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Amine Marrakchi

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Stability of products of equivalence relations

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Abstract

An ergodic probability measure preserving (p.m.p.) equivalence relation $\mathcal{R}$ is said to be stable if $\mathcal{R} \cong \mathcal{R} \times \mathcal{R}_0$ where $\mathcal{R}_0$ is the unique hyperfinite ergodic type II_1 equivalence relation. We prove that a direct product $\mathcal{R} \times \mathcal{S}$ of two ergodic p.m.p. equivalence relations is stable if and only if one of the two components $\mathcal{R}$ or $\mathcal{S}$ is stable. This result is deduced from a new local characterization of stable equivalence relations. The similar question on McDuff II_1 factors is also discussed and some partial results are given.

1. Introduction

An ergodic type II_1 equivalence relation $\mathcal{R}$ is *stable* if $\mathcal{R} \cong \mathcal{R} \times \mathcal{R}_0$ where $\mathcal{R}_0$ is the unique hyperfinite ergodic type II_1 equivalence relation. This notion was introduced and studied in [JS87], by analogy with its von Neumann algebraic counterpart [McD70]. In particular, the following characterization of stability was obtained (for the notation, see the end of this section): an ergodic type II_1 equivalence $\mathcal{R}$ is stable if and only if, for every finite set $K \subset [[\mathcal{R}]]$ and every $\varepsilon > 0$, there exists $v \in [[\mathcal{R}]]$ such that $v^2 = 0$, $vv^* + v^*v = 1$ and

$$\forall u \in K, \quad \|vu - uv\|_2 < \varepsilon.$$

Our first theorem strengthens this characterization by showing that the condition $vv^* + v^*v = 1$ can be removed, thus allowing v to be arbitrarily small.

Theorem A. *An ergodic type* II_1 *equivalence relation* $\mathcal{R}$ *is stable if and only if, for every finite set* $K \subset [[\mathcal{R}]]$ *and every* $\varepsilon > 0$, *there exists* $v \in [[\mathcal{R}]]$ *such that* $v^2 = 0$ *and*

$$\forall u \in K, \quad \|vu - uv\|_2 < \varepsilon \|v\|_2.$$

As an application of Theorem A, we obtain the following rigidity result.

Theorem B. *Let* $\mathcal{R}$ *and* $\mathcal{S}$ *be two ergodic type* II_1 *equivalence relations. Then the product equivalence relation* $\mathcal{R} \times \mathcal{S}$ *is stable if and only if* $\mathcal{R}$ *is stable or* $\mathcal{S}$ *is stable.*

As we said before, the study of stable equivalence relations was inspired by its von Neumann algebraic counterpart: the so-called *McDuff* property. Recall that a II_1 factor M is called McDuff if $M \cong M \mathbin{\overline{\otimes}} R$ where R is the hyperfinite II_1 factor. In [McD70] it is shown that a II_1 factor M

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is McDuff if and only if, for every finite set $K \subset M$ and every $\varepsilon > 0$, there exists $v \in M$ such that $v^2 = 0$, $vv^* + v^*v = 1$ and

$$\forall a \in K, \quad \|va - av\|_2 < \varepsilon.$$

Similarly to the equivalence relation case, we can strengthen this characterization by removing the condition $vv^* + v^*v = 1$, and we obtain the following analog of Theorem A.

Theorem C. *A type* II_1 *factor* M *is McDuff if and only if, for every finite set* $K \subset M$ *and every* $\varepsilon > 0$, *there exists* $x \in M$ *such that* $x^2 = 0$ *and,*

$$\forall a \in K, \quad \|xa - ax\|_2 < \varepsilon \|x\|_2.$$

With regard to this result and the similarity between the theory of stable equivalence relations and the theory of McDuff factors, we strongly believe that the following analog of Theorem B should be true.

Conjecture D. Let M and N be type II_1 factors. Then $M \mathbin{\overline{\otimes}} N$ is McDuff if and only if M is McDuff or N is McDuff.

Even though the proof of Theorem B does not admit a straightforward generalization to the von Neumann algebraic case, we can still provide some partial solutions to Conjecture D by using a different approach. We fix ω, a free ultrafilter on $\mathbf{N}$. Given a II_1 factor M, we denote by M^ω its ultrapower and by $M_\omega = M' \cap M^\omega$ its *asymptotic centralizer* (see [McD70]). Recall from [McD70] that M is McDuff if and only if M_ω is non-commutative.

Our first partial result strengthens [WY14, Theorem 2.1].

Theorem E. *Let* M *be a non-McDuff* II_1 *factor and suppose that there exists an abelian subalgebra* $A \subset M$ *such that* $M_\omega \subset A^\omega$. *Then, for every* II_1 *factor* N, *we have that* $M \mathbin{\overline{\otimes}} N$ *is McDuff if and only if* N *is McDuff.*

As far as the author knows, all concrete examples of non-McDuff factors in the literature do satisfy the assumption of Theorem E (in fact, this is how we show that they are not McDuff). Deciding whether or not this property holds for all non-McDuff factors is an interesting open question.

The second result solves Conjecture D under the additional assumption that $(M \mathbin{\overline{\otimes}} N)_\omega$ is a factor.

Theorem F. *Let* M *and* N *be type* II_1 *factors and suppose that* $(M \mathbin{\overline{\otimes}} N)_\omega$ *is a factor. Then both* M_ω *and* N_ω *are factors. If* $M \mathbin{\overline{\otimes}} N$ *is McDuff, then* M *is McDuff or* N *is McDuff.*

Examples of factors with factorial asymptotic centralizers are obtained by taking infinite tensor products of non-Gamma II_1 factors (see Proposition 5.4). These factors were studied in [Pop10]. By combining [Pop10, Theorem 4.1] and Theorem F, we obtain the following corollary which is not related to Conjecture D. It provides the first example of a McDuff II_1 factor that does not admit any McDuff decomposition in the sense of [HMV16].

Corollary G. *Let* $M = \overline{\bigotimes}_{n \in \mathbf{N}} M_n$ *be an infinite tensor product of non-Gamma type* II_1 *factors* M_n, $n \in \mathbf{N}$. *Let* N *be a type* II_1 *factor such that* $M \cong N \mathbin{\overline{\otimes}} R$. *Then* $M \cong N$.

