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SEMI-ORTHOGONAL FRAME WAVELETS AND FRAME MULTI-RESOLUTION ANALYSES

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We first characterise semi-orthogonal frame wavelets by generalising the characterisation of orthonormal wavelets. We then characterise those semi-orthogonal frame wavelets that are associated with frame multi-resolution analyses. This is a generalisation of a result of Wang and another result of Papadakis. Finally, we illustrate our results by an example.

1. INTRODUCTION

It is well-known that most wavelets are associated with multi-resolution analyses, whereas there exist some 'pathological' wavelets that are not associated with any multi-resolution analyses. We are going to be more clear about what we mean. Let $\psi \in L^2(\mathbb{R})$ be an *orthonormal wavelet* if it generates a *wavelet orthonormal basis*, that is, $\{\psi_{jk} := D^jT_k\psi : j,\ k \in \mathbb{Z}\}$ is an orthonormal basis of $L^2(\mathbb{R})$, where $D : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is the unitary dilation operator defined by $Df(x) := 2^{1/2}f(2x)$, and T_t is the translation operator defined by $T_tf(x) := f(x-t)$ for $t \in \mathbb{R}$. The following useful commutation relation holds:

$$D^nT_t = T_{2^{-n}t}D^n, \quad \text{or} \quad T_tD^n = D^nT_{2^nt}. \tag{1}$$

We recall the characterisation of orthonormal wavelets in [**5, 6, 7, 14**]:

THEOREM 1. *$\psi \in L^2(\mathbb{R})$ is an orthonormal wavelet if and only if*

(a) $\|\psi\|_{L^2(\mathbb{R})} = 1$;

(b) $\sum_{j\in\mathbb{Z}} \left|\widehat{\psi}(2^jx)\right|^2 = 1$ *for almost everywhere* $x \in \mathbb{R}$;

(c) $\sum_{j=0}^{\infty} \widehat{\psi}(2^jx)\overline{\widehat{\psi}}\big(2^j(x+2m\pi)\big) = 0$ *for almost everywhere* $x \in \mathbb{R}, m \in 2\mathbb{Z}+1$.

We use the following form of the Fourier transform: For $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ define $\widehat{f}(x) := \int_{\mathbb{R}} f(t)e^{-ixt}\,dt$ and extend the Fourier transform $\wedge$ to be $\sqrt{2\pi}$ times a unitary operator from $L^2(\mathbb{R})$ onto $L^2(\mathbb{R})$. The most efficient way to construct an orthonormal wavelet is to construct it from an orthonormal multi-resolution analysis ([**7**]).

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DEFINITION 2: A family $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R})$ is said to be a multi-resolution analysis if

(i) $V_j \subset V_{j+1}$ for each $j \in \mathbb{Z}$;
(ii) $D(V_j) = V_{j+1}$ and $T_1(V_0) = V_0$;
(iii) $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$ and $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$;
(iv) There exists $\varphi \in V_0$ such that $\{T_k\varphi : k \in \mathbb{Z}\}$ is an orthonormal basis for V_0.

It is well-known that given a multi-resolution analysis there exists $\psi \in V_1 \ominus V_0$ such that $\{\psi_{jk} : j,\ k \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{R})$ ([7]). On the other hand, suppose that an orthonormal wavelet ψ is given. Let $V_j := \overline{\operatorname{span}}\{\psi_{lk} : k \in \mathbb{Z},\ l < j\}$ for $j \in \mathbb{Z}$. Then it is easy to see that if there exists $\varphi \in V_0$, called the *scaling function*, such that $\{T_k\varphi : k \in \mathbb{Z}\}$ is an orthonormal basis for V_0, then $\{V_j\}_{j\in\mathbb{Z}}$ is a multi-resolution analysis. In this case we say that ψ is *associated with a multi-resolution analysis*. It is established that most 'nice' wavelets are associated with multi-resolution analyses [**7**, Chapter 7]. For example, any compactly supported orthonormal wavelet is associated with a multi-resolution analysis ([**7**, Corollary 3.15, Chapter 7]). On the other hand, there are some 'pathological' orthonormal wavelets that are not associated with multi-resolution analyses ([**14**, p. 77], [**6**]). Hernández and Weiss along with Wang ([**7, 14**]) characterised those orthonormal wavelets that are associated with multi-resolution analyses. Let $\mathbb{T}$ denote the circle group which can be identified with $[-\pi, \pi)$.

