron rather than a cut. We shall begin this chapter by discussing the effect of the running of the coupling.

Before we do so a caveat is in order. The effect of the running of the coupling is a part of the corrections beyond the leading logarithm approximation which were referred to in the preceding chapter. It is, strictly speaking, inconsistent to take this into account without all the other sub-leading logarithm corrections. The hope and expectation that higher order corrections are dominated by the effect of the running of the coupling has been used before in several branches of high energy physics such as the study of infra-red renormalons or corrections to the gap equation for dynamically generated spontaneous chiral symmetry breaking in Technicolour theories. We now add the study of the BFKL Pomeron to this list.

5.1 Diffusion

At first sight it may appear unnecessary to account for the running of the coupling in the BFKL equation. The argument goes like this. The scale of typical transverse momenta involved in the (Mellin transform of the) BFKL amplitude, $f(\omega, \mathbf{k_1}, \mathbf{k_2}, \mathbf{q})$, is set by the impact factors at the top and bottom of the gluon ladder. This transverse momentum, $\mathbf{k_h}$ (we assume it is the same for both the impact factors), comes from the 'primordial' transverse momentum of partons inside the scattering hadrons. Now since the BFKL equation is infra-red safe there is no need to introduce any other momentum scale and so the integrations over transverse momenta in all sections as we go down the ladder must be dominated by $\mathbf{k} \approx \mathbf{k_h}$, and so the correct value to take for the coupling constant is simply $\alpha_s(\mathbf{k_h^2})$.

This is *almost* correct but not quite. The correct statement is that in any section of the ladder the integrand of the transverse momentum integral has a maximum at $\mathbf{k} \approx \mathbf{k_h}$, but as we go further away from the top or bottom of the ladder, where the $\mathbf{k_h}$ is set, then a wider and wider range of transverse momenta becomes significant and consequently the running of the coupling becomes important.

This broadening in the range of typical $\mathbf{k}$ values involved in the loop integrals as we move along the ladder is a diffusion effect which we will now discuss in some detail. It is an important property of the BFKL amplitude to which we shall continually return in the following chapters.

Consider the BFKL amplitude for zero momentum transfer, $F(s, \mathbf{k_1}, \mathbf{k_2}, \mathbf{0})$, as a function of s (rather than its Mellin transform). The asymptotic solution is given by Eq.(4.34). To simplify our notation, let us now define

$$\begin{aligned} y &= \ln \frac{s}{\mathbf{k}^2} \\ \tau &= \ln \frac{\mathbf{k_1^2}}{\mathbf{k_2^2}}. \end{aligned}$$

and

$$\Psi(y, \tau) = \sqrt{\mathbf{k_1^2 k_2^2}} F(s, \mathbf{k_1}, \mathbf{k_2}, \mathbf{0}).$$

For large s we may use the asymptotic solution of Eq.(4.34). In which case, $\Psi(y, \tau)$ satisfies the diffusion equation:

$$\frac{\partial \Psi(y, \tau)}{\partial y} = \omega_0 \Psi(y, \tau) + a^2 \frac{\partial^2 \Psi(y, \tau)}{\partial \tau^2}. \tag{5.1}$$

Starting from the boundary condition, $\Psi(0, \tau) = \pi \delta(\tau)$, we can solve for $\Psi(y', \tau)$. The diffusion equation tells us that as y' increases so the τ-distribution broadens and so the important range of τ-values increases.

More quantitatively, we would like to know: (a) what is the mean $\ln \mathbf{k}^2$ at some point along the ladder; (b) what is the RMS spread of the $\ln \mathbf{k}^2$ distribution at this point. To answer these questions we need first to appreciate that†

$$F(s, \mathbf{k_1}, \mathbf{k_2}, \mathbf{0}) = \int d^2\mathbf{k}' F(s', \mathbf{k_1}, \mathbf{k}', \mathbf{0}) F(s/s', \mathbf{k}', \mathbf{k_2}, \mathbf{0}), \tag{5.2}$$

for arbitrary $s' \leq s$, i.e. we can view the BFKL amplitude as a convolution of two other BFKL amplitudes with an *arbitrary* partitioning of the total energy s. We define $y' = \ln s'/\mathbf{k}^2$. For a

† This can be seen by inverting the Mellin transform of Eq.(4.28) and using the orthonormality relations of the eigenfunctions.

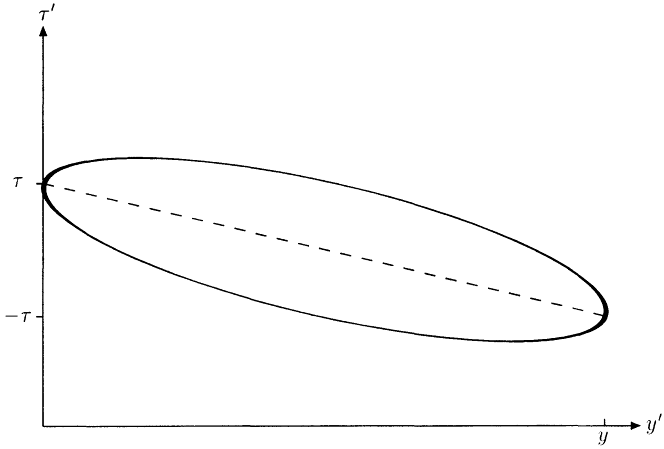

Fig. 5.1. Diffusion in the τ'–y plane.

given y' we can now ask for the mean of

$$\tau' \equiv \ln \frac{\mathbf{k}'^2}{\sqrt{\mathbf{k}_1^2 \mathbf{k}_2^2}}.$$

This is what we mean when we ask for the typical transverse momentum at some point along the ladder. It is a simple matter of Gaussian integration to compute

$$\begin{aligned} \langle \tau' \rangle &= \int d^2\mathbf{k}' \tau' \frac{F(s', \mathbf{k_1}, \mathbf{k}', \mathbf{0}) F(s/s', \mathbf{k}', \mathbf{k_2}, \mathbf{0})}{F(s, \mathbf{k_1}, \mathbf{k_2}, \mathbf{0})} \\ &= \frac{\tau}{2}\left(1 - 2\frac{y'}{y}\right). \end{aligned} \tag{5.3}$$

The RMS deviation, σ, is similarly computed:

$$\begin{aligned} \sigma^2 &= \int d^2\mathbf{k}' (\tau' - \langle \tau' \rangle)^2 \frac{F(s', \mathbf{k_1}, \mathbf{k}', \mathbf{0}) F(s/s', \mathbf{k}', \mathbf{k_2}, \mathbf{0})}{F(s, \mathbf{k_1}, \mathbf{k_2}, \mathbf{0})} \\ &= 2a^2 y' \left(1 - \frac{y'}{y}\right). \end{aligned} \tag{5.4}$$

In Fig. 5.1 we show a plot which illustrates the diffusion in τ'. The dotted straight line represents $\langle \tau' \rangle$ whilst the solid curves are of the functions $\langle \tau' \rangle \pm \sigma$, i.e. they represent the RMS deviations about the mean.