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Before we conclude this introduction, let us say a few words about the methods used to obtain these results. The proof of Theorem A (and Theorem C) is based on a so-called *maximality* argument. This technique involves patching 'microscopic' elements satisfying a given property in order to obtain a 'macroscopic' element satisfying this same property. The term 'maximality' is a reference to Zorn's lemma which is used in the patching procedure. Maximality arguments in the theory of von Neumann algebras were initiated in [MvN43]. Since then, they have been used fruitfully in many of the deepest results of the theory, reaching higher and higher levels of sophistication in [Con76, CS76, Con85, Haa86, Pop86] and culminating in the *incremental patching* method of [Pop87, Pop95, Pop14]. See also [Mar16, HMV17, Mar18] for other recent applications of maximality arguments. On the other hand, the proofs of Theorems E and F are based on a completely different technique which appears in [IV15] and which is inspired by an averaging trick of Haagerup [Haa85]. By using this technique, one can reduce some problems on arbitrary tensor products $M \mathbin{\overline{\otimes}} N$ to the much easier case where one of the two algebras is abelian. This very elementary transfer principle is surprisingly powerful and Theorems E and F are two applications among many others.

Notation

For simplicity, in this paper, all probability spaces are standard and all von Neumann algebras have separable predual (except ultraproducts). We fix some non-principal ultrafilter $\omega \in \beta\mathbf{N}\backslash\mathbf{N}$ once and for all. We denote by $\mathrm{L}(\mathcal{R})$ the von Neumann algebra of a probability measure preserving (p.m.p.) equivalence relation $\mathcal{R}$ (see [FM77]). We denote by $[\mathcal{R}]$ (respectively, $[[\mathcal{R}]]$) its full group (respectively, full pseudo-group) and we will identify them with the corresponding unitaries (respectively, partial isometries) in the von Neumann algebra $\mathrm{L}(\mathcal{R})$. In particular, if $v, w \in [[\mathcal{R}]]$, then $\|v - w\|_2$ refers to the 2-norm of $\mathrm{L}(\mathcal{R})$.

2. A local characterization of stable equivalence relations

In this section we establish the following more precise version of Theorem A. The proof is inspired by [Con76, Theorem 2.1] and [Con85, Theorem 2].

Theorem 2.1. *Let $\mathcal{R}$ be an ergodic p.m.p. equivalence relation on a probability space (X, μ). Then the following statements are equivalent.*

(i) *$\mathcal{R}$ is stable.*

(ii) *For every finite set $K \subset [[\mathcal{R}]]$ and every $\varepsilon > 0$, there exists $v \in [[\mathcal{R}]]$ such that*

$$\begin{gathered} v^2 = 0, \\ vv^* + v^*v = 1, \\ \forall u \in K, \quad \|vu - uv\|_2 < \varepsilon. \end{gathered}$$

(iii) *For every finite set $K \subset [[\mathcal{R}]]$ and every $\varepsilon > 0$, there exists $v \in [[\mathcal{R}]]$ such that*

$$\begin{gathered} v^2 = 0, \\ \forall u \in K, \quad \|vu - uv\|_2 < \varepsilon \|v\|_2. \end{gathered}$$

(iv) *For every finite set $K \subset [[\mathcal{R}]]$ and every $\varepsilon > 0$, there exists $v \in [[\mathcal{R}]]$ such that*

$$\forall u \in K, \quad \|vu - uv\|_2 < \varepsilon \|vv^* - v^*v\|_2.$$

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Proof. The equivalence (i) $\Leftrightarrow$ (ii) is proved in [JS87]. The implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are clear.

First, we show that (iv) $\Rightarrow$ (iii). Let $K = K^* \subset [[\mathcal{R}]]$ be a finite symmetric set and $\varepsilon > 0$. Choose $v \in [[\mathcal{R}]]$ such that

$$\forall u \in K, \quad \|vu - uv\|_2 < \varepsilon \|vv^* - v^*v\|_2.$$

Let $w = v(1 - vv^*) \in [[\mathcal{R}]]$. Then $w^2 = 0$. Moreover, a simple computation shows that

$$\forall u \in K, \quad \|wu - uw\|_2 < 3\varepsilon \|vv^* - v^*v\|_2$$

and we have $\|vv^* - v^*v\|_2 = \|ww^* - w^*w\|_2 = \sqrt{2}\|w\|_2$. Hence we obtain

$$\forall u \in K, \quad \|wu - uw\|_2 < 3\sqrt{2}\varepsilon \|w\|_2.$$

Next, we prove that if $\mathcal{R}$ satisfies condition (iii) then every corner of $\mathcal{R}$ also satisfies it. Let $Y \subset X$ be a non-zero subset and $p = 1_Y$. Suppose that the corner $\mathcal{R}_Y$ does not satisfy (iii). Then we can find a finite set $K \subset p[[\mathcal{R}]]p$ and a constant $\kappa > 0$ such that, for all $v \in p[[\mathcal{R}]]p$ with $v^2 = 0$, we have

$$\|v\|_2^2 \leqslant \kappa \sum_{u \in K} \|vu - uv\|_2^2.$$

Since $\mathcal{R}$ is ergodic, we can find a finite set $S \subset [[\mathcal{R}]]$ such that $\sum_{w \in S} w^*w = p^{\perp}$ and $ww^* \leqslant p$ for all $w \in S$. Then, for every $v \in [[\mathcal{R}]]$, we have

$$\|v\|_2^2 = \|pv\|_2^2 + \sum_{w \in S} \|wv\|_2^2$$

and, for all $w \in S$, we have

$$\|wv\|_2^2 \leqslant 2(\|wv - vw\|_2^2 + \|vw\|_2^2) \leqslant 2(\|wv - vw\|_2^2 + \|vp\|_2^2).$$

Hence, we obtain

$$\|v\|_2^2 \leqslant 2|S|(\|vp\|_2^2 + \|pv\|_2^2) + 2\sum_{w \in S} \|wv - vw\|_2^2.$$

Moreover, we have

$$\|pv\|_2^2 + \|vp\|_2^2 = \|pv - vp\|_2^2 + 2\|pvp\|_2^2,$$

hence

$$\|v\|_2^2 \leqslant 2|S|\|pv - vp\|_2^2 + 4|S|\|pvp\|_2^2 + 2\sum_{w \in S} \|wv - vw\|_2^2.$$

Now fix $v \in [[\mathcal{R}]]$ such that $v^2 = 0$. Since $(pvp)^2 = 0$, we know, by assumption, that

$$\|pvp\|_2^2 \leqslant \kappa \sum_{u \in K} \|(pvp)u - u(pvp)\|_2^2 \leqslant \kappa \sum_{u \in K} \|vu - uv\|_2^2.$$

Therefore, we finally obtain

$$\|v\|_2^2 \leqslant 2|S|\|pv - vp\|_2^2 + 4|S|\kappa \sum_{u \in K} \|vu - uv\|_2^2 + 2\sum_{w \in S} \|wv - vw\|_2^2.$$

This shows that $\mathcal{R}$ does not satisfy (iii).