THEOREM 3. *An orthonormal wavelet ψ is associated with a multi-resolution analysis if and only if* $\sum_{j=1}^{\infty}\sum_{k\in\mathbb{Z}}\left|\widehat{\psi}(2^j(x+2k\pi))\right|^2 = 1$ *for almost every* $x \in \mathbb{T}$.

A sequence $\{f_i : i \in I\}$ of elements of a Hilbert space $\mathcal{H}$ is said to be a *frame* for $\mathcal{H}$ if there exist positive constants A and B such that for each $f \in \mathcal{H}$ $A \leqslant \sum_i \left|\langle f, f_i\rangle\right|^2 \leqslant B$. If $\{f_i : i \in I\}$ is a frame for $\mathcal{H}$, then there exists another frame $\{\widetilde{f}_i : i \in I\}$ for $\mathcal{H}$, called the *dual frame*, such that for any $f \in \mathcal{H}$ $f = \sum_i \langle f, \widetilde{f}_i\rangle f_i$. Hence we can expand any vector by a frame. Moreover, unlike orthonormal basis, a frame can be redundant. In some situations this redundancy is positively sought after. See [**7**] for more details on frames. Papadakis ([**13**]) proved the following.

THEOREM 4. *Any orthonormal wavelet ψ is associated with a generalised multi-resolution analysis in the sense that there exists a countable (finite or countably infinite) subset Φ of V_0 such that $\{T_k\varphi : k \in \mathbb{Z},\ \varphi \in \Phi\}$ is a frame for V_0.*

In this paper we generalise Theorem 1, Theorem 3 and Theorem 4 to semi-orthogonal wavelet frames and frame multi-resolution analyses (see Theorems 7 and 11). First, let us introduce some definitions in order to clarify what we are going to show. $\psi \in L^2(\mathbb{R})$ is said to be a *frame wavelet* if it generates a *wavelet frame* for $L^2(\mathbb{R})$, that is, $\{\psi_{jk} : j,\ k \in \mathbb{Z}\}$ is a frame for $L^2(\mathbb{R})$. It is said to be a *semi-orthogonal frame wavelet* if the wavelet frame it generates is semi-orthogonal in the sense that $\langle\psi_{jk},\psi_{lm}\rangle = 0$ if $j \neq l$. $\{V_j\}_{j\in\mathbb{Z}}$ is said to be a *frame multi-resolution analysis* if Condition (iv) in Definition 2 is replaced by the following.

(iv)′ There exists $\varphi \in V_0$ such that $\{T_k\varphi : k \in \mathbb{Z}\}$ is a frame for V_0.

It is said to be a *finite frame multi-resolution analysis* if Condition (iv) in Definition 2 is replaced by the following.

(iv)″ There exists a finite subset $\Phi \subset V_0$ such that $\{T_k\varphi : k \in \mathbb{Z}, \varphi \in \Phi\}$ is a frame for V_0.

If Φ is countably infinite we say that $\{V_j\}_{j\in\mathbb{Z}}$ is an *infinite frame multi-resolution analysis*. Frame multi-resolution analyses were introduced in [**1**] with an intention to apply the theory to analyse narrow band signals. The fundamental existence problem concerning frame multi-resolution analyses was solved independently in [**2**] and [**10**], and some extension of the theory can be found in [**11**].

In Section 2 we generalise Theorem 1 in the sense that we find equivalent conditions for ψ to be a semi-orthogonal frame wavelet. Then a generalisation of both Theorem 3 and Theorem 4 is presented in Section 3. The idea is to apply shift-invariant space theory ([**3, 4, 8**]) to the problem of association of a wavelet with a multi-resolution analysis. Our solution to the problem of the association of a Riesz wavelet, that is, $\{\psi_{jk} : j,\ k \in \mathbb{Z}\}$ is a Riesz basis of $L^2(\mathbb{R})$, with a multi-resolution analysis is reported in [**12**]. Finally, we illustrate our results by an example.