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The axis of the 'cigar' is tilted since we chose $\tau \neq 0$, i.e. the virtualities of the external gluons are not equal. In order that we can trust a perturbative calculation, it had better be that the cigar does not dip (or tip!) too far into the region of $\mathbf{k}'^2 \sim \Lambda_{QCD}^2$. Remember we need to convolute $F(s, \mathbf{k_1}, \mathbf{k_2}, \mathbf{0})$ with the relevant impact factors to obtain physical cross-sections. The avoidance of diffusion into the infra-red region is equivalent to demanding that the impact factors $\Phi_i(\mathbf{k_i})/\mathbf{k_i^2}$ be peaked at large $\mathbf{k_i^2}$.

We have just seen that even if we pick the impact factors so that the axis of the cigar is horizontal (i.e. $\tau = 0$), we still have to worry about diffusion. It is therefore more sensible to conclude that the value of α_s which should be used in the BFKL equation is $\alpha_s(\mathbf{k}'^2)$ rather than a fixed value. However, we must remember that the BFKL equation involves an integral over transverse momenta from zero upwards and hence, for sufficiently small arguments, the running coupling becomes far too large for perturbation theory to be valid. It is therefore necessary to freeze the coupling below a certain magnitude of transverse momentum (or perform some other regulating procedure). Of course this is a phenomenological procedure without any fundamental basis in QCD. Nevertheless, we now consider how to deal with such a running coupling, at least within a reasonable approximation.

5.2 Accounting for the running of the coupling

In order to solve the BFKL equation for running coupling† we need to find the solutions of the eigenvalue equation:

$$\bar{\alpha}_s(\mathbf{k}^2) \int \frac{d^2\mathbf{k}'}{(\mathbf{k}-\mathbf{k}')^2} \left[\phi_i(\mathbf{k}') - \frac{\mathbf{k}^2}{[\mathbf{k}'^2 + (\mathbf{k}-\mathbf{k}')^2]}\phi_i(\mathbf{k})\right] = \lambda_i \phi_i(\mathbf{k}). \tag{5.5}$$

The running coupling, $\bar{\alpha}_s(\mathbf{k}^2)$, is given (to leading order) by

$$\bar{\alpha}_s(\mathbf{k}^2) = \frac{4N}{\beta_0 \xi}, \tag{5.6}$$

where $\xi = \ln \mathbf{k}^2/\Lambda_{QCD}^2$ and $\beta_0 = 11N/3 - 2n_f/3$ for n_f light flavours (we shall sometimes write this as $\bar{\alpha}_s(\xi)$). We could have

† For the time being we continue to work at zero momentum transfer, deferring studies of large momentum transfer to the next section.

taken the running coupling *inside* the integral and written it as $\bar{\alpha}_s(\mathbf{k}'^2)$ so that it runs with the integrated transverse momentum. The difference between the two choices depends on whether we take the coupling at a particular rung of the ladder to be controlled by the momentum of the gluon above or below that rung. Strictly speaking, one should take the maximum of the two, i.e. $\bar{\alpha}_s(\max(\mathbf{k}^2, \mathbf{k}'^2))$, but this is not necessary since the transverse momenta of two adjacent sections of the ladder are indeed of the same order.

Equation (5.5) cannot be solved analytically in the same way as we did for the case of a fixed coupling constant. There are two possible approaches to finding an approximate solution. The first is to approximate the integral equation by a large matrix (by discretizing the transverse momentum) and finding the eigenvalues and eigenvectors numerically. This was done by Daniell & Ross (1989). The other is to try analytic approximations, which is the method used by Lipatov (1986) that we discuss here.

The method used is similar to the WKB approximation for solving Schrödinger's equation, in which good approximations are found in different regions and these are matched at the turning points. Once more we restrict ourselves to the azimuthally symmetric solution $n = 0$. Motivated by the fact that for fixed coupling constant the eigenfunctions are

$$\phi_\nu^0 \sim \frac{1}{\sqrt{\mathbf{k}^2}} \exp(i\nu\xi), \tag{5.7}$$

with eigenvalues $\bar{\alpha}_s\chi_0(\nu)$, we try a solution

$$\frac{C(\xi)}{\sqrt{\mathbf{k}^2}} \exp\left(\pm i \int^\xi d\xi' \nu(\xi')\right) \tag{5.8}$$

for the eigenfunction with eigenvalue λ_i. Now, ν is treated as a function and it is related to the inverse of the function χ_0 such that,

$$\chi_0(\nu(\xi)) = \lambda_i / \bar{\alpha}_s(\xi). \tag{5.9}$$

Equation (5.8) reduces to Eq.(5.7) with $C(\xi)$ set to a constant if we take $\bar{\alpha}_s$ to be fixed. Equation (5.8) will be a good approximation as long as the function ν does not vary too much with ξ. We can obtain a good approximation for the prefactor $C(\xi)$ by inserting Eq.(5.8) into the eigenvalue equation (Eq.(5.5)) expressed as a differential equation, i.e.

$$\bar{\alpha}_s(\mathbf{k}^2)\chi_0\left(-i\frac{\partial}{\partial\xi}\right)C(\xi)\exp\left(\pm i\int^{\xi}d\xi'\nu(\xi')\right)$$
$$= \lambda_i\, C(\xi)\exp\left(\pm i\int^{\xi}d\xi'\nu(\xi')\right). \qquad (5.10)$$

Assuming that $C(\xi)$ and $\nu(\xi)$ are slowly varying functions so that we may neglect second and higher order derivatives, Eq.(5.10) is satisfied provided

$$\chi_0'(\nu(\xi))\, C'(\xi) + \frac{1}{2}\chi_0''(\nu(\xi))\nu'(\xi)C(\xi) = 0, \qquad (5.11)$$

which is solved by

$$C(\xi) \propto \frac{1}{\sqrt{|\chi_0'(\nu(\xi))|}}. \qquad (5.12)$$

More precisely if $\nu^n(\xi)$ is the nth derivative of the function ν, then the approximation is good so long as

$$\nu^n(\xi) \ll (\nu(\xi))^n \qquad (5.13)$$

(for $n \geq 1$). This condition may be true for some regions of the integration variable ξ', but it cannot be valid throughout. This is because the function $\nu(\xi)$ has a zero when $\chi_0 = 4\ln 2$, which occurs at some critical value of ξ, depending on the eigenvalue λ_i,

$$\xi_c = \frac{16N\ln 2}{\beta_0\lambda_i} \qquad (5.14)$$

and its derivative becomes infinite at that point. Near $\xi = \xi_c$, ν may be approximated using Eqs. (4.29), (5.6), (5.9) and (5.14) as

$$\nu \approx \left(\frac{\lambda_i\beta_0}{56N\zeta(3)}\right)^{1/2}\sqrt{(\xi_c - \xi)}. \qquad (5.15)$$

For values of ξ larger than ξ_c, ν is imaginary, and from Eq.(5.8) we see that the eigenfunction is no longer an oscillating function of ξ, but an exponentially decreasing function:

$$\phi(\mathbf{k}) = \eta\frac{1}{\sqrt{|\chi_0'(\nu(\xi))|\mathbf{k}^2}}\exp\left(-\int_{\xi_c}^{\xi}d\xi'|\nu(\xi')|\right), \qquad (5.16)$$

where η is a (as yet undetermined) phase. Once again Eq.(5.16) is only valid away from the branch point where the inequality (5.13) is expected to hold. There is also an exponentially increasing solution (the function $\nu(\xi)$ has two branches for $\xi > \xi_c$), but like all good physicists we throw this away since an exponentially increasing eigenfunction is not physically acceptable.