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Finally, we use a maximality argument to show that (iii) $\Rightarrow$ (ii). Let $u_1, \ldots, u_n \in [[\mathcal{R}]]$ be a finite family and let $\varepsilon > 0$ and $\delta = 8\varepsilon$. Consider the set Λ of all $(v, U_1, \ldots, U_n) \in [[\mathcal{R}]]^{n+1}$ such that

- $v^2 = 0$,
- $[U_k, vv^* + v^*v] = 0$ for all $k = 1, \ldots, n$,
- $\|vU_k - U_kv\|_2 \leqslant \varepsilon\|v\|_2$ for all $k = 1, \ldots, n$,
- $\|U_k - u_k\|_1 \leqslant \delta\|v\|_1$.

On Λ put the order relation given by

$$(v, U_1, \ldots, U_n) \leqslant (v', U_1', \ldots, U_n')$$

if and only if $v \leqslant v'$ and $\|U_k' - U_k\|_1 \leqslant \delta(\|v'\|_1 - \|v\|_1)$ for all $k = 1, \ldots, n$. Then Λ is an inductive set (because $[[\mathcal{R}]]$ is inductive and is also complete for the distance given by $\|\cdot\|_1$). By Zorn's lemma, let $v \in \Lambda$ be a maximal element. Suppose that $q = vv^* + v^*v \neq 1$. Since, by the previous step, all corners of $\mathcal{R}$ also satisfy (iii), we can apply it to $K = \{U_kq^\perp \mid k = 1, \ldots, n\} \subset q^\perp[[\mathcal{R}]]q^\perp$ in order to find a non-zero element $w \in q^\perp[[\mathcal{R}]]q^\perp$, with $w^2 = 0$ such that

$$\|wU_k - U_kw\|_2 \leqslant \varepsilon\|w\|_2,$$
$$\|wU_k^* - U_k^*w\|_2 \leqslant \varepsilon\|w\|_2,$$

for all $k = 1, \ldots, n$.

Now let

- $p := ww^* + w^*w$,
- $U_k' := pU_kp + p^\perp U_kp^\perp$,
- $v' := v + w$,
- $q' := v'(v')^* + (v')^*v' = q + p$.

Note that $(v')^2 = 0$ and $[U_k', q'] = 0$ for all k. We also have

$$\|v'U_k' - U_k'v'\|_2^2 \leqslant \|vU_k - U_kv\|_2^2 + \|wU_k - U_kw\|_2^2 \leqslant \varepsilon^2\|v\|_2^2 + \varepsilon^2\|w\|_2^2 = \varepsilon^2\|v'\|_2^2.$$

Moreover, by the Cauchy–Schwarz inequality, we have

$$\begin{aligned}\|U_k' - U_k\|_1 &\leqslant \|pU_kp^\perp\|_1 + \|p^\perp U_kp\|_1 \\ &\leqslant \|p\|_2(\|pU_kp^\perp\|_2 + \|p^\perp U_kp\|_2) \\ &\leqslant \sqrt{2}\|p\|_2\|[U_k, p]\|_2 \\ &\leqslant 2\sqrt{2}\|p\|_2(\|[U_k, w]\|_2 + \|[U_k, w^*]\|_2) \\ &\leqslant 4\sqrt{2}\varepsilon\|p\|_2\|w\|_2 \\ &= 8\varepsilon\|w\|_2^2 \\ &= \delta\|w\|_1.\end{aligned}$$

Since $\|v'\|_1 = \|v\|_1 + \|w\|_1$, this implies that

$$\|U_k' - U_k\|_1 \leqslant \delta(\|v'\|_1 - \|v\|_1)$$

and

$$\|U_k' - u_k\|_1 \leqslant \|U_k' - U_k\|_1 + \|U_k - u_k\|_1 \leqslant \delta\|v'\|_1.$$

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Therefore $v' \in \Lambda$ and $v \leqslant v'$. This contradicts the maximality of v. Hence we must have $v^*v + vv^* = q = 1$. Moreover, since

$$\|vu_k - u_kv\|_2 \leqslant \|vU_k - U_kv\|_2 + 2\|U_k - u_k\|_2,$$
$$\|vU_k - U_kv\|_2 \leqslant \varepsilon$$

and

$$\|U_k - u_k\|_2^2 \leqslant 2\|U_k - u_k\|_1 \leqslant 2\delta = 16\varepsilon,$$

we conclude that

$$\|vu_k - u_kv\|_2 \leqslant \varepsilon + 8\sqrt{\varepsilon}.$$

Since such a v exists for every $\varepsilon > 0$, we have proved (ii). □

3. Proof of Theorem B

In this section we prove Theorem B. We need to introduce some notation which will be useful in order to decompose elements of the full pseudo-group $[[\mathcal{R} \times \mathcal{S}]]$ as functions from $\mathcal{R}$ to $[[\mathcal{S}]]$.

Let $\mathcal{R}$ be a p.m.p. equivalence relation on a probability space (X, μ). We denote by $\tilde{\mu}$ the canonical σ-finite measure on $\mathcal{R}$ induced by μ. Then $\mathrm{L}^2(\mathcal{R}) := \mathrm{L}^2(\mathcal{R}, \tilde{\mu})$ can be identified with the canonical L^2-space of $\mathrm{L}(\mathcal{R})$. For every $x \in \mathrm{L}(\mathcal{R})$, we denote by $\widehat{x}$ the corresponding vector in $\mathrm{L}^2(\mathcal{R})$. If $v \in [[\mathcal{R}]]$, then $\widehat{v}$ is just the indicator function of the graph of v. We denote by $\mathfrak{P}(X)$ the set of projections of $\mathrm{L}^\infty(X, \mu)$. For every $p \in \mathfrak{P}(X)$, we can view $\widehat{p}$ as an indicator function in $\mathrm{L}^2(X)$, where $\mathrm{L}^2(X)$ is embedded into $\mathrm{L}^2(\mathcal{R})$ via the diagonal inclusion.

If $\mathcal{S}$ is a second p.m.p. equivalence relation on (Y, ν), then for any $v \in [[\mathcal{R} \times \mathcal{S}]]$, there exists a unique function $v_{\mathcal{S}} \in \mathrm{L}^0(\mathcal{R}, [[\mathcal{S}]])$ which satisfies

$$\widehat{v}(x, x', y, y') = \widehat{v_{\mathcal{S}}(x, x')}(y, y')$$

for almost every (a.e.) $(x, x', y, y') \in \mathcal{R} \times \mathcal{S}$.

If $p \in \mathfrak{P}(X \times Y)$, then there exists a unique function $p_Y \in \mathrm{L}^0(X, \mathfrak{P}(Y))$ such that

$$\widehat{p}(x, y) = \widehat{p_Y(x)}(y)$$

for a.e. $(x, y) \in X \times Y$.

