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STOCHASTIC STABILITY OF SOME STATE-DEPENDENT GROWTH-COLLAPSE PROCESSES

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Abstract

In this paper we consider a discrete-time process which grows according to a random walk with nonnegative increments between crash times at which it collapses to 0. We assume that the probability of crashing depends on the level of the process. We study the stochastic stability of this growth-collapse process. Special emphasis is given to the case in which the probability of crashing tends to 0 as the level of the process increases. In particular, we show that the process may exhibit long-range dependence and that the crash sizes may have a power law distribution.

Keywords: Growth-collapse process; Markov chain; long-range dependence; heavy-tailed distribution

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1. Introduction

The dynamics of a growth-collapse (GC) process is governed by cycles describing periods of growth followed by abrupt crash events at which the process jumps down to a random level. A large variety of systems exhibit such a pattern. Examples include the transmission control protocol of the Internet and biological and physical systems in self-organized criticality.

The transmission control protocol is the main data transmission protocol in the Internet. It controls the transmission rates of packets sent by a source and verifies the correct delivery of data to the destination. It can be described roughly as follows. Files are broken into packets. The first packet is sent by the source, which waits to receive an acknowledgement from the destination within a specified time window. Then the congestion window increases by one and two more packets are sent. No other packet is sent until an acknowledgement is received for one of these packets. This reception marks the end of the current round and the beginning of the next round. As long as acknowledgements are received from the destination machine, the congestion window size increases by one every round-trip time. Two packet-loss detection mechanisms are used to control the reception of packets by the destination. The first mechanism is able to detect the loss of a single packet from time to time through 'triple-duplicated' acknowledgements. The second mechanism detects heavy losses, in case of severe congestion in the network, through a timeout mechanism. The congestion window size is halved when a loss of packets is detected (congestion-avoidance algorithm), goes to 1 when a timeout occurs (slow-start algorithm), and exponentially increases after a timeout until it reaches half the value it was before the timeout. Hence, the process of the congestion window sizes is a GC process.

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The term 'self-organized critical' is used to describe a system in which a mechanism of slow energy accumulation and fast energy redistribution drives the system toward a critical state [2]. Such systems are generally characterized by the existence of a power law that governs their dynamics. The archetype of a self-organized critical system is a sand pile. Imagine that sand is slowly dropped onto a surface, forming a pile. As more grains are added, the slope of the pile increases until it reaches a critical value such that the addition of one more grain results in an 'avalanche' which carries sand from the top to the bottom of the pile. With the addition of still more grains the surface will 'overflow'. Sand is thus added to and eventually lost from the system: the sand pile has self-organized into a critical state. It has been observed that the distribution of the avalanche sizes follows a power law.

It is not possible to derive a general rule for determining whether or not an arbitrary system displays self-organized criticality. It has nevertheless become established as a strong candidate for explaining a number of natural phenomena, including earthquakes (which were known as a source of power law behavior such as the Gutenberg–Richter law describing the statistical distribution of earthquake sizes and Omori's law describing the frequency of aftershocks [13], [3], [8]), solar flares [6], epidemics, and biological evolution (it has been suggested that evolution in complex communities leads to a self-organized critical state where a small event, like the random extinction of a single species, can generate an extinction set propagating through the network structure [17]).

Purely stochastic approaches to the modeling of continuous-time GC processes have been proposed in [10], [11], and [7]. In [10], Eliazar and Klafter considered GC processes where the inflow to the system is assumed to be a one-sided Lévy process, and where the timing of crashes and the crash magnitude are independent of the state of the system. More precisely, the crash epochs form a renewal process and the crash proportions are independent, identically distributed random variables. On the other hand, in [11] they studied the case in which the crashes occur at a Markovian rate (which is linear in the level of the system) and let the system be in its ground level. In [7], Boxma *et al.* considered GC processes where the inflow to the system is constant and the timing of crashes and the crash magnitude depend on the state of the system. In the three papers mentioned, several characteristics of stationary systems are computed (means, variances, stationary distributions, etc.). The particular case of the transmission control protocol congestion window size process has been investigated in [9] (see also the references therein), where some of the stationary characteristics of the protocol are presented.

In the spirit of [11], in this paper we consider a class of state-dependent GC processes which go back to their initial states after the crashes. They grow according to a discrete-time random walk with nonnegative increments between crash times at which they collapse to 0. It is assumed that the probability of crashing depends on the level of the process. Unlike in [11], we put special emphasis on systems whose probability of crashing is a decreasing function of the level of the system. Our main goal is to study the stochastic stability of these GC processes. For some important cases, we show that these systems can exhibit power law distributions and long-range dependence. Processes with such a probability pattern are widespread in the natural world. An example is given by the evolution of the number of single species over geological time scales, for which it seems reasonable to assume that large extinctions are unlikely since the number of species is large [18]. One may also consider colonial organisms. Groups of such organisms are made up of individuals that are, to varying degrees, coordinated. Many of the Earth's largest and most important organisms, from coral reefs to social insect colonies, are colonial. The extinction of such a 'superorganism' is unlikely since the number of individuals is large.

The paper is organized as follows. In Section 2 we present the class of GC processes and give sufficient conditions for the existence of a stationary distribution by using the Markov chain theory developed in [14]. Since GC processes are regenerative processes, these conditions are strongly linked with the distribution of cycle lengths. In Section 3 we present some properties of the processes. First we study the tail behavior of the stationary distribution and establish connections with the tail behavior of the crash size distribution. Then we turn to the persistence properties of the process. In particular, we give a condition sufficient to observe long-range dependence and power law distributions. Proofs are gathered in Section 4.

2. Stationarity

2.1. Definition of the processes

The class of GC processes $(X_t)_{t\geq 0}$ is characterized by the following features.

- *The growth process.* The inflow to the system is assumed to be a random walk with positive, independent, identically distributed increments ε_t. We denote by F the cumulative probability function of ε_1. We assume that ε_1 has a density f positive on $(0, \infty)$ and admits a Laplace transform $\Psi_{\varepsilon_1}(u) = \mathrm{E}[\mathrm{e}^{-u\varepsilon_1}]$ in a neighborhood of 0. The condition on the Laplace transform implies that the distribution of ε_1 has a light tail, at most exponential. The mean of ε_1 is denoted by μ.
- *The crash epochs.* There exists a function φ such that, at each date t, the process can collapse to 0 with probability $1 - \varphi(X_{t-1})$. Of course, $\varphi(0) = 1$. For the purposes of exposition, we assume that φ is a monotone, continuous function from $(0, \infty)$ to $(0, 1)$ such that $\inf_{x>0} \varphi(x) =: \underline{\varphi} > 0$. We denote by $\bar{\varphi}$ the supremum of φ on $(0, \infty)$.

Thus, the dynamics of $(X_t)_{t\geq 0}$ is governed by the stochastic recurrence rule

$$X_t = \begin{cases} X_{t-1} + \varepsilon_t & \text{with probability } \varphi(X_{t-1}), \\ 0 & \text{with probability } 1 - \varphi(X_{t-1}). \end{cases} \tag{1}$$

As an illustration, Figure 1 shows a path of the GC process. The selected probability of crashing is given by $1 - \varphi(x) = 1 - \exp(-(1 + x)^{-1})$ for $x > 0$, whereas the increment has a standard exponential distribution and the initial value is $X_0 = 0$.

The GC process $(X_t)_{t\geq 0}$ is a homogeneous Markov chain with a transition rule corresponding to a mixture of discrete and continuous distributions:

$$\begin{aligned} \mathrm{P}(X_t = 0 \mid X_{t-1} = x_{t-1}) &= 1 - \varphi(x_{t-1}), && x_{t-1} \geq 0, \\ \mathrm{P}(X_t \in (x_t, x_t + \mathrm{d}x_t) \mid X_{t-1} = x_{t-1}) &= \varphi(x_{t-1}) f(x_t - x_{t-1}) \,\mathrm{d}x_t, && x_t \geq x_{t-1}. \end{aligned}$$

Therefore, the stationary distribution, if it exists and is unique, has a point mass κ at $\{0\}$ and a probability density function l on $(0, \infty)$. They satisfy the conditions

$$\kappa = \int_0^\infty (1 - \varphi(x)) l(x) \,\mathrm{d}x, \qquad l(x) = \kappa f(x) + \int_0^x \varphi(y) f(x - y) l(y) \,\mathrm{d}y. \tag{2}$$

The atom $\{0\}$ is regenerative and can be used to define the following regeneration times: $T_0 = 0$ and $T_n = \inf\{t > T_{n-1} \colon X_t = 0\}$, $n \geq 1$. In this way the GC Markov chain is split up into independent, identically distributed cycles $(X_t)_{T_n \leq t < T_{n+1}}$, $n \geq 1$. We denote the independent,

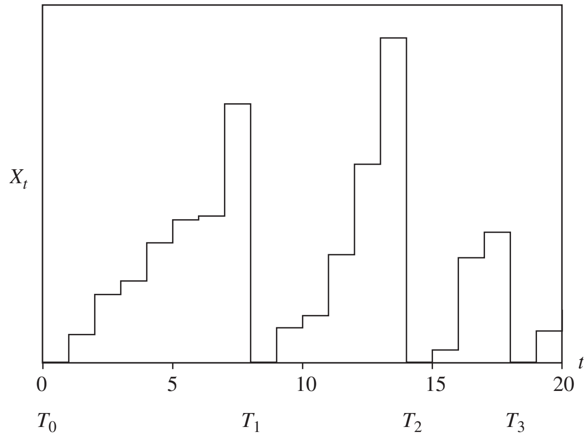

FIGURE 1: Path of a growth-collapse process.

identically distributed cycle lengths by $D_n = T_{n+1} - T_n$, $n \geq 1$. Finally, we define the monotonically decreasing sequence $(p_n)_{n\geq 0}$ by $p_0 = 1$ and

$$p_n = \mathrm{E}\bigg[\prod_{k=1}^{n} \varphi(S_k)\bigg], \qquad n \geq 1,$$

where $S_k = \varepsilon_1 + \cdots + \varepsilon_k$. Note that for $d \geq 1$ we have

$$\mathrm{P}(D_1 \geq d+1) = \mathrm{E}[\varphi(\varepsilon_{T_1+1}) \cdots \varphi(\varepsilon_{T_1+1} + \cdots + \varepsilon_{T_1+d-1})] = \mathrm{E}\bigg[\prod_{k=1}^{d-1} \varphi(S_k)\bigg] = p_{d-1}.$$

2.2. Stochastic stability of the growth-collapse processes

We first assume that ε_1 has an exponential distribution with parameter α, because analytical computations can be performed in this case. We will see that the stationarity condition which is obtained is also suited to the general case.