2. Semi-orthogonal frame wavelets

We first characterise semi-orthogonal frame wavelets as a generalisation of the characterisation of orthonormal wavelets by Gripenberg ([**5**]), Ha, Kang, Lee and Seo ([**6**]), and Hernández and Weiss, and also Wang ([**7, 14**]). The following two propositions are well known. See [**7**, Theorem 1.6, Chapter 7] and [**9**, Theorem A3], respectively.

Proposition 5. *Let $\psi \in L^2(\mathbb{R})$. Then $\{\psi_{j,k} : j,\ k \in \mathbb{Z}\}$ is a tight frame with frame bound 1 for $L^2(\mathbb{R})$, that is,*

$$\sum_{j,k}\left|\langle f,\psi_{j,k}\rangle\right|^2 = \|f\|^2, \quad \text{for all } f \in L^2(\mathbb{R}), \tag{2}$$

if and only if ψ satisfies (b) and (c) of Theorem 1.

Proposition 6. *Let $\psi \in L^2(\mathbb{R})$ and let $W_0 = \overline{\operatorname{span}}\{\psi_{0,k} : k \in \mathbb{Z}\}$. Then $\{\psi_{0,k} : k \in \mathbb{Z}\}$ is a frame for W_0 if and only if there exist positive constants A, B such that*

$$A \leqslant \|\widehat{\psi}_{\|x}\|^2_{\ell^2(\mathbb{Z})} \leqslant B \text{ for almost every } x \in \mathbb{T}\setminus N, \tag{3}$$

where $\widehat{\psi}_{\|x} := \left(\widehat{\psi}(x-2\pi k)\right)_{k\in\mathbb{Z}}$ and $N := \{x \in \mathbb{T} : \widehat{\psi}_{\|x} = 0\}$. In this case, A and B are frame bounds for $\{\psi_{0,k} : k \in \mathbb{Z}\}$.

Now, we state and prove our characterisation of semi-orthogonal frame wavelets.

Theorem 7. *Let $\psi \in L^2(\mathbb{R})$ and define ψ^* by*

$$\widehat{\psi^*}(x) := \begin{cases} \dfrac{\widehat{\psi}(x)}{\|\widehat{\psi}_{\|x}\|_{\ell^2(\mathbb{Z})}}, & \text{if } \widehat{\psi}_{\|x} \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then the following statements are equivalent:

(a) *$\{\psi_{j,k} : j, k \in \mathbb{Z}\}$ is a semi-orthogonal frame wavelet with frame bounds A and B;*

(b) *There exist positive constants A,B such that ψ satisfies (3) and*

$$\sum_{j\in Z} \left|\widehat{\psi^*}(2^j x)\right|^2 = 1, \text{ for almost every } x \in \mathbb{R}, \tag{4}$$

$$\sum_{j\geqslant 0} \widehat{\psi^*}(2^j x)\overline{\widehat{\psi^*}}\big(2^j(x+2p\pi)\big) = 0, \text{ for almost every } x \in \mathbb{R},\ p \in 2\mathbb{Z}+1; \tag{5}$$

(c) *There exist positive constants A,B such that ψ satisfies (3), (5) and*

$$\sum_{k\in\mathbb{Z}} \widehat{\psi}(x+2k\pi)\overline{\widehat{\psi}}\big(2^j(x+2k\pi)\big) = 0, \text{ almost every } x \in \mathbb{R},\ j \geqslant 1, \tag{6}$$

$$A \leqslant \sum_{j\in\mathbb{Z}} \left|\widehat{\psi}(2^j x)\right|^2 \leqslant B, \text{ for almost every } x \in \mathbb{R}. \tag{7}$$

Proof: Let $W_j = \overline{\operatorname{span}}\{\psi_{j,k} : k \in \mathbb{Z}\}$ and $W_j^* = \overline{\operatorname{span}}\{\psi_{j,k}^* : k \in \mathbb{Z}\}$. Note that $W_j = W_j^*$.