What about the 'forbidden' region $\xi \approx \xi_c$. Since ν is small in this region, we may use the expansion Eq.(4.29) to expand χ_0 up to quadratic order and rewrite the integral equation as a second order differential equation:

$$\frac{4N}{\beta_0\xi}\left(4\ln 2 + 14\zeta(3)\frac{\partial^2}{\partial\xi^2}\right)\phi_i(\mathbf{k}) = \lambda_i\phi_i(\mathbf{k}). \tag{5.17}$$

Again using Eq.(5.14), rearranging terms and changing variables from ξ to z where

$$z = \left(\frac{\beta_0\lambda_i}{56N\zeta(3)}\right)^{1/3}(\xi - \xi_c), \tag{5.18}$$

this equation becomes

$$\frac{\partial^2\phi_i}{\partial z^2} - z\phi_i = 0, \tag{5.19}$$

which is Airy's equation. There exists a solution, Ai(z), which has the following asymptotic forms:

$$\begin{aligned}\mathrm{Ai}(z) &\rightarrow \frac{1}{\sqrt{\pi}|z|^{1/4}}\sin\left(\frac{2}{3}|z|^{3/2} + \frac{\pi}{4}\right), \quad z \rightarrow -\infty \\ \mathrm{Ai}(z) &\rightarrow \frac{1}{2\sqrt{\pi}z^{1/4}}\exp\left(-\frac{2}{3}z^{3/2}\right), \quad z \rightarrow \infty\end{aligned}$$

(there is also a solution which grows exponentially as $|z| \rightarrow \infty$ which corresponds to the unphysical discarded solution).

Now for sufficiently small ν where the approximation Eq.(5.15) is valid, we have

$$\frac{2}{3}|z|^{3/2} \approx |\int_{\xi_c}^{\xi} d\xi'\nu(\xi')|$$

and from Eqs.(4.29), (5.15) and (5.18),

$$|z|^{1/4}\sqrt{28\zeta(3)} = \sqrt{|\chi_0'|}\left[\frac{56N\zeta(3)}{\beta_0\lambda_i}\right]^{1/6}.$$

Therefore we can match the region where the Airy function is a good approximation to the regions $\xi \ll \xi_c$ (oscillatory solution) and $\xi \gg \xi_c$ (exponentially decaying solution) and, furthermore, this matching uniquely determines the phase of the oscillatory solution. We may therefore write the approximate solution (up to an overall constant) for all values of ξ:

$$\begin{aligned}
\phi_i(\mathbf{k}) &\approx \sqrt{\frac{28\zeta(3)}{|\chi_0'(\nu(\xi))|\mathbf{k}^2}}\sin\left(\int_\xi^{\xi_c} d\xi'\nu(\xi') + \frac{\pi}{4}\right), \quad \xi \ll \xi_c, \\
\phi_i(\mathbf{k}) &\approx \sqrt{\frac{\pi}{\mathbf{k}^2}}\left(\frac{\lambda_i\beta_0}{56N\zeta(3)}\right)^{-1/6}\mathrm{Ai}\left(\left(\frac{\beta_0\lambda_i}{56N\zeta(3)}\right)^{1/3}(\xi-\xi_c)\right), \\
&\qquad\qquad \xi \sim \xi_c, \\
\phi_i(\mathbf{k}) &\approx \sqrt{\frac{7\zeta(3)}{|\chi_0'(\nu(\xi))|\mathbf{k}^2}}\exp\left(-\int_{\xi_c}^{\xi} d\xi'|\nu(\xi')|\right), \quad \xi \gg \xi_c. \quad (5.20)
\end{aligned}$$

We have introduced one mass scale Λ_{QCD} and this has allowed us to fix the phase of the oscillating solution at the turning point ξ_c. Now we need one more ingredient in order to compute the eigenvalues (or equivalently $\nu(\xi)$). We need to assume that the phase is fixed to some angle ϑ at some value of the (logarithm of the) transverse momentum, ξ_0. For sufficiently large momentum transfer (rather than the zero momentum transfer case that we are considering here) this second scale is provided by the momentum transfer, t, as we shall discuss below. For small or zero momentum transfer processes the value of the phase at the infra-red scale (ξ_0) must be provided by the infra-red features of QCD and cannot be attained from perturbation theory. In other words, we are going beyond perturbation theory in assuming the existence of this infra-red scale which characterizes the non-perturbative behaviour of the gluons in the ladder. Now we have two scales at which the phase is fixed and in analogy with the WKB approximation for solving the Schrödinger equation this sets conditions which can only be met by certain eigenvalues. In this case matching the phase at ξ_0 gives

$$\vartheta = \int_{\xi_0}^{\xi_c} d\xi'\nu(\xi') + \frac{\pi}{4} + (i-1)\pi, \tag{5.21}$$

with i a positive integer. A typical solution is shown in Fig. 5.2. The phase is fixed at the point $\xi = \xi_0$ to the value ϑ and is also fixed in region II, so that the Airy function solution in region III matches the oscillatory solution (region II) (up to a multiple of π) and the exponentially decaying solution (region IV). Below ξ_0 (region I) the solution is dominated by the (non-perturbative)

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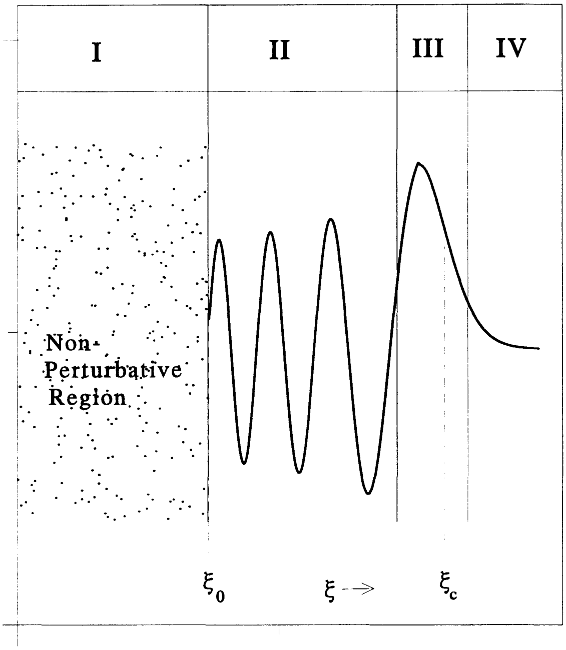

Fig. 5.2. An eigenfunction $\phi_i(\mathbf{k})$ in different regions of ξ (see Eq.(5.20)). Region I is the infra-red region dominated by non-perturbative behaviour. Region II is the oscillatory region ($\xi \ll \xi_c$). Region III is the region given by the Airy function ($\xi \sim \xi_c$) and region IV is the decay region ($\xi \gg \xi_c$). (We have chosen one of the lower eigenfunctions so that the oscillating region can be seen clearly.)