Proposition 1. *Assume that $f(x) = \alpha \exp(-\alpha x)$. The stationary distribution exists and is unique if and only if*

$$\chi = \int_0^\infty \exp\bigg(-\alpha \int_0^x (1 - \varphi(y))\,\mathrm{d}y\bigg)\,\mathrm{d}x < \infty. \tag{3}$$

If (3) is satisfied then

$$l(x) = C_\mathrm{e} \exp\bigg(-\alpha \int_0^x (1 - \varphi(y))\,\mathrm{d}y\bigg), \qquad \kappa = 1 - \chi C_\mathrm{e},$$

where

$$C_\mathrm{e} = \bigg(\int_0^\infty (2 - \varphi(x)) \exp\bigg(-\alpha \int_0^x (1 - \varphi(y))\,\mathrm{d}y\bigg)\,\mathrm{d}x\bigg)^{-1}.$$

Recall that the proof is supplied in Section 4. A stationary distribution exists if φ converges to a constant strictly smaller than 1 or converges to 1 but not too quickly. The idea behind this condition is that the probability of crashing must not be so small that the process cannot come back to 0 sufficiently often.

Remark 1. Let $c := \lim_{x\to\infty}(1-\varphi(x))x$. There exists a stationary distribution if $c > \alpha^{-1} = \mu$, and there exists no stationary distribition if $c < \mu$.

Remark 2. Note that $N(\cdot) = \sum_{k=1}^{\infty} \mathbf{1}_{\{S_k\in(\cdot)\}}$ is an homogeneous Poisson point process with intensity α. By Campbell's theorem,

$$\mathrm{E}\biggl[\exp\biggl(\int_{[0,x]} \ln\varphi\,\mathrm{d}N\biggr)\biggr] = \mathrm{E}\biggl[\prod_{k=1}^{N([0,x])} \varphi(S_k)\biggr] = \exp\biggl(-\alpha\int_0^x (1-\varphi(y))\,\mathrm{d}y\biggr).$$

Let us consider the first cycle, $(X_t)_{T_1\le t<T_2}$. It is easily seen that the survival distribution function of the crash size of this cycle, X_{T_2-1}, is given by

$$\mathrm{P}(X_{T_2-1} > x) = \mathrm{E}\biggl[\prod_{k=1}^{N([0,x])} \varphi(S_{k+T_1} - S_{T_1})\biggr] = \mathrm{E}\biggl[\prod_{k=1}^{N([0,x])} \varphi(S_k)\biggr].$$

It follows that (3) is equivalent to the condition $\chi = \mathrm{E}[X_{T_2-1}] < \infty$.

Now let us introduce independent, identically distributed variables $(Z_t)_{t\ge1}$ independent of the sequence $(\varepsilon_t)_{t\ge1}$. If φ is nondecreasing, assume that the cumulative distribution function of Z_t is given by φ_r with $\varphi_r(x) = \varphi(x)$ if $x > 0$, $\varphi_r(x) = 0$ if $x < 0$, and $\varphi_r(0) = \underline{\varphi}$. If φ is nonincreasing, assume that the survival distribution function of Z_t is given by φ_r with $\varphi_r(x) = \varphi(x)$ if $x > 0$, $\varphi_r(x) = 1$ if $x < 0$, and $\varphi_r(0) = \underline{\varphi}$.

The stochastic recurrence (1) is equivalent to

$$X_t = B_t(X_{t-1} + \varepsilon_t) + C_t(1-B_t)\varepsilon_t,$$

where $B_t = 1$ when $Z_t \le X_{t-1}$ and $B_t = 0$ otherwise if φ is nondecreasing, $B_t = 0$ when $Z_t \le X_{t-1}$ and $B_t = 1$ otherwise if φ is nonincreasing, and $C_t = 1$ when $X_{t-1} = 0$ and $C_t = 0$ otherwise. Let us define the σ-algebra $\mathcal{F}_t = \sigma(\varepsilon_n, Z_n : n \le t)$ and assume, for the purposes of exposition, that $X_0 = 0$. It is easily shown that $\{T_1 = t+1\} \in \mathcal{F}_t$. Therefore, $T_1 - 1$ is a stopping time with respect to $\{\mathcal{F}_t,\ t \ge 1\}$. By Wald's identity, we deduce that

$$\mathrm{E}[X_{T_1-1}] = \mathrm{E}\biggl[\sum_{t=1}^{T_1-1} \varepsilon_t\biggr] = \mathrm{E}[\varepsilon_1](\mathrm{E}[T_1]-1)$$

(see, e.g. Proposition A10.2 of [1]). Since $\mathrm{E}[X_{T_1-1}] = \mathrm{E}[X_{T_2-1}]$ and $\mathrm{E}[T_1] = \mathrm{E}[D_1]$ when $X_0 = 0$, it follows that (3) is also equivalent to the condition that the cycle length distribution have a finite mean, i.e. $\mathrm{E}[D_1] < \infty$.

We now study the stochastic stability of the Markov chain in the general case. We first introduce some notation and definitions (see, e.g. [14]). Let us denote by ν the Lebesgue measure and by $\mathcal{B}$ the Borel σ-algebra on $\mathbb{R}_+$. For any measurable set $A \in \mathcal{B}$, we define the time of entry to A by $\tau_A = \inf\{t \ge 1 : X_t \in A\}$ and the number of visits to A by $\eta_A = \sum_{t=1}^{\infty} \mathbf{1}_{\{X_t\in A\}}$.

Definition 1. (i) A Markov chain $(X_t)_{t\ge0}$ is said to be ν-irreducible if any set of positive Lebesgue measure can be reached in finite time starting from any initial value. This condition can be written as

$$\nu(A) > 0, \quad A \in \mathcal{B} \quad \Longrightarrow \quad L(x, A) := \mathrm{P}(\tau_A < \infty \mid X_0 = x) > 0 \quad \text{for all } x \ge 0.$$

(ii) A Markov chain $(X_t)_{t\geq 0}$ is said to be Harris recurrent if it is ν-irreducible and if, for any set of positive Lebesgue measure, it returns to this set infinitely often starting from any element in the set. This condition can be written as

$$\nu(A) > 0, \quad A \in \mathcal{B} \quad \Longrightarrow \quad \mathrm{P}(\eta_A = \infty \mid X_0 = x) = 1 \quad \text{for all } x \in A.$$

(iii) A Harris recurrent Markov chain admits a unique invariant measure. If the invariant measure is finite then the Markov chain is positive Harris recurrent, and is null recurrent otherwise.

We now characterize the stability properties of the GC Markov chain.

Proposition 2. (i) *The GC Markov chain is ν-irreducible.*

(ii) *The GC Markov chain is positive Harris recurrent if* $c := \lim_{x\to\infty}(1-\varphi(x))x \in (\mu, \infty]$.

(iii) *The GC Markov chain is null recurrent if* $c := \lim_{x\to\infty}(1-\varphi(x))x \in (0, \mu)$.

We see that the condition in part (ii) of Proposition 2 is quite close to the condition derived from the model with exponential increments (see Remark 1).

A Markov chain is highly unstable or transient when it returns only a finite number of times to a given set of 'reasonable size' (see [14] for a mathematical definition). Note that if $\lim_{x\to\infty}(1-\varphi(x))x = 0$, then the GC Markov chain may be either null recurrent or transient.

2.3. The cycle length distribution

A necessary condition to have a GC process which does not diverge to ∞ and may return to any set is that the cycle lengths are almost surely (a.s.) finite.

Proposition 3. *The cycle length D_1 is a.s. finite if $1-\varphi$ is not integrable at ∞.*

It is well known that this last condition is not sufficient for the process to be 'stable', and one often assumes that the cycle length distribution has a finite mean (see Remark 2). Under this condition, $(X_t)_{t\geq 0}$ admits a limiting probability distribution (see, e.g. Theorem 1.2 of [1]).

Therefore, sufficient conditions for the existence of a stationary distribution can also be deduced from the limiting behavior of the probabilities p_n. For $\bar{\varphi} < 1$, we can see that $p_n \leq \bar{\varphi}^n$ and that D_1 has moments of all orders. Let us instead focus on the case $\bar{\varphi} = 1$. We would like to compare conditions on the rate of convergence of φ to 1 with the conditions on the existence of $\mathrm{E}[D_1] = 1 + \sum_{n=0}^{\infty} p_n$. For this reason, we only consider functions φ which tend to 1 at a hyperbolic rate. Let us first recall the definition of regular variation (see, e.g. [5]).

Definition 2. A positive function h on $(0, \infty)$ is regularly varying with index τ if

$$\lim_{t\to\infty} \frac{h(tx)}{h(t)} = x^{\tau}, \qquad x > 0.$$

The proposition below derives asymptotic equivalents for the probabilities p_n when $1-\varphi(x)$ is regularly varying.

Proposition 4. *Assume that $1-\varphi(x)$ is regularly varying with index $\tau > \frac{2}{3}$. Then there exists a constant C such that*

$$p_n = C \exp\biggl(\sum_{k=1}^{n} \log \varphi(k\mu)\biggr)(1 + o(1)) \tag{4}$$

for large n.

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We deduce from this proposition that there exists a stationary distribution if

$$\sum_{n=0}^{\infty} \exp\biggl(\sum_{k=1}^{n} \log \varphi(k\mu)\biggr) < \infty,$$

which is equivalent to the condition, (3), derived from the model with exponential increments.

The choice of $\tau > \frac{2}{3}$ is purely technical. When comparing (4) to the definition of p_n, we see that, for large n, S_n is approximated by its mean, $n\mu$. For $\tau > \frac{2}{3}$, the rate of convergence of φ to 1 is sufficiently fast for the fluctuations of S_n around $n\mu$ to be neglected. This is not the case when $\tau \le \frac{2}{3}$.

Example 1. Let us assume that

$$\varphi(x) = \exp\biggl(-\frac{c}{(d+x)^{\tau}}\biggr) \quad \text{or, equivalently,} \quad -\log \varphi(x) = c(d+x)^{-\tau},$$

where c and d are positive constants. Thus, $1 - \varphi(x) = cx^{-\tau}(1 + O(x^{-(\tau\wedge 1)}))$ and $\underline{\varphi} > 0$.

(i) If $\tau > 1$ then there exists a positive constant a such that $p_n = a(1 + o(1))$ for large n, and D_1 is not a.s. finite. The GC process is unstable and diverges a.s. to infinity.

(ii) If $\tau = 1$ then there exists a positive constant a such that $p_n = an^{-c/\mu}(1+o(1))$ for large n. The series $(p_n)_{n\ge 0}$ has a hyperbolic rate of decay and D_1 has a power law distribution. If $\mu < c$ then $\mathrm{E}[D_1] < \infty$ and there exists a stationary distribution.

(iii) If $\frac{2}{3} < \tau < 1$ then there exists a positive constant a such that

$$p_n = a \exp\biggl(-\frac{c}{\mu^{\tau}} \frac{(n^{1-\tau} - 1)}{1-\tau}\biggr)(1 + o(1))$$

for large n. The distribution of D_1 has a Weibull tail and $\mathrm{E}[D_1] < \infty$.