(a) $\Rightarrow$ (b): Suppose that ψ is a semi-orthogonal frame wavelet with frame bounds A and B , that is,

$$A\|f\|^2 \leqslant \sum_{j,k\in\mathbb{Z}} \left|\langle f, \psi_{j,k}\rangle\right|^2 \leqslant B\|f\|^2,\ f \in L^2(\mathbb{R}).$$

Take $f \in W_0$. Since $W_j \perp W_{j'}$ for $j \neq j'$ by the semi-orthogonality, we have

$$A\|f\|^2 \leqslant \sum_{k\in\mathbb{Z}} \left|\langle f, \psi_{0,k}\rangle\right|^2 \leqslant B\|f\|^2,$$

which is equivalent to (3) by Proposition 6.

Since W_j^*'s are orthogonal to each other and

$$\sum_{k\in\mathbb{Z}} \left|\widehat{\psi^*}(x-2\pi k)\right|^2 = 1, \text{ for almost everywhere } x \in \mathbb{T}\setminus N, \tag{8}$$

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$\{\psi^*_{j,k}\}_{j,k\in\mathbb{Z}}$ is a tight frame with frame bound 1 for $L^2(\mathbb{R})$. Hence (4) and (5) are satisfied by Proposition 5.

(b) $\Rightarrow$ (c): From (3), we have

$$\frac{1}{B}\left|\widehat{\psi}(x)\right|^2 \leqslant \left|\widehat{\psi^*}(x)\right|^2 \leqslant \frac{1}{A}\left|\widehat{\psi}(x)\right|^2, \text{ for almost every } x \in \mathbb{R}. \tag{9}$$

Hence, by Condition (4) we have

$$A \leqslant \sum_{j\in\mathbb{Z}}\left|\widehat{\psi}(2^j x)\right|^2 \leqslant B, \text{ for almost everywhere } x \in \mathbb{R},$$

which shows (7). From the definition of ψ^*, we see that $\{\psi^*_{0,k} : k \in \mathbb{Z}\}$ is a tight frame for W_0^* with frame bound 1. By Proposition 5, $\{\psi^*_{j,k} : j,\ k \in \mathbb{Z}\}$ is also a tight frame with frame bound 1 for $L^2(\mathbb{R})$. Since ψ^* is in W_0, it follows from the tightness of both $\{\psi^*_{0,k} : k \in \mathbb{Z}\}$ and $\{\psi^*_{j,k} : j,\ k \in \mathbb{Z}\}$ that

$$\begin{aligned} \|\psi^*\|^2_{L^2(\mathbb{R})} &= \sum_{j,\,k\in\mathbb{Z}} \left|\langle\psi^*, \psi^*_{j,k}\rangle\right|^2 \\ &= \sum_{k\in\mathbb{Z}}\left|\langle\psi^*, \psi^*_{0,k}\rangle\right|^2. \end{aligned}$$

Therefore, $\langle\psi^*, \psi^*_{j,k}\rangle = 0$ for $j \neq 0$. We argue as in [**7**, Section 3.1] below:

$$\begin{aligned} 0 = \langle\psi^*, \psi^*_{j,k}\rangle &= \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{\psi^*}(x)2^{-j/2}\overline{\widehat{\psi^*}}(2^{-j}x)e^{i2^{-j}kx}\,dx \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{\psi^*}(2^j x)2^{j/2}\overline{\widehat{\psi^*}}(x)e^{ikx}\,dx. \end{aligned}$$

Thus, we have

$$\begin{aligned} 0 &= \sum_{l\in\mathbb{Z}}\int_{2l\pi}^{2(l+1)\pi} \widehat{\psi^*}(2^j x)\overline{\widehat{\psi^*}}(x)e^{ikx}\,dx \\ &= \int_0^{2\pi}\left\{\sum_{l\in\mathbb{Z}}\overline{\widehat{\psi^*}}(x+2k\pi)\widehat{\psi^*}(2^j(x+2k\pi)\right\}e^{ikx}\,dx \end{aligned}$$

for all $k \in \mathbb{Z}$ when $j \geqslant 1$. This shows that

$$\sum_{l\in\mathbb{Z}}\widehat{\psi^*}(x+2k\pi)\overline{\widehat{\psi^*}}\big(2^j(x+2k\pi)\big) = 0, \text{ for almost every } x \in \mathbb{R},\ j \geqslant 1.$$