All this heavy notation is needed for the following key lemma which allows us to decompose a commutator in $[[\mathcal{R} \times \mathcal{S}]]$ into two parts which we will be able to control independently. The proof is just an easy computation.

Lemma 3.1. *Let $\mathcal{R}$ and $\mathcal{S}$ be two p.m.p. equivalence relations on (X, μ) and (Y, ν), respectively. Let $\mathcal{R} \times \mathcal{S}$ be the product p.m.p. equivalence relation on $(X \times Y, \mu \otimes \nu)$. Let $v \in [[\mathcal{R} \times \mathcal{S}]]$ and $p \in \mathfrak{P}(X \times Y)$. Let $v_1 := v_{\mathcal{R}} \in \mathrm{L}^0(\mathcal{S}, [[\mathcal{R}]])$ and $v_2 := v_{\mathcal{S}} \in \mathrm{L}^0(\mathcal{R}, [[\mathcal{S}]])$ be the two functions defined by v. Let $p_1 := p_X \in \mathrm{L}^0(Y, \mathfrak{P}(X))$ and $p_2 := p_Y \in \mathrm{L}^0(X, \mathfrak{P}(Y))$ be the two functions defined by p.*

Define $\xi_1 \in \mathrm{L}^2(\mathcal{S}, \mathrm{L}^2(\mathcal{R}))$ by

$$\xi_1(y, y') = [v_1(\widehat{y, y'), p_1}(y)] \quad \text{for a.e. } (y, y') \in \mathcal{S},$$

and $\xi_2 \in \mathrm{L}^2(\mathcal{R}, \mathrm{L}^2(\mathcal{S}))$ by

$$\xi_2(x, x') = [v_2(\widehat{x, x'), p_2}(x')] \quad \text{for a.e. } (x, x') \in \mathcal{R}.$$

Then, after identifying $\mathrm{L}^2(\mathcal{S}, \mathrm{L}^2(\mathcal{R})) \cong \mathrm{L}^2(\mathcal{R}, \mathrm{L}^2(\mathcal{S})) \cong \mathrm{L}^2(\mathcal{R} \times \mathcal{S})$, we have $\widehat{[v, p]} = \xi_1 + \xi_2$.

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Proof. For a.e. $(x,x',y,y') \in \mathcal{R}\times\mathcal{S}$, we compute

$$\begin{aligned}(\xi_1(y,y'))(x,x') &= \widehat{v}(x,x',y,y')(\widehat{p}(x,y)-\widehat{p}(x',y)),\\ (\xi_2(x,x'))(y,y') &= \widehat{v}(x,x',y,y')(\widehat{p}(x',y)-\widehat{p}(x',y')),\\ \widehat{[v,p]}(x,x',y,y') &= \widehat{v}(x,x',y,y')(\widehat{p}(x,y)-\widehat{p}(x',y')),\end{aligned}$$

hence the required equality. □

Proof of Theorem B. Clearly, if $\mathcal{R}$ or $\mathcal{S}$ is stable then $\mathcal{R}\times\mathcal{S}$ is also stable. Now suppose that $\mathcal{R}$ and $\mathcal{S}$ are not stable. Then, by Theorem 2.1, we can find a constant $\kappa_1>0$ and a finite set $K_1\subset[[\mathcal{R}]]$ such that, for all $v\in[[\mathcal{R}]]$, we have

$$\|vv^*-v^*v\|_2^2 \leqslant \kappa_1\sum_{u\in K_1}\|vu-uv\|_2^2.$$

Similarly, we can find a constant $\kappa_2>0$ and a finite set $K_2\subset[[\mathcal{S}]]$ such that, for all $v\in[[\mathcal{S}]]$, we have

$$\|vv^*-v^*v\|_2^2 \leqslant \kappa_2\sum_{u\in K_2}\|vu-uv\|_2^2.$$

In order to prove that $\mathcal{R}\times\mathcal{S}$ is not stable, we will show that, for all $v\in[[\mathcal{R}\times\mathcal{S}]]$ with $v^2=0$, we have

$$\|v\|_2^2 \leqslant \kappa\sum_{u\in K}\|vu-uv\|_2^2,$$

where $\kappa=2(\kappa_1+\kappa_2)$ and $K=(K_1\otimes 1)\cup(1\otimes K_2)$.

Indeed, let $v\in[[\mathcal{R}\times\mathcal{S}]]$ with $v^2=0$ and let $p=v^*v$. Using the notation of Lemma 3.1, we can write $\widehat{v}=\widehat{[v,p]}=\xi_1+\xi_2$ and we have the formulas

$$\begin{aligned}\|\xi_1\|_2^2 &= \int_{\mathcal{S}}\|v_1(y,y')p_1(y)-p_1(y)v_1(y,y')\|_2^2\,d\nu_\ell(y,y'),\\ \|\xi_2\|_2^2 &= \int_{\mathcal{R}}\|v_2(x,x')p_2(x')-p_2(x')v_2(x,x')\|_2^2\,d\mu_\ell(x,x').\end{aligned}$$

Since $pv=0$, then for a.e. $(y,y')\in\mathcal{S}$ we have that $p_1(y)v_1(y,y')=0$, hence

$$v_1(y,y')p_1(y)-p_1(y)v_1(y,y')=v_1(y,y')(v_1(y,y')^*v_1(y,y')-v_1(y,y')v_1(y,y')^*)p_1(y).$$

This shows that

$$\begin{aligned}\|v_1(y,y')p_1(y)-p_1(y)v_1(y,y')\|_2^2 &\leqslant \|v_1(y,y')^*v_1(y,y')-v_1(y,y')v_1(y,y')^*\|_2^2\\ &\leqslant \kappa_1\sum_{u\in K_1}\|v_1(y,y')u-uv_1(y,y')\|_2^2.\end{aligned}$$

After integrating over $\mathcal{S}$ and using the formula

$$\forall u\in K_1,\quad \|v(u\otimes 1)-(u\otimes 1)v\|_2^2=\int_{\mathcal{S}}\|v_1(y,y')u-uv_1(y,y')\|_2^2\,d\nu_\ell(y,y'),$$

we obtain

$$\|\xi_1\|_2^2\leqslant\kappa_1\sum_{u\in K_1}\|v(u\otimes 1)-(u\otimes 1)v\|_2^2.$$

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Similarly, since $vp = v$, we have $v_2(x,x')p_2(x') = v_2(x,x')$ for a.e $(x,x') \in \mathcal{R}$, hence

$$v_2(x,x')p_2(x') - p_2(x')v_2(x,x') = p_2(x')^{\perp}(v_2(x,x')v_2(x,x')^* - v_2(x,x')^*v_2(x,x'))v_2(x,x').$$

Then, proceeding as before, we show that

$$\|\xi_2\|_2^2 \leqslant \kappa_2 \sum_{u \in K_2} \|v(1 \otimes u) - (1 \otimes u)v\|_2^2.$$

Finally, since $\widehat{v} = \widehat{[v,p]} = \xi_1 + \xi_2$, we conclude that

$$\|v\|_2^2 \leqslant 2(\|\xi_1\|_2^2 + \|\xi_2\|_2^2) \leqslant \kappa \sum_{u \in K} \|vu - uv\|_2^2$$

as required. □

4. A local characterization of McDuff factors

In this section, we establish Theorem C. The proof is more involved than the proof of Theorem A. We will need the following lemma (for a proof see [Con76, Lemma 1.2.6], [CS76, Proposition 1] and [CS76, Theorem 2]).