infra-red features of QCD. Now recall that ξ_c depends on the eigenvalue λ_i. Thus *only certain discrete values of* λ_i *can be solutions of Eq.(5.21) and hence we find a discrete spectrum of the integral operator* $\mathcal{K}_0$. This means that the Mellin transform is given by a sum of isolated poles, the leading one being the solution of Eq.(5.21) with $i = 1$ and is identified as the Pomeron pole. For ξ_0 large enough for us to assume that ν is always small between ξ_0

and ξ_c we can use Eq.(5.15) for ν. It then follows that

$$\int_{\xi_0}^{\xi_c} \nu(\xi')d\xi' = \left(\frac{\lambda_i\beta_0}{56N\zeta(3)}\right)^{1/2} \frac{2}{3}(\xi_c - \xi_0)^{3/2}.$$

We can now calculate the first correction to the location of the Pomeron pole in terms of the phase angle, ϑ, i.e.

$$\begin{aligned} \alpha_P(0) \;\approx\; & 1 + 4\ln 2\frac{\bar{\alpha}_s(\xi_0)}{N} \\ & \times \left\{1 - \left(\frac{\beta_0\bar{\alpha}_s(\xi_0)}{4N}\right)^{2/3} \left[\frac{7\zeta(3)}{2\ln 2}\right]^{1/3} \left(\frac{3(\vartheta - \pi/4)}{2}\right)^{2/3}\right\}. \end{aligned} \quad (5.22)$$

Unfortunately, this is only the first term in a slowly convergent series and provides a very poor approximation over sensible values of ξ_0. For a full numerical solution of Eq.(5.21) we refer to the literature (Hancock & Ross (1992)). We shall shortly turn to a study of the running coupling in the case of large momentum transfer. In this case the phase is fixed by perturbation theory and we are able to quantify the location of the Pomeron (and sub-leading) poles.

However, before leaving the $t = 0$ case we wish to remark that (despite the fact that an exact analytic solution of the BFKL equation with running coupling is not possible) Collins & Kwiecinski (1989) have established upper and lower limits for the intercept of the leading trajectory. They found that if the running of the coupling is 'frozen' at some infra-red scale, $k_0^2 = \Lambda_{QCD}^2 \exp\xi_0$, then the intercept obeys the inequalities

$$1 + 1.2\bar{\alpha}_s(k_0^2) \leq \alpha_P(0) \leq 1 + 4\ln 2\,\bar{\alpha}_s(k_0^2). \quad (5.23)$$

5.3 Large momentum transfer

For non-zero momentum transfer, we proceed in the same way except that we work in impact parameter space and the eigenfunctions are functions of $\mathbf{b}, \mathbf{b}', \mathbf{c}$. From Eq.(4.49) we see that the quantity which is raised to the power $i\nu$ is

$$\left(\frac{(\mathbf{b} - \mathbf{b}')^2}{((\mathbf{b} - \mathbf{c})^2(\mathbf{b}' - \mathbf{c})^2}\right),$$

so we replace $\mathbf{k}^2$ in the preceding section with this expression for the argument of the running of the coupling. The impact parameter $\mathbf{c}$ is integrated over and to the approximation to which we are

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working (leading order in the running) we can set $\mathbf{c} = \mathbf{0}$ inside the running coupling and take the quantity ξ to be

$$\xi = \ln\left(\frac{(\mathbf{b}-\mathbf{b}')^2}{\mathbf{b}^2\mathbf{b}'^2\Lambda_{QCD}^2}\right).$$

If $\mathbf{b} \ll \mathbf{b}'$ or $\mathbf{b}' \ll \mathbf{b}$ then the coupling is controlled by the smaller of the two impact parameters, as one would expect.

As discussed by Kirschner & Lipatov (1990) it turns out to be convenient to work in the 'mixed representation' where we keep explicit the dependence upon the momentum transfer $\mathbf{q}^2$. In other words we invert one of the two Fourier transforms that were performed to get from an eigenfunction which depended on $\mathbf{k}$ and $\mathbf{q}-\mathbf{k}$ to $\mathbf{b}$ and $\mathbf{b}'$. More precisely, we perform the inverse Fourier transform of the right hand side of Eq.(4.49) in the variable $\mathbf{b}+\mathbf{b}'$ which is conjugate to $\mathbf{q}$ and keep the remaining combination $\hat{\mathbf{b}} \equiv \mathbf{b}-\mathbf{b}'$. This leads to an eigenfunction which is a function of $\mathbf{q}$ and $\hat{\mathbf{b}}$. In this case the running of the coupling is controlled by the larger of $\mathbf{q}^2$ and $1/\hat{\mathbf{b}}^2$. When $\hat{\mathbf{b}}$ becomes larger than $1/\mathbf{q}^2$ the coupling stops running and is 'frozen' at $\alpha_s(\mathbf{q}^2)$.† For larger values of $\hat{\mathbf{b}}$ the solution continues to oscillate but with a fixed (angular) frequency ν_0, where

$$\lambda_i = \bar{\alpha}_s(\mathbf{q}^2)\chi_n(\nu_0). \tag{5.24}$$

The Fourier transform is straightforward but tedious. The details are given by Lipatov & Kirschner (1990). The result is

$$\phi_i(\mathbf{q},\hat{\mathbf{b}}) \propto \sin\left(\frac{\pi}{2} - \nu\ln\left(\hat{\mathbf{b}}^2\mathbf{q}^2/4\right) + \frac{\delta(n,\nu)}{2} + n\theta_b\right) + \mathcal{O}(\hat{\mathbf{b}}\mathbf{q}), \tag{5.25}$$

where θ_b is the angle between $\hat{\mathbf{b}}$ and some fixed direction, and the phase $\delta(n,\nu)$ is given by

$$e^{i\delta(n,\nu)} = \frac{\Gamma^2((n+1)/2+i\nu)\Gamma(n+1-2i\nu)\Gamma(-2i\nu)}{\Gamma^2((n+1)/2-i\nu)\Gamma(n+1+2i\nu)\Gamma(2i\nu)}. \tag{5.26}$$

Equation (5.25) is valid in the region $\hat{\mathbf{b}}\mathbf{q} \ll 1$, and we have not written down the constant since it is only the phase matching that is important for the determination of the permitted eigenvalues.

† We assume that $\mathbf{q}^2$ is sufficiently large that $\alpha_s(\mathbf{q}^2)$ is small enough to be a valid expansion parameter in perturbation theory.

In this case we set ξ to

$$\xi = -\ln\left(4\hat{\mathbf{b}}^2\Lambda_{QCD}^2\right)$$

and once again for $\hat{\mathbf{b}} < 1/\mathbf{q}$ the coupling runs as

$$\bar{\alpha}_s(\hat{\mathbf{b}}) = \frac{4N}{\beta_0\xi}$$

and we have to replace $\nu\ln(\hat{\mathbf{b}}^2)$ in Eq.(5.25) by

$$\int^{\xi} \nu(\xi')d\xi'.$$

For sufficiently small values of $\hat{\mathbf{b}}$ (where $\xi > \xi_c$), ν becomes imaginary and we obtain a solution which decays exponentially with $\hat{\mathbf{b}}$. The region $\nu \approx 0$ can again be solved in terms of an Airy function and the matching of the phase tells us that for $\xi \ll \xi_c$ we have

$$\phi_i(\mathbf{q}, \hat{\mathbf{b}}) \propto \sin\left(\frac{\pi}{4} + \int_{\xi}^{\xi_c} \nu(\xi')d\xi'\right). \qquad (5.27)$$