Therefore, we have

$$\sum_{k\in\mathbb{Z}}\widehat{\psi}(x+2k\pi)\overline{\widehat{\psi}}\big(2^j(x+2k\pi)\big) = \|\widehat{\psi}_{\|x}\|_{\ell^2(\mathbb{Z})}\|\widehat{\psi}_{\|2^j x}\|_{\ell^2(\mathbb{Z})}\sum_{k\in\mathbb{Z}}\widehat{\psi^*}(x+2k\pi)\overline{\widehat{\psi^*}}\big(2^j(x+2k\pi)\big) = 0$$

Thus, ψ satisfies Condition (6).

(c) $\Rightarrow$ (a): Condition (3) implies that $\{\psi_{j,k} : k \in \mathbb{Z}\}$ is a frame for W_j by Proposition

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6, and Condition (6) shows that W_0 is orthogonal to W_j for $j \neq 0$. By means of change of variables, $\langle \psi_{j,k}, \psi_{l,m} \rangle = \langle \psi_{0,k-2^{j-l}m}, \psi_{l-j,0} \rangle$, from which $W_j \perp W_l$ follows for $j \neq l$. Therefore, $\{\psi_{j,k} : j,\ k \in \mathbb{Z}\}$ is a frame of $W := \overline{\operatorname{span}}\{\psi_{j,k} : j,\ k \in \mathbb{Z}\}$. We claim that $W = L^2(\mathbb{R})$. It suffices to show that $\{\psi^*_{j,k}\}$ is a frame for $L^2(\mathbb{R})$. As in [**7**, Proposition 1.19, Chapter 7],

$$\begin{aligned}
&\sum_{j,\,k\in\mathbb{Z}} \left|\langle f, \psi^*_{j,k}\rangle\right|^2 \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\widehat{f}(x)\right|^2 \sum_{j\in\mathbb{Z}} \left|\widehat{\psi^*}(2^j x)\right|^2 dx \\
&\quad + \frac{1}{2\pi}\sum_{n\in\mathbb{Z}}\sum_{p\in 2\mathbb{Z}+1}\int_{-\infty}^{\infty} \widehat{f}(x)\overline{\widehat{f}}\,(x + 2\pi p 2^n)\theta_p(2^{-n}x)\,dx \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\widehat{f}(x)\right|^2 \sum_{j\in\mathbb{Z}} \left|\widehat{\psi^*}(2^j x)\right|^2 dx, \qquad (10)
\end{aligned}$$

since $\theta_p(x) := \sum_{l=0}^{\infty} \widehat{\psi^*}(2^l x)\overline{\widehat{\psi^*}}\big(2^l(x + 2p\pi)\big) = 0$ by (5). From (7) and (9), we have

$$A/B \leqslant \sum_{j\in\mathbb{Z}} \left|\widehat{\psi^*}(2^j x)\right|^2 \leqslant B/A.$$

Thus we obtain from (10),

$$A/B\|f\|^2 \leqslant \sum_{j,\,k\in\mathbb{Z}} \left|\langle f, \psi^*_{j,k}\rangle\right|^2 \leqslant B/A\|f\|^2.$$

That is, $\{\psi^*_{j,k}\}$ is a frame for $L^2(\mathbb{R})$ and hence spans $L^2(\mathbb{R})$. Therefore, $W = \overline{\cup W^*_j} = L^2(\mathbb{R})$. ∎