Lemma 4.1. *Let (M,τ) be a tracial von Neumann algebra. For every $x \in M$ and every $t \geqslant 0$, let*

$$u_t(x) = u1_{[t,+\infty)}(|x|),$$

where $x = u|x|$ is the polar decomposition of x.

(i) *For all $x \in M$, we have*

$$\int_0^\infty \|u_{t^{1/2}}(x)\|_2^2 \, dt = \|x\|_2^2.$$

(ii) *For all $x, y \in M^+$, we have*

$$\|x - y\|_2^2 \leqslant \int_0^\infty \|u_{t^{1/2}}(x) - u_{t^{1/2}}(y)\|_2^2 \, dt.$$

(iii) *For all $x \in M$ and all $a \in M^+$, we have*

$$\int_0^\infty \|u_{t^{1/2}}(x)a - au_{t^{1/2}}(x)\|_2^2 \, dt \leqslant 4\|xa - ax\|_2\|xa + ax\|_2.$$

Now we can prove the following more precise version of Theorem C. Note that even if one is only interested in item (iii), one still needs first to prove that it is equivalent to (iv)′.

Theorem 4.2. *Let M be a factor of type II_1 with separable predual. Then the following statements are equivalent.*

(i) *M is McDuff.*

(ii) *For every finite set $F \subset M$ and every $\varepsilon > 0$, there exists a partial isometry $v \in M$ such that*

$$vv^* + v^*v = 1,$$
$$\forall a \in F, \quad \|va - av\|_2 < \varepsilon.$$

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(iii) *For every finite set $F \subset M$ and every $\varepsilon > 0$, there exists a partial isometry $v \in M$ such that*

$$v^2 = 0,$$
$$\forall a \in F, \quad \|va - av\|_2 < \varepsilon \|v\|_2.$$

(iii)′ *For every finite set $F \subset M$ and every $\varepsilon > 0$, there exists $x \in M$ such that*

$$x^2 = 0,$$
$$\forall a \in F, \quad \|xa - ax\|_2 < \varepsilon \|x\|_2.$$

(iv) *For every finite set $F \subset M$ and every $\varepsilon > 0$, there exists a partial isometry $v \in M$ such that*

$$\forall a \in F, \quad \|va - av\|_2 < \varepsilon \|vv^* - v^*v\|_2.$$

(iv)′ *For every finite set $F \subset M$ and every $\varepsilon > 0$, there exists $x \in M$ such that*

$$\forall a \in F, \quad \|x\|_2 \cdot \|xa - ax\|_2 < \varepsilon \||x| - |x^*|\|_2^2.$$

Proof. The equivalence (i) $\Leftrightarrow$ (ii) is already known [McD70]. First, we show that (iii) $\Leftrightarrow$ (iii)′ $\Leftrightarrow$ (iv) $\Leftrightarrow$ (iv)′. For this, we will prove the implications (iii) $\Rightarrow$ (iv)′ $\Rightarrow$ (iv) $\Rightarrow$ (iii)′ $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (iv)′. If v satisfies (iii) then $x := v$ also satisfies (iv)′ since $\||x| - |x^*|\|_2 = \sqrt{2}\|x\|_2$.

(iv)′ $\Rightarrow$ (iv). Suppose, by contradiction, that there exist a finite set $F \subset M$ and a constant $\kappa > 0$ such that, for all partial isometries $v \in M$, we have

$$\|vv^* - v^*v\|_2^2 \leqslant \kappa \sum_{a \in F} \|va - av\|_2^2.$$

We can assume that $F \subset M^+$. Let $x \in M$. Then the above inequality applied to $v := u_t(x)$ yields

$$\|u_t(|x^*|) - u_t(|x|)\|_2^2 \leqslant \kappa \sum_{a \in F} \|u_t(x)a - au_t(x)\|_2^2$$

for all $t \geqslant 0$. Therefore, by Lemma 4.1, we obtain

$$\begin{aligned} \||x^*| - |x|\|_2^2 &\leqslant \int_0^\infty \|u_{t^{1/2}}(|x^*|) - u_{t^{1/2}}(|x|)\|_2^2 \, dt \\ &\leqslant \kappa \sum_{a \in F} \int_0^\infty \|u_{t^{1/2}}(x)a - au_{t^{1/2}}(x)\|_2^2 \, dt. \\ &\leqslant \kappa \sum_{a \in F} 4\|xa - ax\|_2 \|xa + ax\|_2 \\ &\leqslant 8\kappa \Big(\max_{a \in F} \|a\|_\infty\Big) \|x\|_2 \sum_{a \in F} \|xa - ax\|_2 \end{aligned}$$

and this contradicts (iv)′.

(iv) $\Rightarrow$ (iii)′. Let $F = F^* \subset M$ be a finite self-adjoint set and $\varepsilon > 0$. Pick $v \in M$, a partial isometry such that

$$\forall a \in F, \quad \|va - av\|_2 < \varepsilon \|vv^* - v^*v\|_2.$$

Let $x_1 = (1 - v^*v)v$ and $x_2 = v(1 - vv^*)$. Note that $x_1^2 = x_2^2 = 0$. Let $x := x_1$ if $\|x_1\| \geqslant \|x_2\|$ and $x := x_2$ otherwise. Then

$$\|vv^* - v^*v\|_2^2 = \|x_1\|_2^2 + \|x_2\|_2^2 \leqslant 2\|x\|_2^2.$$

Moreover,

$$\forall a \in F, \quad \|xa - ax\|_2 \leqslant 2\|va - av\|_2 + \|v^*a - av^*\|_2 = 2\|va - av\|_2 + \|a^*v - va^*\|_2.$$

Therefore, since F is self-adjoint, we obtain

$$\forall a \in F, \quad \|xa - ax\|_2 < 3\varepsilon \|vv^* - v^*v\|_2 \leqslant 3\sqrt{2}\varepsilon \|x\|_2.$$