It has been observed that for many complex physical systems the distribution of the waiting times between events larger than a certain size follows a power law distribution. In [13] the authors found that the distribution of the time interval, τ_{e}, between a large earthquake (the main shock of a given seismic sequence) and the next one is a power law distribution: $\mathrm{P}(\tau_{\mathrm{e}} > t) = A_{\mathrm{e}} t^{-d_{\mathrm{e}}}$ with exponent $d_{\mathrm{e}} = 1.06$. In [6] the statistics of the times, τ_{s}, between successive bursts of solar flare activity also displays a power law distribution: $\mathrm{P}(\tau_{\mathrm{s}} > t) = A_{\mathrm{s}} t^{-d_{\mathrm{s}}}$ with exponent $d_{\mathrm{s}} = 2.04$.

Note that these observed scaling behaviors are theoretically in contradiction with models of self-organized criticality which predict exponential waiting time distributions [6]. However, as explained in [15], waiting time statistics cannot be used to discard self-organized critical behavior in real physical systems when all other signatures suggest its existence because the definition of waiting time may be ‘contaminated’ by there being other, differently distributed time scales in the problem (see [22] and [12]). Finally, let us underline that the power law behavior of waiting times could also be explained in terms of a Poisson process with a time-varying rate (see [21] and [20] for a discussion of this in the context of solar flares).

3. Properties of the growth-collapse processes

In this section we assume that $c = \lim_{x\to\infty}(1-\varphi(x))x \in (\mu, \infty]$, and we denote by X a random variable with the unique stationary distribution. Moreover, we assume that the distribution of X_0 is the stationary distribution. Let us first give a moment characterization of this distribution.

Proposition 5. *Let g be a measurable function. Then*

$$\mathrm{E}[g(X)] = \kappa\biggl(g(0) + \sum_{n=1}^{\infty}\mathrm{E}\biggl[g(S_n)\prod_{k=0}^{n-1}\varphi(S_k)\biggr]\biggr),$$

where $S_0 = 0$ and $\kappa = \mathrm{P}(X = 0)$.

When it exists, any moment of the stationary distribution can be written as an infinite sum of specific moments of the trajectories of the random walk $(S_n)_{n\geq 0}$.

Example 2. (i) If $g = 1 - \varphi$ then

$$\mathrm{E}[g(X)] = \kappa\sum_{n=1}^{\infty}\mathrm{E}\biggl[(1-\varphi(S_n))\prod_{k=0}^{n-1}\varphi(S_k)\biggr] = \kappa\biggl(\sum_{n=1}^{\infty}(p_{n-1} - p_n)\biggr) = \kappa,$$

which is equivalent to the left-hand condition in (2).

(ii) If $g = 1$ then

$$\mathrm{E}[g(X)] = 1 = \kappa\biggl(1 + \sum_{n=1}^{\infty}\mathrm{E}\biggl[\prod_{k=0}^{n-1}\varphi(S_k)\biggr]\biggr) = \kappa\biggl(1 + \sum_{n=1}^{\infty}p_{n-1}\biggr) = \kappa\,\mathrm{E}[D_1].$$

We deduce from Example 2(ii) that

$$\mathrm{E}[g(X)] = \frac{1}{\mathrm{E}[D_1]}\biggl(g(0) + \sum_{n=1}^{\infty}\mathrm{E}\biggl[g(S_n)\prod_{k=0}^{n-1}\varphi(S_k)\biggr]\biggr) = \frac{1}{\mathrm{E}[D_1]}\,\mathrm{E}\biggl[\sum_{t=T_1}^{T_2-1}g(X_t)\biggr].$$

This is the well-known stationary equation of a regenerative process (see, e.g. Theorem 1.2 of [1]).

Let us now focus on the tails of some particular distributions.

3.1. Tail analysis

We begin with the tail behavior of the stationary distribution. Let us recall some definitions (see, e.g. [5]).

Definition 3. (i) A positive function h on $(0, \infty)$ is rapidly varying with index $-\infty$ if

$$\lim_{t\to\infty}\frac{h(tx)}{h(t)} = \begin{cases} 0 & \text{for } x > 1, \\ \infty & \text{for } 0 < x < 1. \end{cases}$$

(ii) A positive function h on $(0, \infty)$ is slowly varying if it is regularly varying with index 0.

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Proposition 6. (i) *Assume that* $\varphi_1 := \lim_{x\to\infty}\varphi(x) < 1$. *Let* (a,∞) *be the maximal open interval (where* $a < 0$*) such that*

$$\Psi_{\varepsilon_1}(u) < \infty \quad \text{for } u \in (a,\infty).$$

If a *is infinite then assume that* $f\circ\log$ *is rapidly varying. If* a *is finite then assume that* $f\circ\log$ *is regularly varying with index* a *and that* $\lim_{\delta\searrow 0}\Psi_{\varepsilon_1}(a+\delta)=\infty$. *Let* γ *be the unique positive scalar such that* $\Psi_{\varepsilon_1}(-\gamma)=\varphi_1^{-1}$. *Then*

$$\mathrm{P}(X > x) = \psi(x)\exp(-\gamma x),$$

where $\psi\circ\log$ *is slowly varying.*

(ii) *Assume that* $1-\varphi(x) = x^{-\tau}L(x)$, *where* L *is a slowly varying function for* $\frac{2}{3} < \tau < 1$ *and* $L(x) = c(1+o(1))$ *for* $\tau = 1$. *Then*

$$\mathrm{P}(X > x) = (1+o(1))\kappa \sum_{n=\lfloor x/\mu\rfloor}^{\infty} p_n$$

for large x, *where* $\lfloor x\rfloor$ *denotes the integer part of* x.

Example 3. (*Example 1(ii) continued.*) If

$$\varphi(x) = \exp\biggl(-\frac{c}{d+x}\biggr) \quad \text{or, equivalently,} \quad -\log\varphi(x) = c(d+x)^{-1},$$

where $c > \mu$ and $d > 0$, then we deduce from Proposition 6(ii) that the survival distribution function of X has a hyperbolic rate of decay:

$$\mathrm{P}(X > x) = (1+o(1))\frac{a\kappa\mu^{c/\mu}}{c-\mu}x^{1-c/\mu}.$$

First, note that if $\varphi(x) = \varphi_{\mathrm{c}} < 1$ is constant, then l satisfies the integral equation

$$l(x) = \kappa f(x) + \varphi_{\mathrm{c}}\int_0^x f(y)l(x-y)\,\mathrm{d}y.$$

Let γ_{c} be the positive constant such that $\Psi_\varepsilon(-\gamma_{\mathrm{c}}) = \varphi_{\mathrm{c}}^{-1}$, and define the probability measure $G(\mathrm{d}x) = \varphi_{\mathrm{c}}\exp(\gamma_{\mathrm{c}}x)f(x)\,\mathrm{d}x$. We have

$$h(x) = \kappa\exp(\gamma_{\mathrm{c}}x)f(x) + \int_0^x h(x-y)G(\mathrm{d}y), \tag{5}$$

where $h(x) = \exp(\gamma_{\mathrm{c}}x)l(x)$. Equation (5) defines a renewal equation. From the renewal theorem (see, e.g. [1, p. 155]), we deduce that there exists a positive constant D_{c} such that $\lim_{x\to\infty}h(x) = D_{\mathrm{c}}$. Then we have

$$\mathrm{P}(X > x) = (1+o(1))\gamma_{\mathrm{c}}^{-1}D_{\mathrm{c}}\exp(-\gamma_{\mathrm{c}}x)$$

for large x. Therefore, when φ is constant, the tail of the stationary distribution decreases at an exponential rate and $\psi(x)$ converges to a positive constant as x tends to ∞. However, this is not always the case. Consider the function

$$\varphi(x) = \varphi_{\mathrm{b}}\biggl(1-\frac{\delta}{1+x}\biggr),$$

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where $0 < \varphi_{\mathrm{b}} < 1$ and $0 < \delta < 1$, and assume that ε_1 has an exponential distribution with parameter α. We have

$$l(x) = C_{\mathrm{e}} \exp\biggl(-\alpha \int_0^x (1-\varphi(y))\,\mathrm{d}y\biggr) = \frac{C_{\mathrm{e}}}{(1+x)^{\alpha\delta\varphi_{\mathrm{b}}}} \exp(-\alpha(1-\varphi_{\mathrm{b}})x),$$

and $\psi(x)$ converges to 0 as x tends to ∞.

Second, note that, in the case of Proposition 6(ii), X is a.s. finite if $\mathrm{E}[D_1] < \infty$, which again demonstrates the strong link between the existence of a stationary distribution and the existence of the first moment of D_1. Moreover, we have

$$\mathrm{P}(X > \mu k) = (1+o(1)) \frac{\mathrm{E}[(D_1-k)_+]}{\mathrm{E}[D_1]}$$

for large integer values k, where $(D_1 - k)_+$ denotes the positive part of $D_1 - k$. This is the result that we would have obtained if we had assumed that $\varepsilon_1 = \mu$ a.s., which also means that, in this case, the rate of convergence of φ to 1 is sufficiently fast for the fluctuations of S_n around $n\mu$ to be neglected.

The following proposition characterizes the correlation structure of the stationary process $(\varphi(X_t))_{t\geq 0}$.

Proposition 7. *The autocovariance function of the process* $(\varphi(X_t))_{t\geq 0}$ *is given by*

$$\operatorname{cov}(\varphi(X_t), \varphi(X_{t+k})) = \kappa^2 \sum_{n=k}^{\infty} p_n.$$

It follows that, in the case of Proposition 6(ii), the autocovariance function of $(\varphi(X_t))_{t\geq 0}$ and the tail of the stationary distribution of $(X_t)_{t\geq 0}$ admit the same asymptotical decay rate,

$$\lim_{k\to\infty} \frac{\mathrm{P}(X > \mu k)}{\operatorname{cov}(\varphi(X_t), \varphi(X_{t+k}))} = \frac{1}{\kappa}.$$

We explore the persistence properties of the GC process more deeply in the next subsection.

Let us now denote by $A_n = X_{T_{n+1}-1}$ the crash size of the nth cycle and turn to studying the tail behavior of its distribution. Of course, A_n, $n \geq 1$, are independent, identically distributed random variables.

Proposition 8. (i) *Assume that* $\varphi_{\mathrm{l}} = \lim_{x\to\infty} \varphi(x) < 1$ *and that* φ *is a nondecreasing function. Then*

$$\mathrm{P}(A_1 > x) = (1+o(1))(1-\varphi_{\mathrm{l}})\kappa^{-1}\,\mathrm{P}(X > x)$$

for large x.

(ii) *Assume that* $1-\varphi(x) = x^{-\tau}L(x)$*, where* L *is a slowly varying function for* $\frac{2}{3} < \tau < 1$ *and* $L(x) = c(1+o(1))$ *for* $\tau = 1$*. Then*

$$\mathrm{P}(A_1 > x) = (1+o(1))\,p_{\lfloor x/\mu \rfloor}$$

for large x.