Now for consistency we must match the phases in Eqs.(5.25) and (5.27) which puts a constraint on the allowed values for ξ_c and consequently also on the allowed eigenvalues, λ_i. We will impose this phase matching at $\xi = \xi_0$, where

$$\xi_0 = \ln\left(\frac{\mathbf{q}^2}{\Lambda_{QCD}^2}\right) = \frac{4N}{\beta_0\bar{\alpha}_s(\mathbf{q}^2)}. \qquad (5.28)$$

At this point $\hat{\mathbf{b}}^2 = \hat{\mathbf{b}}_0^2 = 4/\mathbf{q}^2$ and the coupling freezes (i.e. for larger impact parameters than $\hat{\mathbf{b}}_0$ the coupling is determined by $\mathbf{q}^2$ and not $\hat{\mathbf{b}}^2$). Strictly, we cannot push Eq.(5.25) as far as this because there are corrections of order $\hat{\mathbf{b}}\mathbf{q}$. However, since the coupling varies only logarithmically with $\hat{\mathbf{b}}$ we can go to a value of $\hat{\mathbf{b}}$ where $\hat{\mathbf{b}}\mathbf{q}$ is still small but $\alpha_s(\hat{\mathbf{b}}) \approx \alpha_s(\mathbf{q})$. Again we confine ourselves to the azimuthally symmetric solution ($n = 0$). Setting $\hat{\mathbf{b}}$ to $\hat{\mathbf{b}}_0$ in Eq.(5.25) (where $\ln(\hat{\mathbf{b}}^2\mathbf{q}^2/4) = 0$) and in Eq.(5.27) and matching the phases we obtain

$$\int_{\xi_0}^{\xi_c} \nu(\xi')d\xi' = \frac{(1+4i)\pi}{4} + \frac{\delta(0,\nu_0)}{2}. \qquad (5.29)$$

This fixes the allowed eigenvalues (it is directly analogous to Eq.(5.21) for the $t = 0$ case) in terms of a *perturbative* phase,

$\delta(0,\nu_0)$ ($\approx \pi$ for small ν_0). As before, for large enough $\mathbf{q}^2$, we can find the approximate solution:

$$\lambda_i \approx 4\ln 2\,\bar{\alpha}_s(\mathbf{q}^2) \times \left\{1 - \left(\frac{\beta_0\alpha_s(\mathbf{q}^2)}{4\pi}\right)^{2/3}\left[\frac{7\zeta(3)}{2\ln 2}\right]^{1/3}\left(\frac{3\pi(i+3/4)}{2}\right)^{2/3}\right\}. \quad (5.30)$$

So again, running the coupling has discretized the cut to a semi-infinite series of poles (again the analytic result is a poor approximation for attainable values of t). As we shall see in Figs. 5.4 and 5.5, the leading pole is shifted significantly downwards after taking asymptotic freedom into account. However, it is still too large to account for the behaviour of hadronic total cross-sections. Moreover, the trajectory is very flat in t (we will discuss the t-dependence further in Chapter 7), which is not consistent with (for example) the observed shrinkage of the forward diffraction peak. We can conclude, therefore, that although it is indeed possible to obtain an isolated Pomeron trajectory purely from perturbative considerations (for sufficiently large values of $\mathbf{q}^2$), non-perturbative effects are likely to be essential if we are to have any chance of reproducing the Pomeron identified in the study of soft hadron physics.

5.4 The Landshoff–Nachtmann model

We shall spend the rest of this chapter discussing various attempts that have been made to incorporate non-perturbative effects into the construction of the Pomeron in the hope of reproducing at least some of the phenomenological properties of the 'soft' Pomeron.

There are two orthogonal approaches to this. In the first approach it is assumed that the 'hard' Pomeron that we have been considering so far is heavily attenuated at small transverse momenta so that it becomes subdominant and the 'soft' Pomeron, an entirely different object which has nothing to do with gluon ladders and belongs completely to the non-perturbative realm of QCD, takes over as the dominant contribution to diffractive processes. In the second approach the 'hard' Pomeron converts smoothly into the 'soft' Pomeron at sufficiently low transverse

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momenta. Whereas the first approach provides an adequate explanation of why we have so far failed to reproduce any of the phenomenological properties of the 'soft' Pomeron, it offers no explanation of how this 'soft' Pomeron might arise from QCD. The second approach is more optimistic, although, as we shall see, it has so far only made very small steps towards its ultimate goal of providing a complete description of the 'soft' Pomeron.

Landshoff & Nachtmann (1987) developed a model based firmly on the Low–Nussinov picture, namely the exchange of two gluons in a colour singlet state. However, they argued that since the 'soft' Pomeron is very much controlled by the non-perturbative (infra-red) aspects of QCD, one should not expect the exchanged gluons to have a propagator which at low k^2 behaves like

$$\frac{1}{k^2},$$

particularly as gluons are supposed to be confined and so the propagator cannot have a pole. These non-perturbative gluons would have a propagator, $D_{np}(k^2)$, with a much softer k^2-dependence. This non-perturbative propagator can be related to the vacuum expectation value of the square of the gluon operator:

$$\langle 0| : G_{\mu\nu}(x) G^{\mu\nu}(x) : |0\rangle = -i \int \frac{d^4k}{(2\pi)^4} 6k^2 D_{np}(k^2). \qquad (5.31)$$

In order for the integral on the right hand side of Eq.(5.31) to converge it is necessary that $D_{np}(k^2)$ falls with increasing k^2 at least as fast as $1/k^6$ (the perturbative propagator takes over at large k^2). Therefore, from dimensional analysis the non-perturbative propagator must depend on some length scale, a, provided by the infra-red region of QCD.

Landshoff and Nachtmann were concerned with the problem of quark counting in the coupling of the Pomeron to hadrons (e.g. the Pomeron coupling to a baryon with three valence quarks is depicted in Fig. 5.3). A straightforward calculation shows that the contribution from the graph of Fig. 5.3(a), where both gluons couple to the same quark, dominates over the contribution from the graph in Fig. 5.3(b) provided a is small compared with the typical hadron radius, R. If this condition can be achieved then the quark-counting rule follows with corrections of order a^2/R^2.

The model was confined to the consideration of an Abelian

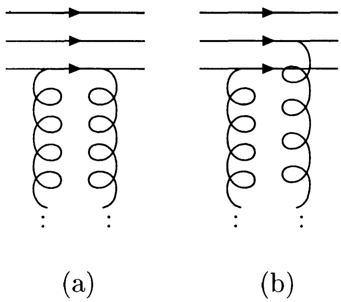

Fig. 5.3. Graphs contributing to the coupling of the Pomeron to the three valence quarks of a baryon. Graphs of type (a) must dominate those of type (b) in order to reproduce the quark-counting rule.

gauge theory to describe the gluons and did not address the question of obtaining the Pomeron trajectory through gluon ladders. In a non-Abelian theory one would expect the scales a and R to be of the same order of magnitude since they are both generated by the same mechanism, i.e. the infra-red properties of QCD. Nevertheless, factors of 2 and π would certainly arise and it is perfectly possible that the non-perturbative gluon propagator does behave (at least qualitatively) in the manner suggested by Landshoff and Nachtmann and that a is sufficiently small compared with R to account for the observed quark-counting rule.

5.5 The effect of non-perturbative propagators

The Landshoff–Nachtmann model described above immediately poses two important questions for those who wish to relate the 'soft' Pomeron to QCD.