(iii)$'$ $\Rightarrow$ (iii). Suppose, by contradiction, that there exist a finite set $F \subset M$ and a constant $\kappa > 0$ such that, for all partial isometries $v \in M$ with $v^2 = 0$, we have

$$\|v\|_2^2 \leqslant \kappa \sum_{a \in F} \|va - av\|_2^2.$$

We can assume that $F \subset M^+$. Let $x \in M$ such that $x^2 = 0$. Then, for every $t > 0$, we have $u_t(x)^2 = 0$. Hence, by Lemma 4.1, we have

$$\|x\|_2^2 = \int_0^\infty \|u_{t^{1/2}}(x)\|_2^2 \, dt \leqslant \kappa \sum_{a \in F} \int_0^\infty \|u_{t^{1/2}}(x)a - au_{t^{1/2}}(x)\|_2^2 \, dt.$$

Since, for every $a \in F$, we have

$$\int_0^\infty \|u_{t^{1/2}}(x)a - au_{t^{1/2}}(x)\|_2^2 \, dt \leqslant 4\|xa - ax\|_2 \|xa + ax\|_2 \leqslant 8\|a\|_\infty \|x\|_2 \|xa - ax\|_2,$$

we obtain

$$\|x\|_2 \leqslant 8\Big(\max_{a \in F} \|a\|_\infty\Big) \kappa \sum_{a \in F} \|xa - ax\|_2$$

and this contradicts (iii)$'$.

This finishes the proof of the equivalences (iii) $\Leftrightarrow$ (iii)$'$ $\Leftrightarrow$ (iv) $\Leftrightarrow$ (iv)$'$. Next, we will prove that if M satisfies (iii) then pMp also satisfies (iii) for every non-zero projection $p \in M$. Suppose, by contradiction, pMp does not satisfy (iii). Then pMp does not satisfy (iv)$'$. Hence we can find a constant $\kappa > 0$ and a finite set $F \subset pMp$ such that

$$\forall x \in pMp, \quad \||x| - |x^*|\|_2^2 \leqslant \kappa \|x\|_2 \sum_{a \in F} \|ax - xa\|_2.$$

Take $S \subset M$ a finite set of partial isometries such that $\sum_{w \in S} w^*w = p^\perp$ and $ww^* \leqslant p$ for all $w \in S$. Now take a partial isometry $v \in M$ with $v^2 = 0$ and let $x := pvp$. Then we have

$$\|v\|_2^2 = \|pv\|_2^2 + \sum_{w \in S} \|wv\|_2^2$$

and, for all $w \in S$,

$$\|wv\|_2^2 \leqslant 2(\|wv - vw\|_2^2 + \|vw\|_2^2) \leqslant 2(\|wv - vw\|_2^2 + \|vp\|_2^2).$$

Hence, we obtain

$$\|v\|_2^2 \leqslant 2|S|(\|vp\|_2^2 + \|pv\|_2^2) + 2\sum_{w \in S} \|wv - vw\|_2^2.$$

Moreover, we have

$$\|pv\|_2^2 + \|vp\|_2^2 = 2\|x\|_2^2 + \|pv - vp\|_2^2,$$

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hence
$$\|v\|_2^2 \leqslant 4|S|\|x\|_2^2 + 2|S|\|pv - vp\|_2^2 + 2\sum_{w\in S}\|wv - vw\|_2^2.$$
Now, by assumption, we have
$$\||x| - |x^*|\|_2^2 \leqslant \kappa\|x\|_2 \sum_{a\in F}\|ax - xa\|_2.$$
Moreover, we have
$$\||x| - |pv|\|_2 \leqslant \|x - pv\|_2 \leqslant \|vp - pv\|_2$$
and
$$\||x^*| - |vp|\|_2 \leqslant \|x^* - vp\|_2 \leqslant \|vp - pv\|_2.$$
Hence, by using the fact that $v^2 = 0$, we get
$$\|x\|_2 \leqslant \|pv\|_2 \leqslant \||pv| - |vp|\|_2 \leqslant \||x| - |x^*|\|_2 + 2\|vp - pv\|_2$$
which implies that
$$\|x\|_2^2 \leqslant 2\kappa\|x\|_2 \sum_{a\in F}\|ax - xa\|_2 + 8\|pv - vp\|_2^2.$$
Therefore, we obtain
$$\|v\|_2^2 \leqslant 8|S|\kappa\|x\|_2 \sum_{a\in F}\|ax - xa\|_2 + 34|S|\|pv - vp\|_2^2 + 2\sum_{w\in S}\|wv - vw\|_2^2.$$
Finally, using the fact that
$$\|x\|_2 \sum_{a\in F}\|ax - xa\|_2 \leqslant \|v\|_2 \sum_{a\in F}\|av - va\|_2,$$
$$\|pv - vp\|_2^2 \leqslant 2\|v\|_2\|pv - vp\|_2$$
and
$$\|wv - vw\|_2^2 \leqslant 2\|v\|_2\|wv - vw\|_2,$$
we can conclude that
$$\|v\|_2 \leqslant \kappa' \sum_{a\in F'}\|av - va\|_2,$$
for some $\kappa' > 0$, some finite set $F' \subset M$ and all partial isometries $v \in M$ with $v^2 = 0$. This shows that M does not satisfy (iii), as required.

Finally, one can prove (iii) $\Rightarrow$ (ii) by using exactly the same maximality argument that we used in the proof of Theorem 2.1. □

5. Another approach to Question D

The following lemma is extracted from [IV15] and is inspired by a trick used in [Haa85]. Recall that if M is a von Neumann algebra, then $\mathrm{L}^2(M^\omega)$ is in general much smaller than the ultraproduct Hilbert space $\mathrm{L}^2(M)^\omega$ (see [Con76, Proposition 1.3.1]).

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Lemma 5.1. *Let M and N be finite von Neumann algebras. Fix a tracial state τ on M and pick an orthonormal basis $(e_n)_{n\in\mathbf{N}}$ of (M,τ). Let $A=\mathrm{L}^\infty(\mathbb{T}^{\mathbf{N}})=\mathrm{L}^\infty(\mathbb{T})^{\overline{\otimes}\,\mathbf{N}}$ and, for each $n\in\mathbf{N}$, let $u_n\in\mathcal{U}(A)$ be the canonical generator of the nth copy of $\mathrm{L}^\infty(\mathbb{T})$. Let $V:\mathrm{L}^2(M)\to\mathrm{L}^2(A)$ be the unique (non-surjective) isometry which sends e_n to u_n for every $n\in\mathbf{N}$.*

Then the naturally defined ultraproduct isometry

$$(V\otimes 1)^\omega:\mathrm{L}^2(M\mathbin{\overline{\otimes}}N)^\omega\to\mathrm{L}^2(A\mathbin{\overline{\otimes}}N)^\omega$$

sends $\mathrm{L}^2((M\mathbin{\overline{\otimes}}N)^\omega)$ into $\mathrm{L}^2((A\mathbin{\overline{\otimes}}N)^\omega)$.