Note that in Proposition 8(ii) we have

$$\mathrm{P}(A_1 > \mu x) = (1 + o(1))\,\mathrm{P}(D_1 > x)$$

for large x. This means that the crash size distribution and the cycle length distribution have the same hyperbolic tail behavior (see Example 1(ii)). This relation has been observed in the study of, e.g. earthquakes. Despite the apparent complexity of the dynamics of earthquakes, the probability distribution of the energy, E_{e}, of an earthquake follows a simple power law distribution known as the Gutenberg–Richter law: $\mathrm{P}(E_{\mathrm{e}} > v) = B_{\mathrm{e}} v^{-c_{\mathrm{e}}}$, where B_{e} is a positive constant and the exponent c_{e} is a universal exponent (in the sense that it does not depend on a particular geographic area) close to 1. Also, as mentioned in Section 2, the distribution of the time interval, τ_{e}, between two large earthquakes is also a power law distribution: $\mathrm{P}(\tau_{\mathrm{e}} > t) = A_{\mathrm{e}} t^{-d_{\mathrm{e}}}$ with exponent $d_{\mathrm{e}} = 1.06$ [13]. Of course, a GC process is too simple to model such a complex phenomenon fully, but its study may provide some intuition about the mechanism of energy accumulation.

3.2. Persistence analysis

Let us now assume that $c = \lim_{x\to\infty}(1 - \varphi(x))x \in (3\mu, \infty]$, such that the second moment and the autocorrelations of the stationary process exist.

The GC structure can lead to a sustained correlation phenomenon which is called long-range dependence. Physicists also denote this phenomenon by the term '$1/f$ noise' because such a process has a spectral density f_{s} satisfying $f_{\mathrm{s}}(\omega) \propto \omega^{-d}$ with $0 < d < 1$. Intuitively, if the spectral density diverges to ∞ at a certain rate, then the covariance function converges to 0 at an appropriate slow rate (and vice versa). More specifically, the definition of long-range dependence that we use is the following.

Definition 4. A stationary process $(X_t)_{t\ge 0}$ is said to be long-range dependent if the absolute values of its autocorrelations,

$$\rho(k) = \frac{\mathrm{E}[X_t X_{t+k}] - \mathrm{E}[X_t]\,\mathrm{E}[X_{t+k}]}{\sqrt{\mathrm{var}(X_t)}\sqrt{\mathrm{var}(X_{t+k})}},$$

sum to infinity, i.e. if $\sum_{k=0}^{\infty} |\rho(k)| = \infty$.

Another common definition is to have autocorrelations that are regularly varying at ∞ with exponent less than 1 (see, e.g. Section 4 of [19]). In the following proposition we give a lower bound on the sum of the autocorrelations.

Proposition 9. *Assume that $1 - \varphi(x) = x^{-\tau} L(x)$, where $L(x)$ is a slowly varying function for $\frac{2}{3} < \tau < 1$ and $L(x) = c(1 + o(1))$ for $\tau = 1$. Then there exists a constant B such that*

$$\lim_{n\to\infty} \sum_{k=0}^{n} |\rho(k)| \ge B\,\mathrm{E}[D_1^4].$$

We now derive a condition sufficient to observe long-range dependence.

Corollary 1. *Let us assume that $c = \lim_{x\to\infty}(1 - \varphi(x))x \in (3\mu, 4\mu)$. Then, for each k, $|\rho(k)| < \infty$ and $\sum_{k=0}^{\infty} |\rho(k)| = \infty$.*

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Remark 3. The dynamical processes underlying evolution over geological time scales have highlighted a possible signature of long-range dependence [17], [18]. The fluctuations in the evolutionary record have been proposed to result from intrinsic nonlinear dynamics for which self-organized criticality provides an appropriate theoretical framework [16]. Our model suggests that the decreasing probability of having large extinctions may also be a relevant ingredient.

4. Proofs

4.1. Proofs of results in Section 2

Proof of Proposition 1. If it exists, the probability density function of the stationary distribution satisfies the integral equation

$$l(x) = \kappa\alpha \exp(-\alpha x) + \alpha \int_0^x \varphi(y) \exp(-\alpha(x-y)) l(y)\, \mathrm{d}y.$$

Let $h(x) = \exp(\alpha x) l(x)$. The equation then becomes

$$h(x) = \kappa\alpha + \alpha \int_0^x \varphi(y) h(y)\, \mathrm{d}y.$$

By differentiating both sides of the equation, we obtain $h'(x) = \alpha\varphi(x)h(x)$, and we deduce that

$$h(x) = C_{\mathrm{e}} \exp\biggl(\alpha \int_0^x \varphi(y)\, \mathrm{d}y\biggr)$$

for some positive constant C_{e}. Therefore,

$$l(x) = C_{\mathrm{e}} \exp\biggl(-\alpha \int_0^x (1-\varphi(y))\, \mathrm{d}y\biggr)$$

and the stationary distribution exists if and only if

$$\int_0^\infty \exp\biggl(-\alpha \int_0^x (1-\varphi(y))\, \mathrm{d}y\biggr)\, \mathrm{d}x < \infty.$$

Since

$$\kappa + \int_0^\infty l(x)\, \mathrm{d}x = 1 = \int_0^\infty (2-\varphi(x)) l(x)\, \mathrm{d}x,$$

we deduce that

$$C_{\mathrm{e}} = \biggl(\int_0^\infty (2-\varphi(x)) \exp\biggl(-\alpha \int_0^x (1-\varphi(y))\, \mathrm{d}y\biggr)\, \mathrm{d}x\biggr)^{-1},$$
$$\kappa = 1 - C_{\mathrm{e}} \int_0^\infty \exp\biggl(-\alpha \int_0^x (1-\varphi(y))\, \mathrm{d}y\biggr)\, \mathrm{d}x,$$

which completes the proof.

Proof of Proposition 2. (i) Since the density of ε_1 is positive and $0 < \varphi(x) < 1$ for all $x > 0$, for any $A \in \mathcal{B}$ such that $\nu(A) > 0$ we have

$$\begin{aligned} L(x, A) &\geq \mathrm{P}(\tau_A = 1 \mid X_0 = x) + \mathrm{P}(\tau_A = 2 \mid X_0 = x) \\ &\geq \varphi(x)\,\mathrm{P}(\varepsilon_1 + x \in A) + (1 - \varphi(x))\,\mathrm{P}(\varepsilon_2 \in A) > 0. \end{aligned}$$

(ii) From Theorem 11.0.1 of [14], the Markov chain is positive Harris recurrent if there exist a petite set C, a constant $b < \infty$, and a Lyapunov function $V \geq 0$, finite at some $x \geq 0$, such that

$$\Delta V(x) := \mathrm{E}[V(X_1) \mid X_0 = x] - V(x) \leq -1 + b\,\mathbf{1}_C(x).$$

Recall that a set C is petite if the chain satisfies

$$\mathrm{P}(X_1 \in A \mid X_0 = x) \geq \tilde{\nu}(A)$$

for any $x \in C$ and any $A \in \mathcal{B}$, where $\tilde{\nu}$ is a nontrivial measure on $\mathcal{B}$.

Let $M > 0$. Note that, for any $A \in \mathcal{B}$,

$$\mathrm{P}(X_1 \in A \mid X_0 = x) \geq \underline{\varphi}\,\mathrm{P}(\varepsilon_1 + x \in A) \geq \underline{\varphi}\,\mathrm{P}(\varepsilon_1 + x \in A \cap (M, \infty)).$$

Since f is positive and the Laplace transform is well defined in a neighborhood of 0,

$$\eta(M) := \inf_{\substack{A \in \mathcal{B},\, \nu(A \cap (M,\infty)) > 0 \\ 0 \leq x \leq M}} \frac{\mathrm{P}(\varepsilon_1 + x \in A \cap (M, \infty))}{\mathrm{P}(\varepsilon_1 \in A \cap (M, \infty))} > 0.$$

Then, for any x, $0 \leq x \leq M$, and any $A \in \mathcal{B}$, we have

$$\mathrm{P}(X_1 \in A \mid X_0 = x) \geq \underline{\varphi}\eta(M)\,\mathrm{P}(\varepsilon_1 \in A \cap (M, \infty)) =: \tilde{\nu}(A).$$

This means that every set of the form $[0, M]$ is petite.

Let us consider the function $V(x) = ax$, $x \geq 0$, $a > 0$. We have

$$a^{-1}\Delta V(x) = \varphi(x)\,\mathrm{E}[x + \varepsilon_1] - x = (\varphi(x) - 1)x + \varphi(x)\mu.$$

If c is finite, let us consider a $\delta > 0$ such that $c - \mu - \delta > 0$ and define $a^{-1} = c - \mu - \delta$. If c is not finite, let us choose $a = 1$. Then there exists an $M > 0$ such that, for $x > M$, $\Delta V(x) \leq -1$. Finally, we have

$$\Delta V(x) \leq -1 + b\,\mathbf{1}_C(x),$$

where $C = \{x : 0 \leq x \leq M\}$ and $b = \sup\{\Delta V(x) + 1 : 0 \leq x \leq M\}$.

(iii) From Theorem 11.5.1 of [14], the Markov chain is null recurrent if there exist a petite set C and a Lyapunov function $V \geq 0$ such that

$$\Delta V(x) \geq 0 \qquad \text{for all } x \in \mathbb{R}_+ \setminus C$$

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and

$$\sup_{x\geq 0} \mathrm{E}[|V(X_1) - V(X_0)| \mid X_0 = x] < \infty.$$

Let us again consider the function $V(x) = ax$. From (ii) we have

$$\lim_{x\to\infty} a^{-1}\Delta V(x) = \mu - c > 0,$$

and we can use the same arguments as in (ii) to obtain the first condition. Let us now focus on the second condition. We have

$$a^{-1}\mathrm{E}[|V(X_1) - V(X_0)| \mid X_0 = x] = \varphi(x)\mu + (1-\varphi(x))x.$$

Since $c = \lim_{x\to\infty}(1-\varphi(x))x < \mu$, we deduce that

$$\sup_{x\geq 0} \mathrm{E}[|V(X_1) - V(X_0)| \mid X_0 = x] < \infty,$$

and the result follows.

Proof of Proposition 3. Since $A_k := \prod_{i=1}^k \varphi(S_i)$ is a nonincreasing sequence of nonnegative random variables, the a.s. convergence of A_k to 0 implies that the condition $\lim_{k\to\infty} p_k = 0$ holds, by Beppo Levi's theorem (see [4, p. 209]).

Let us now check the convergence of A_k to 0. From the law of the iterated logarithm, it is known that, for any $\lambda > 1$ (see [4, p. 155]), $S_n > n\mu + \lambda(2\sigma^2 n \log\log n)^{1/2}$ a finite number of times a.s., where $\sigma^2 = \mathrm{var}(\varepsilon_1)$. We deduce that a sufficient condition for the a.s. convergence of A_k to 0 is

$$\sum_{n\geq 3} \log \varphi(n\mu + \lambda(2\sigma^2 n \log\log n)^{1/2}) = -\infty$$

or, equivalently, $\sum_{n\geq 0}(1-\varphi(n)) = \infty$.