3. Frame multiresolution analyses

In this section we characterise those semi-orthogonal frame wavelets which are associated with frame multi-resolution analyses. This association problem can best be understood by the theory of shift-invariant spaces. We first introduce briefly those parts of shift-invariant space theory that will be used directly in this paper. The theory has a rich history, and is well-known to approximation theorists. The interested reader may consult [**3, 4, 8**] and the references therein. A closed subspace S of $L^2(\mathbb{R})$ is said to be *shift-invariant* if $T_k f \in S$ for any $f \in S$ and $k \in \mathbb{Z}$. Let $\Phi \subset L^2(\mathbb{R})$. Then $S := \mathcal{S}(\Phi) := \overline{\operatorname{span}}\{T_k\varphi : \varphi \in \Phi,\ k \in \mathbb{Z}\}$ is clearly shift-invariant. The *length* of S is defined to be $\min\{\#\Phi : S = \mathcal{S}(\Phi), \Phi \subset L^2(\mathbb{R})\}$, where $\#\Phi$ means the cardinality of Φ. It is established in [**3**, Section 3] that the length of a shift-invariant subspace of $L^2(\mathbb{R})$ is at most countable. For $f \in L^2(\mathbb{R})$, let $\widehat{f}_{\|x} := \big(\widehat{f}(x - 2\pi k)\big)_{k\in\mathbb{Z}}$, which is in $\ell^2(\mathbb{Z})$ for almost every $x \in \mathbb{T}$. For $x \in \mathbb{T}$, $A \subset L^2(\mathbb{R})$ we let $\widehat{A}_{\|x} := \{\widehat{f}_{\|x} : f \in A\}$.

Lemma 8. *Let S be a shift-invariant subspace of $L^2(\mathbb{R})$, and λ its length which may be infinite. Then there exists $\Phi \subset L^2(\mathbb{R})$, with cardinality λ, such that $\{T_k f : k \in \mathbb{Z},\ f \in \Phi\}$ is a frame for S. Moreover, if $S = \mathcal{S}(\Psi)$ for some $\Psi \subset L^2(\mathbb{R})$, then*

$$\begin{aligned}\lambda &= \text{ess-sup}\{\dim \widehat{S}_{\|x} : x \in \mathbb{T}\}\\ &= \text{ess-sup}\{\dim \overline{\text{span}}\ \widehat{\Psi}_{\|x} : x \in \mathbb{T}\}.\end{aligned}$$

Proof: The first part of the theorem follows from [**4**, Theorem 3.3] and the remark following it. The equations concerning λ follow from [**3**, Theorem 3.5] and [**4**, Proposition 1.5]. ∎

Suppose that ψ generates a semi-orthogonal wavelet frame, that is, $\{D^j T_k \psi : j,\ k \in \mathbb{R}\}$ is a frame for $L^2(\mathbb{R})$ and $\langle D^j T_k \psi, D^l T_m \psi \rangle = 0$ if $j \neq l$. Let $W_l := \overline{\text{span}}\{D^l T_k \psi : k \in \mathbb{Z}\}$, and $V_j := \bigoplus_{l<j} W_l$ for $j,\ l \in \mathbb{Z}$. Then it is easy to see that $\{D^l T_k \psi : k \in \mathbb{Z}\}$ is a frame for W_l for each $l \in \mathbb{Z}$, and that $L^2(\mathbb{R}) = \bigoplus_{l\in\mathbb{Z}} W_l$. It is also easy to see that ψ is associated with a frame multi-resolution analysis if and only if there exists $\varphi \in V_0$ such that $\{T_k \varphi : k \in \mathbb{Z}\}$ is a frame for V_0; ψ is associated with a finite frame multi-resolution analyses if and only if there exists $\{\varphi_1, \varphi_2, \dots, \varphi_n\} \subset V_0$ such that $\{T_k \varphi_i : k \in \mathbb{Z},\ 1 \leqslant i \leqslant n\}$ is a frame for V_0; ψ is associated with an infinite frame multi-resolution analysis if and only if there exists $\{\varphi_i : i \in \mathbb{N}\}$ such that $\{T_k \varphi_i : i \in \mathbb{Z}\}$ is a frame for V_0.