1. Can non-perturbative propagators with the required low momentum properties be extracted from QCD?
2. Do gluon ladders with non-perturbative propagators for the vertical gluons simulate the 'soft' Pomeron?

There have been several attempts to extract soft gluon propagators from various non-perturbative approaches to QCD, varying from lattice techniques to solutions of the Dyson–Schwinger

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equations. Some of these investigations do indeed give propagators which are finite as $k^2 \to 0$ or at least have a softer singularity than a pole – others give propagators which have an even steeper singularity as $k^2 \to 0$ and such behaviour has been hailed as a signal for a confining gluon potential. Recently Büttner & Pennington (1995) have argued that, at least in the Landau gauge, a propagator with a small momentum behaviour softer than $1/k^2$ is inconsistent with the Dyson–Schwinger equation. They argue that the Pomeron cannot be explained in terms of the Landshoff–Nachtmann model and that its behaviour is controlled by the coupling of soft gluons to off-shell quarks inside the hadron.

Notwithstanding this, we shall investigate the effect of soft propagators, $D(k^2)$, which do *not* have a pole at $k^2 = 0$ (i.e. propagators which represent confined as opposed to confining gluons) but which, for large k^2, reduce to the usual perturbative propagators.

A complete non-perturbative treatment of the Pomeron would require knowledge about all the gluon Green functions, not just the propagator. Clearly, this is impossible and so we have to compromise. One possible approach is to make the assumption that the non-perturbative features of QCD manifest themselves mainly by the effect of the propagators for soft gluons, whereas for the vertices we may continue to use the perturbative expressions. An approach along these lines is that used by Hancock & Ross (1992, 1993) in which such non-perturbative propagators are inserted directly into the BFKL equation (with running coupling). Thus, for example, at zero momentum transfer the kernel, $\mathcal{K}_0$, of Eq.(4.18) is replaced by

$$\begin{aligned} \mathcal{K}_0 \bullet f(\omega, \mathbf{k_1}, \mathbf{k_2}, \mathbf{q}) &= \frac{\bar{\alpha}_s(\mathbf{k}_1^2)}{\pi} \int d^2\mathbf{k}' D((\mathbf{k_1} - \mathbf{k}')^2) \\ &\times \left[f(\omega, \mathbf{k}', \mathbf{k_2}, \mathbf{q}) \right. \\ &\quad \left. - \frac{D(\mathbf{k}_1^2) D(\mathbf{k}'^2) D((\mathbf{k_1} - \mathbf{k}')^2)}{D(\mathbf{k}'^2) + D((\mathbf{k_1} - \mathbf{k}')^2)} f(\omega, \mathbf{k_1}, \mathbf{k_2}, \mathbf{q}) \right]. \end{aligned} \tag{5.32}$$

Clearly, the eigenvalues of this operator have to be found by numerical techniques which involve discretizing the transverse momenta $\mathbf{k_1}$ and $\mathbf{k}'$ and diagonalizing the resulting matrix. The eigenvalues are discrete since there is an 'ultra-violet' scale, Λ_{QCD}, encoded in the running of the coupling, as well as an infra-red scale, a, contained in the non-perturbative gluon propagator.† In other words the infra-red scale is set to $\xi_0 = -\ln(a^2\Lambda_{QCD}^2)$ and the fact that a discrete spectrum of eigenvalues is obtained means that the infra-red behaviour which has been introduced by replacing the propagators with non-perturbative propagators fixes the phase ϑ (see Eq.(5.21)) at this infra-red scale, although it is difficult to understand from analytic considerations exactly how this phase fixing mechanism works.

It turns out that the leading eigenvalue for zero momentum transfer (i.e. the intercept of the Pomeron) does not depend on the exact nature of the non-perturbative propagator but only on the infra-red scale. Therefore, in Fig. 5.4 we take the simplest possible example in which it is assumed that the infra-red effects introduce an effective mass $1/a$ for the gluon (i.e. we take $D(\mathbf{k}^2) = a^2/(1 + a^2\mathbf{k}^2)$), and plot the intercept of the Pomeron against $1/a$. We observe that there is a reduction of this intercept as the effective mass is increased. The intercept is still a long way from the observed value of 1.08 for the 'soft' Pomeron, but it is clear that this, albeit naive, attempt to take non-perturbative effects into consideration has the effect of pushing the intercept in the right direction.

One can also insert non-perturbative propagators into the BFKL equation for non-zero momentum transfer and solve numerically. The result of such a procedure (taking $1/a$ to be 0.25 GeV) for the leading trajectory and first two sub-leading trajectories is shown in Fig. 5.5. We also show (dashed lines) the result of the purely perturbative trajectories discussed in the preceding section (i.e. the solutions of Eq.(5.30)). We note that these perturbative solutions have a very small slope indicating a very small t-dependence of the perturbative trajectories. The trajectories obtained using non-perturbative propagators (solid lines) deviate substantially from the perturbative trajectories at sufficiently small values of $-t$.

The slope of the trajectories at the origin increases as the infra-red scale $1/a$ increases. We can see from Fig. 5.6 that a slope at

† The running of the coupling is assumed to stop at the infra-red scale, i.e. $\alpha_s(\mathbf{q}^2 < 1/a^2) = \alpha_s(1/a^2)$.

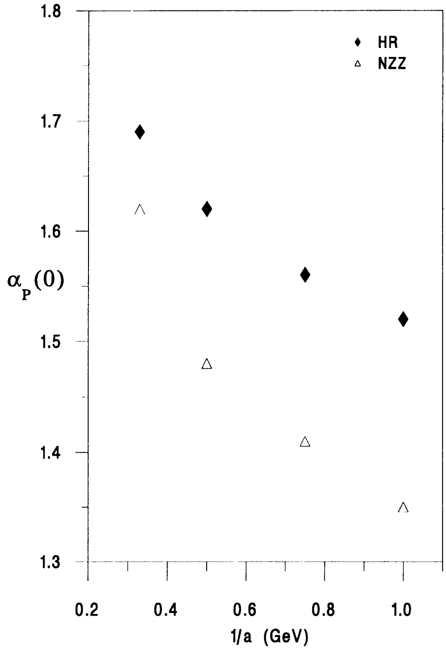

Fig. 5.4. Pomeron intercept against infra-red scale (effective gluon mass) $1/a$ for the Hancock and Ross approach (HR) and the Nikolaev, Zakharov and Zoller approach (NZZ).

the origin of 0.25 GeV as suggested by experiment would require an infra-red scale of about 0.8 GeV. On the other hand, it is quite clear from Fig. 5.5 that the trajectories are very far from linear and that the asymptotic (perturbative) solution has been reached at $-t = 1$ GeV^2.