Proof of Proposition 4. Let us first consider the case in which $\frac{2}{3} < \tau \leq 1$, and write $q_n = p_n / \prod_{i=1}^n \varphi(i\mu) = \mathrm{E}[\prod_{i=1}^n \varphi(S_i)/\varphi(i\mu)]$. We will prove the convergence of the sequence $(\sum_{k=1}^n |q_n - q_{n-1}|)_{n\geq 1}$ to deduce the convergence of the sequence $(q_n)_{n\geq 1}$. Let us now recall a large deviation principle which will be used intensively in the proof (see Theorem 9.4 of [4]).

Proposition 10. *If ε_1 admits a Laplace transform in a neighborhood of* 0 *then, for any sequence $(a_n)_{n\geq 1}$ satisfying $a_n \to \infty$ and $a_n/\sqrt{n} \to 0$,*

$$\mathrm{P}(S_n \geq n\mu + a_n\sqrt{n}) = \exp\biggl(-\frac{a_n^2(1+\zeta_{1,n})}{2\sigma^2}\biggr),$$
$$\mathrm{P}(S_n \leq n\mu - a_n\sqrt{n}) = \exp\biggl(-\frac{a_n^2(1+\zeta_{2,n})}{2\sigma^2}\biggr)$$

for two sequences $(\zeta_{1,n})_{n\geq 1}$ and $(\zeta_{2,n})_{n\geq 1}$ going to 0.

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Let $0 < 2\nu < \min(-2 + 3\tau, \frac{2}{3})$. Proposition 10 will be used further with the sequence $a_n = n^{\tau-1/2-\nu}$. Let us note that

$$q_n - q_{n-1} = \frac{1-\varphi(n\mu)}{\varphi(n\mu)} \mathrm{E}\bigg[\bigg(\prod_{i=1}^{n-1} \frac{\varphi(S_i)}{\varphi(i\mu)}\bigg)\bigg(1 - \frac{1-\varphi(S_n)}{1-\varphi(n\mu)}\bigg)\bigg],$$

and introduce the following notation:

$$\begin{aligned} H_{\mathrm{u},n} &= \{i \le n \colon S_i \ge i\mu + a_i\sqrt{i}\}, & C_{\mathrm{u},n} &= \mathrm{card}(H_{\mathrm{u},n}), & C_{\mathrm{u}} &= C_{\mathrm{u},\infty}, \\ H_{\mathrm{d},n} &= \{i \le n \colon 0 < S_i \le i\mu - a_i\sqrt{i}\}, & C_{\mathrm{d},n} &= \mathrm{card}(H_{\mathrm{d},n}), & C_{\mathrm{d}} &= C_{\mathrm{d},\infty}. \end{aligned}$$

Note that C_{u} is a.s. finite because

$$\mathrm{E}[C_{\mathrm{u}}] = \mathrm{E}\bigg[\sum_{i=1}^{\infty} \mathbf{1}_{\{S_i \ge i\mu + a_i\sqrt{i}\}}\bigg] = \sum_{i=1}^{\infty} \mathrm{P}(S_i \ge i\mu + a_i\sqrt{i}) \le \sum_{i=1}^{\infty} \exp\bigg(-\frac{a_i^2(1+\zeta_{1,i})}{2\sigma^2}\bigg) < \infty,$$

and that C_{d} is a.s. finite because $\mathrm{E}[C_{\mathrm{d}}] < \infty$.

To derive the asymptotic equivalents, we will determine upper and lower bounds for the sequence $q_n - q_{n-1}$, and compare the asymptotic behaviors of both bounds.

Step 1: Upper bound for the integrand. Let $[[1; n-1]] = \{1, 2, \ldots, n-1\}$. We have

$$\begin{aligned}
&\bigg(\prod_{i=1}^{n-1} \frac{\varphi(S_i)}{\varphi(i\mu)}\bigg)\bigg(1 - \frac{1-\varphi(S_n)}{1-\varphi(n\mu)}\bigg) \\
&\quad = \exp\bigg(\sum_{i=1}^{n-1} \log\varphi(S_i) - \sum_{i=1}^{n-1}\log\varphi(i\mu)\bigg)\bigg(1 - \frac{1-\varphi(S_n)}{1-\varphi(n\mu)}\bigg) \\
&\quad \le \exp\bigg(\sum_{i\in[[1;n-1]]\setminus H_{\mathrm{u},n}} \log\varphi(i\mu + a_i\sqrt{i}) - \sum_{i=1}^{n-1}\log\varphi(i\mu)\bigg) \\
&\qquad \times \bigg(1 - \frac{1-\varphi(n\mu + a_n\sqrt{n})}{1-\varphi(n\mu)}\bigg)\mathbf{1}_{\{n\notin H_{\mathrm{u},n}\}} \\
&\qquad + \exp\bigg(\sum_{i\in[[1;n-1]]\setminus H_{\mathrm{u},n}} \log\varphi(i\mu + a_i\sqrt{i}) - \sum_{i=1}^{n-1}\log\varphi(i\mu)\bigg)\mathbf{1}_{\{n\in H_{\mathrm{u},n}\}} \\
&\quad \le \exp\bigg(\sum_{i=C_{\mathrm{u},n}+1}^{n-1} \log\varphi(i\mu + a_i\sqrt{i}) - \sum_{i=1}^{n-1}\log\varphi(i\mu)\bigg)\bigg(1 - \frac{1-\varphi(n\mu + a_n\sqrt{n})}{1-\varphi(n\mu)}\bigg) \\
&\qquad + \exp\bigg(-\sum_{i=1}^{n-1}\log\varphi(i\mu)\bigg)\mathbf{1}_{\{n\in H_{\mathrm{u},n}\}} \\
&\quad \le \exp\bigg(-\sum_{i=1}^{C_{\mathrm{u}}} \log\varphi(i\mu + a_i\sqrt{i})\bigg)\exp\bigg(\sum_{i=1}^{n-1}\log\varphi(i\mu + a_i\sqrt{i}) - \sum_{i=1}^{n-1}\log\varphi(i\mu)\bigg) \\
&\qquad \times \bigg(1 - \frac{1-\varphi(n\mu + a_n\sqrt{n})}{1-\varphi(n\mu)}\bigg) + \exp\bigg(-\sum_{i=1}^{n-1}\log\varphi(i\mu)\bigg)\mathbf{1}_{\{n\in H_{\mathrm{u},n}\}},
\end{aligned}$$

which provides an upper bound for the integrand. Note that this upper bound is stochastic because of C_{u} and the indicator function $\mathbf{1}_{\{n\in H_{\mathrm{u},n}\}}$.

Step 2: Lower bound for the integrand. Similarly, we have

$$
\begin{aligned}
&\biggl(\prod_{i=1}^{n-1}\frac{\varphi(S_i)}{\varphi(i\mu)}\biggr)\biggl(1-\frac{1-\varphi(S_n)}{1-\varphi(n\mu)}\biggr)\\
&\quad=\exp\biggl(\sum_{i=1}^{n-1}\log\varphi(S_i)-\sum_{i=1}^{n-1}\log\varphi(i\mu)\biggr)\biggl(1-\frac{1-\varphi(S_n)}{1-\varphi(n\mu)}\biggr)\\
&\quad\geq \underline{\varphi}^{C_{\mathrm{d},n}}\exp\biggl(\sum_{i\in[[1;n-1]]\setminus H_{\mathrm{d},n}}\log\varphi(i\mu-a_i\sqrt{i})-\sum_{i=1}^{n-1}\log\varphi(i\mu)\biggr)\\
&\qquad\times\biggl(1-\frac{1-\varphi(n\mu-a_n\sqrt{n})}{1-\varphi(n\mu)}\biggr)\mathbf{1}_{\{n\notin H_{\mathrm{d},n}\}}\\
&\qquad-\underline{\varphi}^{C_{\mathrm{d},n}}\exp\biggl(\sum_{i\in[[1;n-1]]\setminus H_{\mathrm{d},n}}\log\varphi(i\mu-a_i\sqrt{i})-\sum_{i=1}^{n-1}\log\varphi(i\mu)\biggr)\\
&\qquad\times(1-\underline{\varphi})(1-\varphi(n\mu))^{-1}\,\mathbf{1}_{\{n\in H_{\mathrm{d},n}\}}\\
&\quad\geq\exp(C_{\mathrm{d}}\log\underline{\varphi})\exp\biggl(\sum_{i=1}^{n-1}\log\varphi(i\mu-a_i\sqrt{i})-\sum_{i=1}^{n-1}\log\varphi(i\mu)\biggr)\\
&\qquad\times\biggl(1-\frac{1-\varphi(n\mu-a_n\sqrt{n})}{1-\varphi(n\mu)}\biggr)\mathbf{1}_{\{n\notin H_{\mathrm{d},n}\}}\\
&\qquad-\exp\biggl(-\sum_{i=1}^{n-1}\log\varphi(i\mu)\biggr)(1-\underline{\varphi})(1-\varphi(n\mu))^{-1}\,\mathbf{1}_{\{n\in H_{\mathrm{d},n}\}},
\end{aligned}
$$

which provides a lower bound for the integrand. Note that this lower bound is stochastic because of C_{d} and the indicator function $\mathbf{1}_{\{n\in H_{\mathrm{d},n}\}}$.

Step 3: Analysis of the deterministic components of the bound. Let us consider the sequences $(m_{1,n})$ and $(m_{2,n})$ defined by

$$
\begin{aligned}
m_{1,n}&=\exp\biggl(\sum_{i=1}^{n}\log\varphi(i\mu+a_i\sqrt{i})-\sum_{i=1}^{n}\log\varphi(i\mu)\biggr)\biggl(1-\frac{1-\varphi(n\mu+a_n\sqrt{n})}{1-\varphi(n\mu)}\biggr),\\
m_{2,n}&=\exp\biggl(\sum_{i=1}^{n}\log\varphi(i\mu-a_i\sqrt{i})-\sum_{i=1}^{n}\log\varphi(i\mu)\biggr)\biggl(1-\frac{1-\varphi(n\mu-a_n\sqrt{n})}{1-\varphi(n\mu)}\biggr).
\end{aligned}
$$

Let $\eta>0$. For large n, we have

$$
\begin{aligned}
0&\leq 1-\frac{1-\varphi(n\mu+a_n\sqrt{n})}{1-\varphi(n\mu)}\leq\frac{\tau+\eta}{\mu}\frac{a_n}{\sqrt{n}},\\
0&\geq 1-\frac{1-\varphi(n\mu-a_n\sqrt{n})}{1-\varphi(n\mu)}\geq-\frac{\tau+\eta}{\mu}\frac{a_n}{\sqrt{n}}.
\end{aligned}
$$

Moreover, for large n we have

$$\begin{aligned}|\log\varphi(i\mu+a_i\sqrt{i})-\log\varphi(i\mu)| &= (1-\varphi(i\mu))\left|1-\frac{1-\varphi(i\mu+a_i\sqrt{i})}{1-\varphi(i\mu)}\right|(1+o(1))\\ &\le \frac{(\tau+\eta)a_i}{\mu^{1+\tau}i^{1/2+\tau}}L(i\mu)(1+o(1))\\ &= \frac{\tau+\eta}{\mu^{1+\tau}i^{1+\nu}}L(i\mu)(1+o(1))\end{aligned}$$

and

$$\begin{aligned}|\log\varphi(i\mu-a_i\sqrt{i})-\log\varphi(i\mu)| &= (1-\varphi(i\mu))\left|1-\frac{1-\varphi(i\mu-a_i\sqrt{i})}{1-\varphi(i\mu)}\right|(1+o(1))\\ &\le \frac{(\tau+\eta)a_i}{\mu^{1+\tau}i^{1/2+\tau}}L(i\mu)(1+o(1))\\ &= \frac{\tau+\eta}{\mu^{1+\tau}i^{1+\nu}}L(i\mu)(1+o(1)).\end{aligned}$$

We deduce that $m_{1,n} \le O(a_n/\sqrt{n}) = O(n^{\tau-1-\nu})$ and that $m_{2,n} \ge O(a_n/\sqrt{n}) = O(n^{\tau-1-\nu})$.