Lemma 9. *V_0 is shift-invariant.*

Proof: First note that $V_0^{\perp} = \bigoplus_{l\geqslant 0} W_l$. Equation (1) implies that, for each $l \in \mathbb{Z}$, $f \in W_l$ if and only if $T_{2^{-l}m} f \in W_l$ for each $m \in \mathbb{Z}$, that is, W_l is $2^{-l}\mathbb{Z}$-shift-invariant space. In particular, W_l is shift-invariant for $l \geqslant 0$. This implies that $V_0^{\perp}$ is shift-invariant. Hence so is V_0 by [**3**, Corollary 3.4]. ∎

Lemma 10. *$V_0 = \mathcal{S}(\{D^j \psi : j < 0\})$.*

Proof: Let $V_0' := \mathcal{S}(\{D^j \psi : j < 0\})$. Note that $V_0 = \overline{\text{span}}\{D^j T_k \psi : j < 0,\ k \in \mathbb{Z}\}$ by the definition of V_0, and that $V_0' = \overline{\text{span}}\{T_k D^j \psi : j < 0,\ k \in \mathbb{Z}\} = \overline{\text{span}}\{D^j T_{2^j k} \psi : j < 0,\ k \in \mathbb{Z}\}$ by the definition of the shift-invariant space and Equation (1). V_0, however, is shift-invariant by Lemma 9. Hence

$$\begin{aligned}V_0 &= \overline{\text{span}}\{T_l D^j T_k \psi : j < 0, k, l \in \mathbb{Z}\}\\ &= \overline{\text{span}}\{D^j T_{2^j l + k} \psi : j < 0, k, l \in \mathbb{Z}\}\\ &= \overline{\text{span}}\{D^j T_{2^j l} \psi : j < 0, l \in \mathbb{Z}\} = V_0'.\end{aligned}$$

∎

The following theorem gives a generalisation of both [**7**, Theorem 3.2, Chapter 7] and the main result in [**13**]. We note that the last part of the following theorem is Theorem 3.

THEOREM 11. *Suppose ψ generates a semi-orthogonal wavelet frame. Let, for $x \in \mathbb{T}$,*

$$D(x) := \dim \overline{\operatorname{span}} \{(D^j\psi)^{\wedge}_{\|x} : j < 0\},$$

and

$$\lambda := \text{ess-sup}\{D(x) : x \in \mathbb{T}\},$$

which may be infinite. Then ψ is associated with a frame multi-resolution analysis if and only if $\lambda = 1$; and it is associated with a finite frame multi-resolution analysis if and only if $\lambda < \infty$. In this case there exists $\{\varphi_1, \varphi_2, \dots, \varphi_\lambda\} \subset V_0$ such that $\{T_k\varphi_i : k \in \mathbb{Z}, 1 \leqslant i \leqslant \lambda\}$ is a frame for V_0. It is associated with an infinite frame multi-resolution analysis if $\lambda = \infty$. Suppose, furthermore, that ψ generates an orthonormal basis. Then

$$D(x) = \sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}} \left| \widehat{\psi}(2^j(x + 2\pi k)) \right|^2, \tag{11}$$

and it is associated with an orthonormal multi-resolution analysis if and only if $D(x) = 1$ for almost everywhere $x \in \mathbb{T}$.

PROOF: First note that λ is the length of the shift-invariant space V_0 by Lemma 10 and Lemma 8. Suppose that ψ is associated with a frame multi-resolution analysis. Then there exists $\varphi \in V_0$ such that $\{T_k\varphi : k \in \mathbb{Z}\}$ is a frame for V_0. Hence $V_0 = \mathcal{S}(\varphi)$. Hence $\lambda = 1$ by Lemma 8. Suppose, on the other hand, that $\lambda = 1$. Then there exists $\varphi \in V_0$ such that $\{T_k\varphi : k \in \mathbb{Z}\}$ is a frame for V_0 by Lemma 8. The statements about finite and infinite frame multi-resolution analyses follow similarly. Now suppose that ψ generates an orthonormal basis. Equation (11) follows from [**7**, Equation (3.8), Chapter 7]. Suppose that ψ is associated with an orthonormal multi-resolution analysis. Then there exists $\varphi \in V_0$ whose translates form an orthonormal basis of V_0. Hence $V_0 = \mathcal{S}(\varphi)$. Moreover, $D(x) = \dim \operatorname{span}\{\widehat{\varphi}_{\|x}\}$ by [**4**, Proposition 1.5]. It is well-known that $\|\widehat{\varphi}_{\|x}\|^2_{\ell^2(\mathbb{Z})} = 2\pi \neq 0$ for almost every $x \in \mathbb{T}$. Hence $D(x) = 1$ for almost every $x \in \mathbb{T}$. Suppose, on the other hand, that $D(x) = 1$ for almost every $x \in \mathbb{T}$. Then there exists $\varphi \in V_0$ whose translates form an orthonormal basis for V_0 by [**3**, Theorem 3.2]. ∎