The above treatment tells us that the inclusion of non-perturbative gluon propagators directly into the BFKL equation produces qualitatively desirable effects as far as the reproduction of the 'soft' Pomeron phenomenology is concerned. However, this approach is clearly far too cavalier. A somewhat more subtle procedure has been carried out by Nikolaev, Zakharov & Zoller

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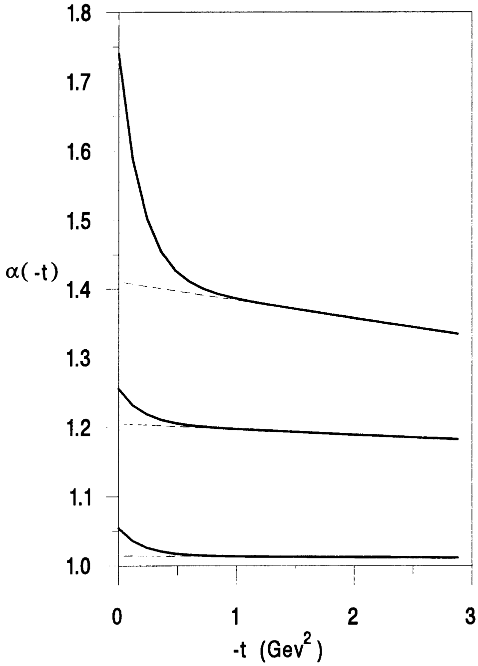

Fig. 5.5. The Pomeron and the two sub-leading trajectories as functions of momentum transfer, $-t$. The solid lines are the solutions with the inclusion of a massive gluon propagator with $1/a$ set to 0.25 GeV and the dashed lines are the results obtained using perturbative gluon propagators.

(1994a,b), using the Fock space expansion, which was briefly mentioned in the preceding chapter. In this procedure all the (BFKL) radiative corrections are incorporated in the impact factors, which are determined by considering the Fock space expansion for the wavefunction of the scattering hadron (e.g. for a meson the lowest order Fock space state is simply a quark–antiquark pair; the next is a quark–antiquark–gluon state, etc.). For each of these states a convolution is taken between the square wavefunction and the cross-section (calculated at Born level only) for the scattering pro-

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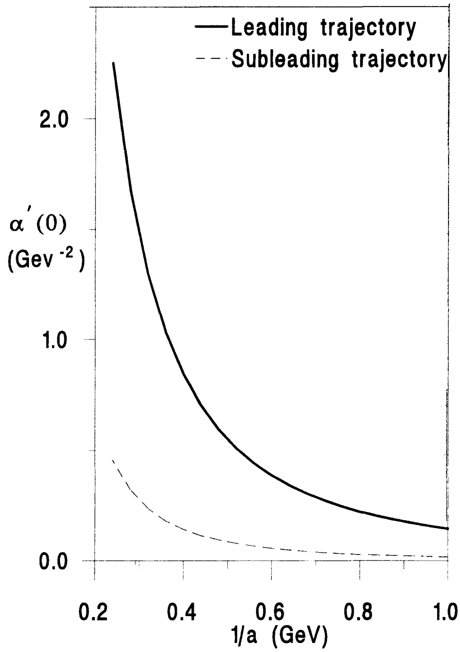

Fig. 5.6. The slopes of the Pomeron and the first sub-leading trajectories as functions of the infra-red scale (effective gluon mass) $1/a$.

cess between the Fock states. Thus, for example, the leading term for meson–meson forward scattering, where both mesons are considered to be quark–antiquark pairs, is given by

$$\mathcal{A}(s,0) = \int d^2\mathbf{b_1} d^2\mathbf{b_2} dz_1 dz_2 \, |\Psi(z_1,\mathbf{b_1})|^2 |\Psi(z_2,\mathbf{b_2})|^2 \sigma(\mathbf{b_1},\mathbf{b_2}), \tag{5.33}$$

where $\Psi(z_i,\mathbf{b_i})$ is the amplitude for meson i, with momentum p_i to consist of a quark–antiquark pair with longitudinal momenta $z_i p_i$, $(1-z_i)p_i$ and be separated by $\mathbf{b_i}$ in impact parameter space. The 'cross-section' , $\sigma(\mathbf{b_1},\mathbf{b_2})$, is the lowest order amplitude for a process consisting of the exchange of two gluons between two quark–antiquark pairs with impact parameter separations $\mathbf{b_1}$ and $\mathbf{b_2}$ respectively. Up to a colour factor this is

$$8\alpha_s^2 \int d^2\mathbf{k} \frac{(1 - e^{i\mathbf{k}.\mathbf{b_1}})(1 - e^{i\mathbf{k}.\mathbf{b_2}})}{\mathbf{k}^4}. \tag{5.34}$$

This procedure lends itself very easily to the incorporation of non-perturbative propagators for the long-range gluons, since the long-range gluons only appear in the cross-section. We therefore simply replace the two factors of $1/\mathbf{k}^2$ in Eq.(5.34) by $D(\mathbf{k}^2)$. Once again the simplest non-perturbative propagator is obtained by introducing an effective gluon mass $1/a$. The results for the intercept of the Pomeron obtained by Nikolaev, Zakharov & Zoller (1994a,b) are also shown in Fig. 5.4. We note that this procedure leads to a larger reduction of the intercept than the treatment by Hancock & Ross (1992). In fact, the results they obtained are lower than the lower limits given by Collins & Kwiecinski (1989) (Eq.(5.23)). However, these bounds were obtained within the context of the perturbative theory with a running coupling. Incorporation of any non-perturbative effects such as an effective gluon mass can lead to a violation of these bounds.

Collins & Landshoff (1992) have taken a somewhat different approach to the low transverse momentum behaviour of the BFKL kernel. They cut the transverse momentum off below some $\mathbf{k_0}$ in the integral for the part of the kernel which accounts for real gluon emission (the first term of Eq.(4.18)). They found that this did not shift the position of the leading singularity in the Mellin transform of the amplitude. However, they observed that there should also be an upper limit to the transverse momentum integration (from the kinematic limits) which should be of order $\sqrt{s}$. In the derivation of the BFKL equation this upper cut-off was ignored since it does not affect the leading logarithm results. Restoring the upper cut-off effectively takes into account some of the sub-leading logarithm corrections. Collins & Landshoff showed that if this upper cut-off is introduced (along with the infra-red cut-off) then the intercept of the Pomeron is reduced. McDermott, Forshaw & Ross (1995) showed that the shift downwards of the leading eigenvalue is less than 20% for $s/\mathbf{k_0^2} > 10^4$, i.e. the leading eigenvalue is shifted to

$$\alpha_P(0) = 4\ln 2\, \bar{\alpha}_s \frac{1}{1 + \pi^2/[2 + \frac{1}{2}\ln(s/\mathbf{k_0^2})]^2}.$$

Furthermore, in the Collins & Landshoff analysis the strong coupling was kept fixed. The running of the coupling (which is also a sub-leading logarithm effect) plays a similar role to the imposition of an upper cut-off to the transverse momentum integration. If the coupling is allowed to run then the effect of imposing an upper cut-off on the transverse momentum integral is diminished.

5.6 The heterotic Pomeron

Levin & Tan (1992) have given some consideration to the question of how one might interpolate between the 'hard' and 'soft' Pomerons. They postulate a 'heterotic Pomeron' which tends to the BFKL Pomeron when the virtuality of the external gluons is sufficiently large, and tends to the 'soft' Pomeron for near on-shell external gluons.