Step 4: Analysis of the stochastic components of the upper bound. (i) Let us first prove that $\mathrm{E}[\exp(-\sum_{i=1}^{C_\mathrm{u}}\log\varphi(i\mu+a_i\sqrt{i}))] < \infty$. We have

$$\mathrm{E}\left[\exp\left(-\sum_{i=1}^{C_\mathrm{u}}\log\varphi(i\mu+a_i\sqrt{i})\right)\right] = \sum_{n=1}^{\infty}\mathrm{P}(C_\mathrm{u}=n)\exp\left(-\sum_{i=1}^{n}\log\varphi(i\mu+a_i\sqrt{i})\right).$$

On the one hand, there exists a constant B_1 such that if $\frac{2}{3} < \tau < 1$ then

$$\begin{aligned}\exp\left(-\sum_{i=1}^{n}\log\varphi(i\mu+a_i\sqrt{i})\right) &\le \exp\left(-\int_0^{n-1}\log\varphi(x\mu+x^{\tau-\nu})\,\mathrm{d}x\right)\\ &\le \exp(B_1L(n)n^{1-\tau}),\end{aligned}$$

by using Karamata's theorem (see, e.g. [5]), and that if $\tau = 1$ then

$$\exp\left(-\sum_{i=1}^{n}\log\varphi(i\mu+a_i\sqrt{i})\right) \le \exp\left(-\int_0^{n-1}\log\varphi(x\mu+x^{\tau-\nu})\,\mathrm{d}x\right) \le \exp(B_1n^{\nu}).$$

On the other hand, from Proposition 10, we have

$$\mathrm{P}(C_\mathrm{u}=n) \le \mathrm{P}(C_\mathrm{u}\ge n) \le \mathrm{P}\left(\bigcup_{i=n}^{\infty}\{S_i \ge i\mu+a_i\sqrt{i}\}\right) \le \sum_{i=n}^{\infty}\exp\left(-\frac{a_i^2(1+\zeta_{1,i})}{2\sigma^2}\right).$$

Since $a_i^2 = i^{2\tau-1-2\nu}$, there exists another constant, B_2, such that

$$\mathrm{P}(C_\mathrm{u}=n) \le \exp(-B_2n^{2\tau-1-2\nu}).$$

Then $\mathrm{E}[\exp(-\sum_{i=1}^{C_\mathrm{u}}\log\varphi(i\mu+a_i\sqrt{i}))] < \infty$ if $2\tau-1-2\nu > 1-\tau \Leftrightarrow 2\nu < -2+3\tau$ for $\frac{2}{3} < \tau < 1$ and if $1-2\nu > \nu \Leftrightarrow 2\nu < \frac{2}{3}$ for $\tau = 1$.

(ii) Let us now consider the second stochastic component of the upper bound. By the same arguments as previously, for $\frac{2}{3} < \tau < 1$ we have

$$\begin{aligned}\frac{\sqrt{n}}{a_n}\exp\biggl(-\sum_{i=1}^{n-1}\log\varphi(i\mu)\biggr)\mathrm{E}[\mathbf{1}_{\{n\in H_{\mathrm{u},n}\}}] &\le \frac{\sqrt{n}}{a_n}\exp(D_1L(n)n^{1-\tau})\,\mathrm{P}(S_n \ge n\mu + a_n\sqrt{n})\\ &= o(1),\end{aligned}$$

since $\sqrt{n}/a_n = n^{-(\tau-1-\nu)}$. The arguments for $\tau = 1$ run similarly. Then we deduce that, for large n,

$$\mathrm{E}\biggl[\biggl(\prod_{i=1}^{n-1}\frac{\varphi(S_i)}{\varphi(i\mu)}\biggr)\biggl(1-\frac{1-\varphi(S_n)}{1-\varphi(n\mu)}\biggr)\biggr] \le \text{const.}\times\frac{a_n}{\sqrt{n}}.$$

Step 5: Analysis of the stochastic components of the lower bound. (i) Let us first note that, for large n,

$$\begin{aligned}\mathrm{E}[\exp(C_{\mathrm{d}}\log\underline{\varphi})\,\mathbf{1}_{\{n\notin H_{\mathrm{d},n}\}}] &\ge \mathrm{E}[\exp(C_{\mathrm{d}}\log\underline{\varphi})\,\mathbf{1}_{\{1\notin H_{d,1}\}}]\\ &= \mathrm{E}[\exp(C_{\mathrm{d}}\log\underline{\varphi});\ \varepsilon_1 \le \mu+1].\end{aligned}$$

We have $\{\varepsilon_1 \le \mu+1\} \ne \varnothing$ and $\mathrm{P}(\varepsilon_1 \le \mu+1) > 0$. Moreover, C_{d} is a.s. finite and, thus,

$$\mathrm{E}[\exp(C_{\mathrm{d}}\log\underline{\varphi});\ \varepsilon_1 \le \mu+1] > 0.$$

(ii) The arguments for the second stochastic component of the lower bound are similar to the analogous arguments for the upper bound, and we deduce that

$$\frac{\sqrt{n}}{a_n(1-\varphi(n\mu))}\exp\biggl(-\sum_{i=1}^{n-1}\log\varphi(i\mu)\biggr)\mathrm{E}[\mathbf{1}_{\{n\in H_{\mathrm{d},n}\}}] = o(1),$$

since $(1-\varphi(n\mu))^{-1} = (n\mu)^{\tau}L(n\mu)$. Then we deduce that

$$\mathrm{E}\biggl[\biggl(\prod_{i=1}^{n-1}\frac{\varphi(S_i)}{\varphi(i\mu)}\biggr)\biggl(1-\frac{1-\varphi(S_n)}{1-\varphi(n\mu)}\biggr)\biggr] \ge -\text{const.}\times\frac{a_n}{\sqrt{n}}.$$

Step 6. By gathering the lower and upper bounds we obtain

$$\biggl|\mathrm{E}\biggl[\biggl(\prod_{i=1}^{n-1}\frac{\varphi(S_i)}{\varphi(i\mu)}\biggr)\biggl(1-\frac{1-\varphi(S_n)}{1-\varphi(n\mu)}\biggr)\biggr]\biggr| \le O\biggl(\frac{a_n}{\sqrt{n}}\biggr),$$

where $a_n/\sqrt{n} = n^{\tau-1-\nu}$, and, thus,

$$\begin{aligned}|q_n - q_{n-1}| &= \frac{1-\varphi(n\mu)}{\varphi(n\mu)}\biggl|\mathrm{E}\biggl[\biggl(\prod_{i=1}^{n-1}\frac{\varphi(S_i)}{\varphi(i\mu)}\biggr)\biggl(1-\frac{1-\varphi(S_n)}{1-\varphi(n\mu)}\biggr)\biggr]\biggr|\\ &\le O(L(n\mu)n^{-(1+\nu)})\\ &\le O(n^{-(1+\nu/2)}).\end{aligned}$$

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Therefore, the sequence $q_n = \mathrm{E}[\prod_{i=1}^n \varphi(S_i)/\varphi(i\mu)]$ converges. Moreover,

$$\liminf_{n\to\infty} q_n \geq \liminf_{n\to\infty} \exp\biggl(\sum_{i=1}^n \log\varphi(i\mu - a_i\sqrt{i}) - \sum_{i=1}^n \log\varphi(i\mu)\biggr)\mathrm{E}[\exp(C_\mathrm{d}\log\underline{\varphi})] > 0$$

(see steps 2–4), and the limit is not equal to 0, which provides the first equivalent to Proposition 4.

Let us now consider the case $\tau > 1$. Note that $\lim_{n\to\infty}\sum_{i=1}^n \log\varphi(i\mu) < \infty$. Let $(a_n)_{n\geq 1}$ be such that $a_n \to \infty$ and $a_n/\sqrt{n} \to 0$. Then

$$\begin{aligned} 1 \geq p_n &= \mathrm{E}\biggl[\prod_{i=1}^n \varphi(S_i)\biggr] \\ &\geq \mathrm{E}[\underline{\varphi}^{C_{\mathrm{d},n}}]\exp\biggl(\sum_{i=1}^{n-1}\log\varphi(i\mu - a_i\sqrt{i})\biggr) \\ &\geq \mathrm{E}[\exp(C_\mathrm{d}\log\underline{\varphi})]\exp\biggl(\sum_{i=1}^{n-1}\log\varphi(i\mu - a_i\sqrt{i})\biggr). \end{aligned}$$

By using the same arguments as previously (see step 5), it is easily seen that $\lim_{n\to\infty} p_n > 0$ and, therefore, that the sequence $(p_n)_{n\geq 1}$ converges to a constant.

4.2. Proofs of results in Section 3

Proof of Proposition 5. Since the distribution of X_0 is the stationary distribution, we can equivalently assume that the GC Markov chain is defined for any $t \in \mathbb{Z}$. We have

$$\begin{aligned} \mathrm{E}[g(X)] &= \sum_{n=0}^\infty \mathrm{E}[g(X_t)\,\mathbf{1}_{\{X_t>0,\,\dots,\,X_{t-n+1}>0,\,X_{t-n}=0\}}] \\ &= \sum_{n=0}^\infty \mathrm{E}[g(X_t);\ X_t > 0,\ \dots,\ X_{t-n+1} > 0,\ X_{t-n} = 0] \\ &= \mathrm{P}(X=0)\biggl(g(0) + \sum_{n=1}^\infty \mathrm{E}[g(X_t);\ X_t > 0,\ \dots,\ X_{t-n+1} > 0 \mid X_{t-n} = 0]\biggr) \\ &= \kappa\biggl(g(0) + \sum_{n=1}^\infty \mathrm{E}\biggl[g(S_n)\prod_{k=0}^{n-1}\varphi(S_k)\biggr]\biggr), \end{aligned}$$

where $S_0 = 0$.