Lemma 5.1 is useful because it allows us to reduce many problems on sequences in tensor products $M\mathbin{\overline{\otimes}}N$ to the case where M is abelian. We now present two applications of this principle.

The first one slightly generalizes [IV15, Corollary]. We will need it for Theorem E.

Proposition 5.2. *Let M and N be finite von Neumann algebras. For any von Neumann subalgebras $Q,P\subset N$ such that $Q'\cap N^\omega\subset P^\omega$, we have*

$$(1\otimes Q)'\cap(M\mathbin{\overline{\otimes}}N)^\omega\subset(M\mathbin{\overline{\otimes}}P)^\omega.$$

Proof. First, we deal with the case where M is abelian, that is, $M=\mathrm{L}^\infty(T,\mu)$ for some probability space (T,μ). Take $(x_n)^\omega$ in the unit ball of $(1\mathbin{\overline{\otimes}}Q)'\cap(M\mathbin{\overline{\otimes}}N)^\omega$ and write $x_n=(t\mapsto x_n(t))\in M\mathbin{\overline{\otimes}}N=\mathrm{L}^\infty(T,\mu,N)$ for every $n\in\mathbf{N}$. Let $\varepsilon>0$ and choose a finite set $F\subset Q$ and $\delta>0$ such that, for every x in the unit ball of N, we have

$$(\forall a\in F,\|[x,a]\|_2\leqslant\delta)\Longrightarrow\|x-E_P(x)\|_2\leqslant\varepsilon.$$

Since $(x_n)^\omega\in(1\mathbin{\overline{\otimes}}Q)'\cap(M\mathbin{\overline{\otimes}}N)^\omega$, we have

$$\lim_{n\to\omega}\mu(\{t\in T\mid\forall a\in F,\|[x_n(t),a]\|_2\leqslant\delta\})=1.$$

Hence, we have

$$\lim_{n\to\omega}\mu(\{t\in T\mid\|x_n(t)-E_P(x_n(t))\|_2\leqslant\varepsilon\})=1.$$

This means that

$$\lim_{n\to\omega}\|x_n-E_{M\mathbin{\overline{\otimes}}P}(x_n)\|_2\leqslant\varepsilon,$$

and since this holds for every $\varepsilon>0$, we conclude that $(x_n)^\omega\in(M\mathbin{\overline{\otimes}}P)^\omega$.

We now extend to the general case where M is not necessarily abelian. Let $\xi\in\mathrm{L}^2((M\mathbin{\overline{\otimes}}N)^\omega)$ be a Q-central vector. We want to show that $\xi\in\mathrm{L}^2((M\mathbin{\overline{\otimes}}P)^\omega)$. By Lemma 5.1, we know that $\eta=(V\otimes 1)^\omega(\xi)\in\mathrm{L}^2((A\mathbin{\overline{\otimes}}N)^\omega)$. Since $(V\otimes 1)^\omega$ is N-bimodular, we know that η is Q-central. Hence, by the abelian case, we obtain that $\eta\in\mathrm{L}^2((A\mathbin{\overline{\otimes}}P)^\omega)$. But this clearly implies that $\xi\in\mathrm{L}^2((M\mathbin{\overline{\otimes}}P)^\omega)$. □

Proof of Theorem E. Suppose that $M\mathbin{\overline{\otimes}}N$ is McDuff, that is, $(M\mathbin{\overline{\otimes}}N)_\omega$ is non-commutative. By Proposition 5.2, we know that $(M\mathbin{\overline{\otimes}}N)_\omega\subset(A\mathbin{\overline{\otimes}}N)_\omega$, so that $(A\mathbin{\overline{\otimes}}N)_\omega$ is also non-commutative. Therefore, we can find $x=(x_n)^\omega$ and $y=(y_n)^\omega$ in $(A\mathbin{\overline{\otimes}}N)_\omega$ with $\|x_n\|_\infty$, $\|y_n\|_\infty\leqslant 1$ for all n, such that $\|[x,y]\|_2=\delta>0$. Let $A=\mathrm{L}^\infty(T,\mu)$ with (T,μ) a probability space. Write $x_n=(t\mapsto x_n(t))\in A\mathbin{\overline{\otimes}}N=\mathrm{L}^\infty(T,\mu,N)$ with $\|x_n(t)\|_\infty\leqslant 1$ for all n and t. Similarly, let $y_n=(t\mapsto y_n(t))$. Fix $F\subset N$ a finite subset and $\varepsilon>0$. Since $x,y\in(A\mathbin{\overline{\otimes}}N)_\omega$, we know that

$$\lim_{n\to\omega}\mu(\{t\in T\mid\forall a\in F,\|[x_n(t),a]\|_2\leqslant\varepsilon\})=1$$

and

$$\lim_{n\to\omega} \mu(\{t \in T \mid \forall a \in F, \|[y_n(t), a]\|_2 \leqslant \varepsilon\}) = 1.$$

Moreover, since $\|[x, y]\|_2 = \delta > 0$, we have

$$\lim_{n\to\omega} \mu(\{t \in T \mid \|[x_n(t), y_n(t)]\|_2 \geqslant \delta/2\}) > 0.$$

Hence, for n large enough, the intersection of these three sets is non-empty, that is, there exists t such that

$$\begin{aligned} &\forall a \in F, \quad \|[x_n(t), a]\|_2 \leqslant \varepsilon, \\ &\forall a \in F, \quad \|[y_n(t), a]\|_2 \leqslant \varepsilon, \\ &\|[x_n(t), y_n(t)]\|_2 \geqslant \delta/2. \end{aligned}$$

Hence, by iterating this procedure, we can extract a sequence $a_k = x_{n_k}(t_k)$, $k \in \mathbf{N}$, and $b_k = y_{n_k}(t_k)$, $k \in \mathbf{N}$, such that $a = (a_k)^\omega$ and $b = (b_k)^\omega$ are in N_ω and $\|[a, b]\|_2 \geqslant \delta/2$. Thus N_ω is not commutative, that is, N is McDuff as we wanted. □

The second application is the following lemma which we will need in the proof of Theorem F.