We introduce the impact parameter $\bar{\mathbf{b}}$, conjugate to $\mathbf{q}$, and define $\tilde{F}(s, \mathbf{k_1}, \mathbf{k_2}, \bar{\mathbf{b}})$ by

$$F(\omega, \mathbf{k_1}, \mathbf{k_2}, \mathbf{q}) = \int \frac{d^2\bar{\mathbf{b}}}{2\pi} \, e^{i\mathbf{q}\cdot\bar{\mathbf{b}}} \tilde{F}(s, \mathbf{k_1}, \mathbf{k_2}, \bar{\mathbf{b}}).$$

We have shown that even with the running of the coupling the dependence of the trajectories with the momentum transfer, $t = -\mathbf{q}^2$, is very small, so to a good approximation we may neglect the diffusion of the impact parameter $\bar{\mathbf{b}}$ as we go down the ladder. Therefore $\tilde{F}(s, \mathbf{k_1}, \mathbf{k_2}, \bar{\mathbf{b}})$ also (approximately) obeys the $t = 0$ BFKL equation, Eq.(4.17), which we can write (after inverting the Mellin transform) as

$$\tilde{F}(s, \mathbf{k_1}, \mathbf{k_2}, \bar{\mathbf{b}}) = \int_0^s \frac{ds'}{s'} \, d^2\mathbf{k}' \, \mathcal{K}_0(\mathbf{k}', \mathbf{k_1}) \tilde{F}(s', \mathbf{k}', \mathbf{k_2}, \bar{\mathbf{b}}). \quad (5.35)$$

We have explicitly written the argument of the kernel. This can be generalized to include the case where there *is* indeed substantial diffusion in the impact parameter $\bar{\mathbf{b}}$ and we can also allow for a more general energy (s) dependence. The generalized equation is then

$$\begin{aligned} \tilde{F}(s, \mathbf{k_1}, \mathbf{k_2}, \bar{\mathbf{b}}) \;\; &= \;\; \int_0^s \frac{ds'}{s'} \, d^2\mathbf{k}' \, d^2\bar{\mathbf{b}}' \, \mathcal{K}(s/s', \mathbf{k}', \mathbf{k_1}, (\bar{\mathbf{b}} - \bar{\mathbf{b}}')) \\ &\quad \times \; \tilde{F}(s', \mathbf{k_1}, \mathbf{k_2}, \bar{\mathbf{b}}'). \end{aligned} \quad (5.36)$$

For the 'hard' Pomeron which obtains at sufficiently large $\mathbf{k_1}, \mathbf{k}'$ the kernel is

$$\mathcal{K}(s/s', \mathbf{k}', \mathbf{k_1}, (\bar{\mathbf{b}} - \bar{\mathbf{b}}')) = \mathcal{K}_0(\mathbf{k}' - \mathbf{k_1})\delta^2(\bar{\mathbf{b}} - \bar{\mathbf{b}}'). \qquad (5.37)$$

On the other hand, for small $\mathbf{k}', \mathbf{k_1}$ (below $\mathbf{k_0}$) the Pomeron has a significant $\bar{\mathbf{b}}$-dependence, but does not depend much on the gluon virtuality, so we expect $\mathbf{k_1}$ to remain fixed at around $\mathbf{k_0}$. In this limit the generalized kernel has the form

$$\mathcal{K}(s/s', \mathbf{k}', \mathbf{k_1}, (\bar{\mathbf{b}} - \bar{\mathbf{b}}')) = \delta^2(\mathbf{k}' - \mathbf{k_0}) \left(\frac{s}{s'}\right)^c B(\bar{\mathbf{b}} - \bar{\mathbf{b}}'), \quad (5.38)$$

where B is a function which vanishes as its argument becomes large, but has a non-zero width. The dynamics which determine this function are not yet understood. It cannot be derived from usual perturbation theory, but alternative techniques such as the $1/N$ expansion may shed some light on it.

The kernel Eq.(5.38) should lead to the 'soft' Pomeron, which can be described in terms of a 'ladder' in some sense (although it may not be a ladder of gluons) and as we go down the ladder we have diffusion in $\bar{\mathbf{b}}$ but not in virtuality $\mathbf{k}$.

Levin and Tan considered the case where the function $B(\bar{\mathbf{b}} - \bar{\mathbf{b}}')$ was determined by a random walk in impact parameter space as one goes down the ladder. In such a case the diffusion equation becomes

$$\frac{\partial \tilde{F}}{\partial \ln s} = c\tilde{F} + c_b \partial^2_{\bar{\mathbf{b}}} \tilde{F}. \qquad (5.39)$$

The solution to this equation has a dependence on impact parameter, $\bar{\mathbf{b}}$, which is

$$\sim \frac{\exp(-\bar{\mathbf{b}}^2/4c_b \ln s)}{(\ln s)^{(1-c)}},$$

and which, when Fourier transformed, gives the t-dependence

$$\sim s^{c_b t}.$$

Comparing this with the experimental value for the slope of the 'soft' Pomeron trajectory we must have

$$c_b = 0.25 \text{ GeV}^{-2}.$$

The heterotic Pomeron would therefore be determined by a kernel which interpolates between the two expressions (5.37) and (5.38) as the virtuality of the external gluons varies from $\mathbf{k_1} \ll \mathbf{k_0}$

to $\mathbf{k_1} \gg \mathbf{k_0}$. The soft QCD physics which gives rise to this kernel has not so far been identified. Nevertheless, the existence of a kernel, which interpolates between the 'two' Pomerons, is an intriguing possibility.

We have been discussing various attempts to explain the soft Pomeron within the context of QCD. So far none of these attempts has been particularly successful.

Nevertheless, we have the hard Pomeron which is derived from perturbative QCD using no further assumptions about the infrared behaviour. Of course, this hard Pomeron is in itself a very interesting object and it is important to put it to experimental test. In the next two chapters we shall be discussing processes such as deep inelastic scattering and large rapidity gap events in which the hard Pomeron can (at least in principle) be isolated, studied, and compared with the predictions of the 'clean' part of QCD.

5.7 Summary

- We can view the $t = 0$ BFKL equation as a diffusion equation in the transverse momentum of the emitted gluons (i.e. which make up the rungs of the ladder). Therefore, a wide range of transverse momenta contributes to the amplitude and hence it becomes necessary to consider the running of the QCD coupling.
- The BFKL equation with running coupling can be solved approximately using a technique analogous to the WKB approximation. The modified solution changes from an oscillating solution to an exponentially decaying solution above some critical transverse momentum.
- If the phase of the oscillations is fixed at some low transverse momentum by the (non-perturbative) infra-red effects of QCD, then it can only be matched for certain values of the Mellin transform variable, ω. This then leads to isolated poles for the Mellin transform of the Pomeron amplitude, as opposed to the cut obtained in the fixed coupling case.
- The infra-red phase fixing can be obtained by inserting non-perturbative gluon propagators into the BFKL equation. The intercept of the Pomeron thus obtained is reduced compared with

the position of the branch point in the fixed coupling case. However, the intercept is still too large to explain, by itself, the phenomena which are so well described by the soft Pomeron.

- Non-perturbative propagators are also required to explain the quark-counting rule within the context of the two-gluon exchange model of the Pomeron. The non-perturbative propagator introduces a length scale which, if small compared to the hadron radius, will suppress quark-counting-violating contributions to the scattering amplitude in which the two gluons land on different quarks within the hadron.
- A kernel which, for large transverse gluon momenta, tends to the BFKL kernel (giving rise to diffusion in s and transverse gluon momenta but no diffusion in impact parameter) and which for small transverse gluon momenta gives rise to diffusion in s and impact parameter but *not* in gluon transverse momentum, could provide a useful interpolation between the seemingly very different 'hard' and 'soft' Pomerons.