Before giving the proof of Proposition 6, we first recall the definition of a function with bounded increase and a version of the Drasin–Shea theorem (see Theorem 5.2.3 of [5]).

Definition 5. Consider a function $h\colon (0,\infty) \to [0,\infty)$. The upper Matuszewska index $\alpha(h)$ is the infimum of those $\alpha \in \mathbb{R}$ for which there exists a constant $C(\alpha)$ such that, for each $\Lambda > 1$,

$$\frac{h(\lambda x)}{h(x)} \leq C(\alpha)(1+o(1))\lambda^\alpha \quad (x\to\infty) \text{ uniformly on } [1,\Lambda].$$

The function h is said to have bounded increase if $\alpha(h) < \infty$.

Note that nonnegative, ultimately decreasing functions have bounded increase.

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Theorem 1. (The Drasin–Shea theorem.) *Consider a function* $k\colon (0,\infty) \to \mathbb{R}$*. Let* (a, b) *be the maximal open interval (where* $a < 0$*) such that*

$$\check{k}(z) := \int_0^\infty t^{-z} \frac{k(t)}{t} \, \mathrm{d}t < \infty \quad \textit{for } a < \operatorname{Re} z < b.$$

If a *is finite then assume that* $\lim_{\delta \to 0^+} \check{k}(a + \delta) = \infty$*. If* b *is finite then assume that* $\lim_{\delta \to 0^+} \check{k}(b - \delta) = \infty$*. Let* $h\colon [0, \infty) \to [0, \infty)$ *be a continuous function with bounded increase. If*

$$\lim_{x\to\infty} \frac{\int_0^\infty k(x/t)h(t)\,\mathrm{d}t/t}{h(x)} = c > 0,$$

then there exists a scalar $\rho \in (a, b)$ *such that*

$$c = \check{k}(\rho), \qquad h(x) = x^\rho L(x),$$

where L *is a slowly varying function.*

Proof of Proposition 6. We first prove part (i). Let us define the functions $\tilde{\varphi}$, $\tilde{f}$, and $\tilde{l}$ from $\mathbb{R}$ to $[0, \infty)$ by

$$\tilde{\varphi}(x) = \mathbf{1}_{\{x\geq 0\}}\, \varphi(x), \qquad \tilde{f}(x) = \mathbf{1}_{\{x\geq 0\}}\, f(x), \qquad \tilde{l}(x) = \mathbf{1}_{\{x\geq 0\}}\, l(x),$$

and the functions $\hat{\varphi}$, $\hat{f}$, and $\hat{l}$ from $(0, \infty)$ to $[0, \infty)$ by

$$\hat{\varphi}(u) = \tilde{\varphi}(\log u), \qquad \hat{f}(u) = \tilde{f}(\log u), \qquad \hat{l}(u) = \tilde{l}(\log u).$$

From (2) we can see that

$$\hat{l}(v) = \kappa \hat{f}(v) + \int_0^\infty \hat{f}(v/u)\hat{\varphi}(u)\hat{l}(u)\frac{\mathrm{d}u}{u}. \tag{6}$$

Step 1: Limit of $f(x)/l(x)$ *as* x *tends to* ∞. Let $M > 0$. First note that, for $x > M$,

$$l(x) \geq f(x) \int_0^M \varphi(y) \frac{f(x-y)}{f(x)} l(y)\,\mathrm{d}y. \tag{7}$$

For $a = -\infty$, $f \circ \log$ is a rapidly varying function and, for all $y \in [0, M]$, $f(x-y)/f(x)$ diverges as x goes to ∞. We deduce from (7) that

$$\lim_{x\to\infty} \frac{f(x)}{l(x)} = 0.$$

For $a < \infty$, $f \circ \log$ is a regularly varying function with index $-a$. It follows that

$$\lim_{x\to\infty} \frac{f(x-y)}{f(x)} = \mathrm{e}^{ay}.$$

We now establish that the Laplace transform of the stationary distribution, $\mathrm{E}[\mathrm{e}^{-uX_t}]$, diverges when $u = -a$. Indeed, for $u < 0$ we have

$$\begin{aligned}
\mathrm{E}[\mathrm{e}^{-uX_t}] &= \mathrm{E}[\mathrm{E}[\mathrm{e}^{-uX_t} \mid X_{t-1}]] \\
&= \mathrm{E}[1 - \varphi(X_{t-1})] + \mathrm{E}[\varphi(X_{t-1})\mathrm{e}^{-uX_{t-1}}\mathrm{e}^{-u\varepsilon_t}] \\
&\geq \mathrm{E}[\varphi(X_{t-1})\mathrm{e}^{-uX_{t-1}}\mathrm{e}^{-u\varepsilon_t}] \\
&\geq \underline{\varphi}\,\mathrm{E}[\mathrm{e}^{-u\varepsilon_t}].
\end{aligned}$$

Since $\mathrm{E}[\mathrm{e}^{a\varepsilon_t}] = \infty$, we deduce that $\mathrm{E}[\mathrm{e}^{aX_t}] = \infty$. It follows that

$$\limsup_{x\to\infty} \frac{f(x)}{l(x)} \le \frac{1}{\int_0^M \varphi(y)\mathrm{e}^{ay} l(y)\,\mathrm{d}y} \quad \text{for any } M.$$

As M tends to ∞, we obtain

$$\lim_{x\to\infty} \frac{f(x)}{l(x)} = 0.$$

Step 2: Drasin–Shea theorem. Let $h(u) = \hat{\varphi}(u)\hat{l}(u)$. It follows from step 1 and (6) that

$$\lim_{v\to\infty} \frac{\int_0^\infty \hat{f}(v/u)h(u)\,\mathrm{d}u/u}{h(v)} = \frac{1}{\varphi_1}.$$

Let us define $k(t) = \hat{f}(t)$. We have $\check{k}(u) = \Psi_{\varepsilon_1}(u)$. Since the density of ε_1 is strictly positive, $b = \infty$ and $\lim_{u\to\infty} \check{k}(u) = 0$. Moreover, Ψ_{ε_1} is convex and continuous on (a, ∞). From the Drasin–Shea theorem, there exist a unique scalar γ such that $\Psi_\varepsilon(-\gamma) = \varphi_1^{-1}$ and a slowly varying function L such that

$$h(u) = u^{-\gamma} L(u).$$

Note that γ is necessarily positive. Therefore, for $u > 1$ we have

$$l(\log u) = \hat{l}(u) = (1 + o(1))\varphi_1^{-1} u^{-\gamma} L(u).$$

Thus,

$$\mathrm{P}(X > \log u) = \int_{\log u}^\infty l(v)\,\mathrm{d}v = \int_u^\infty l(\log s)\frac{\mathrm{d}s}{s} = \int_u^\infty \hat{l}(s)\frac{\mathrm{d}s}{s}$$

and by Karamata's theorem (see, e.g. [5]) we deduce that

$$\mathrm{P}(X > \log u) = (1 + o(1))\frac{\varphi_1^{-1}}{\gamma} u^{-\gamma} L(u).$$

Finally, we have

$$\mathrm{P}(X > x) = (1 + o(1))(\varphi_1\gamma)^{-1} L(\mathrm{e}^x)\mathrm{e}^{-\gamma x},$$

where

$$\lim_{x\to\infty} \frac{L(\mathrm{e}^{x+a})}{L(\mathrm{e}^x)} = 1 \quad \text{for all } a \in \mathbb{R}$$

since L is a slowly varying function.

We now prove part (ii) of the proposition. Let $v(x) = x^\alpha$ with $\frac{1}{2} < \alpha < 1$, and define $\underline{N}(x) := \lfloor (x - v(x))/\mu \rfloor$ and $\overline{N}(x) := \lfloor (x + v(x))/\mu \rfloor$. Let $\underline{w}(\underline{N}(x)) := x - \mu\underline{N}(x)$ and $\overline{w}(\overline{N}(x)) := \mu\overline{N}(x) - x$. It is easily seen that

$$\underline{w}(\underline{N}(x)) = \mu^\alpha \underline{N}(x)^\alpha(1 + o(1)), \qquad \overline{w}(\overline{N}(x)) = \mu^\alpha \overline{N}(x)^\alpha(1 + o(1)).$$

Recall that $S_0 = 0$ and $\varphi(0) = 1$. By using Proposition 5 with $g(s) = \mathbf{1}_{\{s>x\}}$, the stationary survival distribution function can be decomposed as

$$\begin{aligned}
\mathrm{P}(X > x) &= \mathrm{P}(X = 0) \sum_{n=1}^{\infty} \mathrm{E}\Bigg[\prod_{k=0}^{n-1} \varphi(S_k);\ S_n > x\Bigg] \\
&= \mathrm{P}(X = 0) \sum_{n=1}^{\underline{N}(x)} \mathrm{E}\Bigg[\prod_{k=0}^{n-1} \varphi(S_k);\ S_n > x\Bigg] \\
&\quad + \mathrm{P}(X = 0) \sum_{n=\underline{N}(x)+1}^{\lfloor x/\mu \rfloor} \mathrm{E}\Bigg[\prod_{k=0}^{n-1} \varphi(S_k);\ S_n > x\Bigg] \\
&\quad - \mathrm{P}(X = 0) \sum_{n=\lfloor x/\mu \rfloor+1}^{\overline{N}(x)-1} \mathrm{E}\Bigg[\prod_{k=0}^{n-1} \varphi(S_k);\ S_n \le x\Bigg] \\
&\quad - \mathrm{P}(X = 0) \sum_{n=\overline{N}(x)}^{\infty} \mathrm{E}\Bigg[\prod_{k=0}^{n-1} \varphi(S_k);\ S_n \le x\Bigg] \\
&\quad + \mathrm{P}(X = 0) \sum_{n=\lfloor x/\mu \rfloor}^{\infty} p_n \\
&=: T_1(x) + T_2(x) - T_3(x) - T_4(x) + \mathrm{P}(X = 0) \sum_{n=\lfloor x/\mu \rfloor}^{\infty} p_n.
\end{aligned}$$

In a first step we give a lower bound for $\sum_{n=\lfloor x/\mu \rfloor}^{\infty} p_n$. Then we explain how each term $T_i(x)$, $i = 1, \ldots, 4$, can be neglected with respect to $\sum_{n=\lfloor x/\mu \rfloor}^{\infty} p_n$. Let $0 < \eta < 3\tau - 2$.