We illustrate our results by considering an example $\psi_a \in L^2(\mathbb{R})$ defined by $\widehat{\psi_a} = \chi_{[-2a,-a)} + \chi_{[a,2a)}$ for $a > 0$. That is,

$$\psi_a(x) = (2/\pi x)\cos(3ax/2)\sin(ax/2).$$

If $a = \pi$, ψ_π is the well-known Shannon wavelet.

For $0 < a \leqslant \pi/2$, we shall show that that ψ_a is a semi-orthogonal frame wavelet by checking the conditions in Theorem 7 (b). We see that

$$\sum_{j \in \mathbb{Z}} \left| \widehat{\psi_a}(2^j x) \right|^2 = \sum_{j \in \mathbb{Z}} \widehat{\psi_a}(2^j x) = 1, \text{ for almost everywhere } x \in \mathbb{R}.$$

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Since $\widehat{\psi_a}(x)\overline{\widehat{\psi_a}}(2^j x) = 0$ for $j \geqslant 1$,

$$\sum_{k\in\mathbb{Z}} \widehat{\psi_a}(x+2k\pi)\overline{\widehat{\psi_a}}\big(2^j(x+2k\pi)\big) = 0, \text{ for almost everywhere } x \in \mathbb{R},\ j \geqslant 1.$$

We can check that

$$\sum_{k\in\mathbb{Z}} \big|\widehat{\psi_a}(x+2k\pi)\big|^2 = \sum_{k\in\mathbb{Z}} \widehat{\psi_a}(x+2k\pi) = \chi_{\mathbb{T}\setminus N},$$

where $N = [-\pi,-2a) \cup [-a,a) \cup [2a,\pi)$. Finally, we check Condition (5). Let $2^j x \in [-2a,-a) \cup [a,2a)$ for $j \geqslant 0$ and let $p \in 2\mathbb{Z}+1$. If $p \geqslant 1$, then $2^j x + 2p2^j\pi \geqslant 2^j x + 2\pi \geqslant -2a + 2\pi \geqslant \pi \geqslant 2a$. If $p \leqslant -1$, then $2^j x + 2p2^j\pi \leqslant 2^j x - 2\pi < 2a - 2\pi \leqslant -2a$. We have

$$\widehat{\psi_a}(2^j x)\overline{\widehat{\psi_a}}\big(2^j(x+2p\pi)\big) = 0 \text{ for } j \geqslant 0 \text{ and } p \in 2\mathbb{Z}+1,$$

and hence

$$\sum_{j\geqslant 0} \widehat{\psi_a}(2^j x)\overline{\widehat{\psi_a}}\big(2^j(x+2p\pi)\big) = 0,\ p \in 2\mathbb{Z}+1.$$

Therefore, we have shown that ψ_a is a semi-orthogonal frame wavelet for $0 < a \leqslant \pi/2$ by Theorem 7. We can also check that ψ_a is not a semi-orthogonal frame wavelet if $\pi/2 < a < \pi$ or $a > \pi$ by using Theorem 7.

Now, we show that ψ_a is associated with a frame multi-resolution analyses for $0 < a \leqslant \pi/2$ by applying Theorem 11. If $x \in [-\pi,-a) \cup [a,\pi)$, we see that $\widehat{\psi}\big(2^j(x+2\pi k)\big) = 0$ for $k \in \mathbb{Z}$ and $j \geqslant 1$ and so $D(x) = 0$. If $x \in [-a,a)\setminus\{0\}$ then $2^{j_x}x \in [-2a,-a) \cup [a,2a)$ for some $j_x \geqslant 1$ and so $\widehat{\psi}\big(2^j(x+2k\pi)\big) = \delta_{j,j_x}\delta_{0,k}$; hence $D(x) = 1$. Therefore $\lambda = 1$ and so ψ_a is associated with a frame multi-resolution analysis by Theorem 11.

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