Lemma 5.3. *Let M and N be finite von Neumann algebras. Then we have*

$$1 \otimes \mathcal{Z}(N_\omega) \subset \mathcal{Z}(N' \cap (M \mathbin{\overline{\otimes}} N)^\omega).$$

Proof. First, we treat the case where M is abelian, that is, $M = \mathrm{L}^\infty(T, \mu)$ for some probability space (T, μ). Let $(a_k)_{k\in\mathbf{N}}$ be a $\|\cdot\|_2$-dense sequence in $(N)_1$ and let

$$N_k := \{x \in (N)_1 \mid \forall r \leqslant k, \|[x, a_r]\|_2 \leqslant 1/k\}.$$

Let $y = (y_n)^\omega \in \mathcal{Z}(N_\omega)$ with $\|y_n\|_\infty \leqslant 1$ for all n. By [McD70, Lemma 10], there exists a sequence of sets $U_k \in \omega$, $k \in \mathbf{N}$, such that

$$\forall k \in \mathbf{N}, \forall x \in N_k, \forall n \in U_k, \quad \|[y_n, x]\|_2 \leqslant 1/k.$$

Let $x = (x_n)^\omega \in (1 \otimes N)' \cap (M \mathbin{\overline{\otimes}} N)^\omega$ with $\|x_n\|_\infty \leqslant 1$ for all $n \in \mathbf{N}$. We want to show that $x(1 \otimes y) = (1 \otimes y)x$. Write $x_n = (t \mapsto x_n(t)) \in M \mathbin{\overline{\otimes}} N = \mathrm{L}^\infty(T, \mu, N)$ with $\|x_n(t)\|_\infty \leqslant 1$ for all t and all $n \in \mathbf{N}$. Since $x \in (1 \otimes N)' \cap (M \mathbin{\overline{\otimes}} N)^\omega$, there exists a sequence of sets $V_k \in \omega$ such that

$$\mu(\{t \in T \mid x_n(t) \in N_k\}) \geqslant 1 - 1/k^2$$

for all $n \in V_k$.

Therefore, for all $n \in U_k \cap V_k$, we have

$$\mu(\{t \in T \mid \|[y_n, x_n(t)]\|_2 \leqslant 1/k\}) \geqslant 1 - 1/k^2,$$

which implies that

$$\|[1 \otimes y_n, x_n]\|_2^2 = \int_T \|[y_n, x_n(t)]\|_2^2 \, d\mu(t) \leqslant 5/k^2.$$

Since $U_k \cap V_k \in \omega$ for all $k \in \mathbf{N}$, we conclude that $\lim_{n\to\omega} \|[1 \otimes y_n, x_n]\|_2 = 0$ as required.

Finally, we extend to the general case where M is not necessarily abelian. Let $\xi \in \mathrm{L}^2((M \mathbin{\overline{\otimes}} N)^\omega)$ be an N-central vector. We want to show that ξ is $\mathcal{Z}(N_\omega)$-central. By Lemma 5.1, we know that $\eta = (V \otimes 1)^\omega(\xi) \in \mathrm{L}^2((A \mathbin{\overline{\otimes}} N)^\omega)$. Since $(V \otimes 1)^\omega$ is N-bimodular, we know that η is N-central. Hence, by the abelian case, we obtain that η is $\mathcal{Z}(N_\omega)$-central. Since $(V \otimes 1)^\omega$ is N^ω-bimodular, we conclude that ξ is also $\mathcal{Z}(N_\omega)$-central. □

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Proof of Theorem F. By Lemma 5.3, we know that $1 \otimes \mathcal{Z}(N_\omega)$ is contained in the center of $(1 \otimes N)' \cap (M \mathbin{\overline{\otimes}} N)^\omega$, hence it is also contained in the center of $(M \mathbin{\overline{\otimes}} N)_\omega$. Since $(M \mathbin{\overline{\otimes}} N)_\omega$ is a factor, this implies that N_ω is also a factor, and the same argument shows that M_ω is a factor.

Now suppose that $M \mathbin{\overline{\otimes}} N$ is McDuff. Then $(M \mathbin{\overline{\otimes}} N)_\omega$ is non-trivial. Thus M_ω or N_ω is also non-trivial (use [Con76, Corollary 2.2] or Proposition 5.2). This means that M_ω or N_ω is a non-trivial factor. In particular, M or N is McDuff. □

The following fact is well known to experts, but we provide a proof for the reader's convenience.

Proposition 5.4. *Let $M = \overline{\bigotimes}_{n \in \mathbf{N}} M_n$ be an infinite tensor product of non-Gamma type II_1 factors M_n, $n \in \mathbf{N}$. Then M_ω is a factor.*

Proof. For every $n \in \mathbf{N}$, we let

$$Q_n := M_0 \mathbin{\overline{\otimes}} M_1 \mathbin{\overline{\otimes}} \cdots \mathbin{\overline{\otimes}} M_n \otimes 1 \otimes 1 \otimes \cdots \subset M.$$

Suppose that $(x_k)_{k \in \mathbf{N}}$ is a non-trivial central sequence in M with $\|x_k\|_2 = 1$ and $\tau(x_k) = 0$ for all $k \in \mathbf{N}$. Then, since Q_n is non-Gamma, we know by [Con76, Theorem 2.1] that $\lim_k \|x_k - E_{Q_n' \cap M}(x_k)\|_2 = 0$. Hence we can find a sequence $(n_k)_{k \in \mathbf{N}}$ with $n_k \to \infty$ such that $\|E_{Q_{n_k}' \cap M}(x_k)\|_2 \geqslant \frac{1}{2}$ for all $k \in \mathbf{N}$. Let $y_k = E_{Q_{n_k}' \cap M}(x_k)$. Since $Q_{n_k}' \cap M$ is a finite factor, we know that $0 = \tau(x_k) = \tau(y_k)$ is in the weakly closed convex hull of

$$\{u y_k u^* \mid u \in \mathcal{U}(Q_{n_k}' \cap M)\}.$$

Hence, there must exist some unitary $u_k \in \mathcal{U}(Q_{n_k}' \cap M)$ such that

$$\|u_k y_k u_k^* - y_k\|_2 \geqslant \tfrac{1}{2}\|y_k\|_2 \geqslant \tfrac{1}{4}$$

which yields $\|[u_k, x_k]\|_2 \geqslant \frac{1}{4}$. But, by construction, $(u_k)_{k \in \mathbf{N}}$ is a central sequence in M. This shows that $(x_k)_{k \in \mathbf{N}}$ is not in the center of M_ω. Therefore M_ω is a factor. □

Proof of Corollary G. By Proposition 5.4, we know that M_ω is a factor. By Theorem F, we then know that N_ω is also a factor. Since N has property Gamma thanks to [Pop10, Theorem 4.1], we conclude that N_ω is non-commutative or equivalently that N is McDuff. Thus $N \cong N \mathbin{\overline{\otimes}} R \cong M$ as required. □

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Amine Marrakchi amine.marrakchi@math.u-psud.fr

Laboratoire de Mathématiques d'Orsay, Université Paris-Sud, CNRS, Université Paris-Saclay, 91405 Orsay, France

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