Step 1: Lower bound for $\sum_{n=\lfloor x/\mu \rfloor}^{\infty} p_n$. By Proposition 4,

$$p_n = C \exp\Bigg(\sum_{i=1}^{n} \log \varphi(i\mu))(1 + o(1)\Bigg).$$

Thus, for large n,

$$p_n > \begin{cases} \exp(-n^{1-\tau+\eta}), & \frac{2}{3} < \tau < 1, \\ n^{-2c/\mu}, & \tau = 1. \end{cases}$$

For $\frac{2}{3} < \tau < 1$, we obtain a lower bound as follows:

$$\begin{aligned}
\sum_{n=\lfloor x/\mu \rfloor}^{\infty} p_n &> \int_{\lfloor x/\mu \rfloor}^{\infty} \exp(-u^{1-\tau+\eta})\,\mathrm{d}u \\
&= \int_{\lfloor x/\mu \rfloor}^{\infty} \frac{1}{(1-\tau+\eta)u^{-\tau+\eta}} (1-\tau+\eta)u^{-\tau+\eta} \exp(-u^{1-\tau+\eta})\,\mathrm{d}u \\
&= \frac{1}{1-\tau+\eta} \frac{x^{\tau-\eta}}{\mu^{\tau-\eta}} \exp\bigg(-\frac{x^{1-\tau+\eta}}{\mu^{1-\tau+\eta}}\bigg) - \frac{\tau-\eta}{1-\tau+\eta} \int_{\lfloor x/\mu \rfloor}^{\infty} u^{\tau-\eta-1} \exp(-u^{1-\tau+\eta})\,\mathrm{d}u \\
&> \frac{1}{2(1-\tau+\eta)} \frac{x^{\tau-\eta}}{\mu^{\tau-\eta}} \exp\bigg(-\frac{x^{1-\tau+\eta}}{\mu^{1-\tau+\eta}}\bigg).
\end{aligned}$$

For $\tau = 1$, we obtain a lower bound from

$$\sum_{n=\lfloor x/\mu \rfloor}^{\infty} p_n > \frac{\mu^{2c/\mu-1}}{(2c/\mu - 1)x^{2c/\mu-1}}.$$

Step 2: Analysis of $T_1(x)$. For $n \le \underline{N}(x)$, we have

$$\mathrm{E}\Bigg[\prod_{k=0}^{n-1} \varphi(S_k);\ S_n > x\Bigg] \le \mathrm{P}(S_n > x) = \mathrm{P}(S_n > \mu \underline{N}(x) + \underline{w}(\underline{N}(x))).$$

Let us define $K = \inf\{\mathrm{P}(S_n > n\mu) \colon n \ge 1\}$. We have $\mathrm{P}(S_1 > \mu) > 0$ and, by recursion, $\mathrm{P}(S_n > n\mu) > 0$. From the central limit theorem, we deduce that $\lim_{n\to\infty} \mathrm{P}(S_n > n\mu) = \frac{1}{2}$ and, hence, that K is positive. We have

$$\begin{aligned}
\mathrm{P}(S_{n+p} - \mu(n+p) > x) &\ge \mathrm{P}(S_p > p\mu,\ S_{n+p} - S_p > x + n\mu) \\
&= \mathrm{P}(S_p > p\mu)\,\mathrm{P}(S_n > x + n\mu) \\
&\ge K\,\mathrm{P}(S_n - n\mu > x)
\end{aligned}$$

for any $n \ge 1$, $p \ge 1$, and $x > 0$. Then it follows that, for $n \le \underline{N}(x)$,

$$\begin{aligned}
\mathrm{P}(S_n > x) &= \mathrm{P}(S_n - \mu n > \mu(\underline{N}(x) - n) + \underline{w}(\underline{N}(x))) \\
&\le K^{-1}\,\mathrm{P}(S_{\underline{N}(x)} - \mu \underline{N}(x) > \mu(\underline{N}(x) - n) + w(\underline{N}(x))) \\
&\le K^{-1}\,\mathrm{P}(S_{\underline{N}(x)} > \mu \underline{N}(x) + \underline{w}(\underline{N}(x))).
\end{aligned}$$

Let us define $a_n = \underline{w}(n)/\sqrt{n}$. Since $\frac{1}{2} < \alpha < 1$, we have $a_n \to \infty$ and $a_n/\sqrt{n} \to 0$ and can use Proposition 10. For any $n \le \underline{N}(x)$, we have

$$\mathrm{P}(S_n > x) \le K^{-1} \exp\bigg(-\frac{\underline{w}^2(\underline{N}(x))(1 + \zeta_{1,\underline{N}(x)})}{2\sigma^2 \underline{N}(x)}\bigg),$$

where $\zeta_{1,n}$ tends to 0 as n tends to ∞. Thus,

$$\begin{aligned}
T_1(x) &\le \mathrm{P}(X = 0) K^{-1} \underline{N}(x) \exp\bigg(-\frac{\underline{w}^2(\underline{N}(x))(1 + \zeta_{1,\underline{N}(x)})}{2\sigma^2 \underline{N}(x)}\bigg) \\
&\le 2\,\mathrm{P}(X = 0)(\mu K)^{-1} x \exp\bigg(-\frac{x^{2\alpha-1}}{(2\mu)^{2\alpha-2} 2^2 \sigma^2}\bigg),
\end{aligned}$$

since $x/2\mu < \underline{N}(x) < 2x/\mu$ for large x.

For $\frac{2}{3} < \tau < 1$, let us choose α such that $2\alpha - 1 > 1 - \tau + \eta$, that is, $1 - \tau/2 + \eta/2 < \alpha$. For $\tau = 1$, let us choose α such that $2\alpha - 1 > 0$. We then obtain

$$T_1(x) = o\bigg(\sum_{n=\lfloor x/\mu \rfloor}^{\infty} p_n\bigg).$$

Step 3: Analysis of $T_4(x)$. For any $n \geq \overline{N}(x)$ and sufficiently large x, we have

$$\begin{aligned}
\mathrm{E}\left[\prod_{k=0}^{n-1} \varphi(S_k);\ S_n \leq x\right] &\leq \mathrm{P}(S_n \leq x) \\
&= \mathrm{P}(S_n - \mu n \leq \mu(\overline{N}(x) - n) - (\overline{w}(\overline{N}(x)) - \overline{w}(n)) - \overline{w}(n)) \\
&\leq \mathrm{P}(S_n - \mu n \leq -\overline{w}(n)).
\end{aligned}$$

Let $a_n = \overline{w}(n)/\sqrt{n}$. Since $\frac{1}{2} < \alpha < 1$, we again have $a_n \to \infty$ and $a_n/\sqrt{n} \to 0$ and can use Proposition 10. For any $n \geq \overline{N}(x)$, we have

$$\mathrm{P}(S_n \leq x) \leq \exp\left(-\frac{\overline{w}^2(n)(1+\zeta_{2,n})}{2\sigma^2 n}\right),$$

where $\zeta_{2,n}$ tends to 0 as n tends to ∞. Thus, by integrating we obtain

$$\sum_{n=\overline{N}(x)}^{\infty} \mathrm{P}(S_n \leq x) \leq \text{const.} \times \overline{N}(x)^{2(1-\alpha)} \exp\left(-\frac{\overline{w}^2(\overline{N}(x))}{2\sigma^2 \overline{N}(x)}\right).$$

For $\frac{2}{3} < \tau < 1$, let us choose α such that $2\alpha - 1 > 1 - \tau + \eta$, that is, $1 - \tau/2 + \eta/2 < \alpha$. For $\tau = 1$, let us choose α such that $2\alpha - 1 > 0$. Then, as in step 2, we conclude that

$$T_4(x) = o\left(\sum_{n=\lfloor x/\mu \rfloor}^{\infty} p_n\right).$$

Step 4: Analysis of $T_2(x)$ *and* $T_3(x)$. We have

$$\begin{aligned}
T_2(x) + T_3(x) &\leq \mathrm{P}(X=0) \sum_{n=\underline{N}(x)}^{\overline{N}(x)-1} p_n \\
&\leq 2\,\mathrm{P}(X=0) v(x) \mu^{-1} p_{\underline{N}(x)} \\
&= 2\,\mathrm{P}(X=0) \mu^{-1} x^{\alpha} p_{\underline{N}(x)}.
\end{aligned}$$

From Proposition 4,

$$\begin{aligned}
1 &\leq \frac{p_{\underline{N}(x)}}{p_{\lfloor x/\mu \rfloor}} \\
&= (1+o(1)) \exp\left(-\sum_{i=\underline{N}(x)}^{\lfloor x/\mu \rfloor} \log \varphi(i\mu)\right) \\
&\leq (1+o(1)) \exp(\mu^{\tau-1} v(x) \underline{N}(x)^{-\tau} L(\underline{N}(x)\mu)).
\end{aligned}$$

For $\alpha < \tau$, we have $p_{\underline{N}(x)} = (1+o(1)) p_{\lfloor x/\mu \rfloor}$.

By using Karamata's theorem, it can be shown that, for $\frac{2}{3} < \tau < 1$,

$$p_{\lfloor x/\mu \rfloor} = (1 + o(1))x^{-\tau} L(x) \sum_{n=\lfloor x/\mu \rfloor}^{\infty} p_n.$$

For $\tau = 1$,

$$p_{\lfloor x/\mu \rfloor} = (1 + o(1))(c - \mu)x^{-1} \sum_{n=\lfloor x/\mu \rfloor}^{\infty} p_n.$$

We deduce that

$$v(x) p_{\underline{N}(x)} \leq \text{const.} \times x^{\alpha - \tau} L(x) \sum_{n=\lfloor x/\mu \rfloor}^{\infty} p_n.$$

Let us choose α such that $\alpha < \tau$; then we obtain

$$|T_2(x) - T_3(x)| \leq T_2(x) + T_3(x) = o\bigg(\sum_{n=\lfloor x/\mu \rfloor}^{\infty} p_n \bigg).$$

Step 5. Finally, for $\frac{2}{3} < \tau < 1$ let us choose α such that $1 - \tau/2 + \eta/2 < \alpha < \tau$, which is possible since $0 < \eta < 3\tau - 2$, and for $\tau = 1$ let us choose α such that $\frac{1}{2} < \alpha < 1$. The result then follows.

Proof of Proposition 7. Let us first recall that

$$\operatorname{cov}(\varphi(X_t), \varphi(X_{t+k})) = \mathrm{E}[\varphi(X_t)\varphi(X_{t+k})] - \mathrm{E}[\varphi(X_t)] \, \mathrm{E}[\varphi(X_{t+k})].$$

By using Proposition 5 with $g = \varphi$, we have

$$\mathrm{E}[\varphi(X_t)] = \mathrm{E}[\varphi(X_{t+k})] = \kappa \bigg(1 + \sum_{n=1}^{\infty} p_n \bigg) = \kappa \sum_{n=0}^{\infty} p_n = \kappa(\mathrm{E}[D_1] - 1) = 1 - \kappa.$$

Now note that

$$\mathrm{E}[\varphi(X_t)\varphi(X_{t+k})] = \mathrm{E}[\varphi(X_t) \, \mathrm{E}[\varphi(X_{t+k}) \mid X_t]]$$

and

$$\begin{aligned} &\mathrm{E}[\varphi(X_{t+k}) \mid X_t = x_t] \\ &\quad = \kappa \sum_{n=0}^{k-1} p_n + \mathrm{E}[\varphi(X_t + S_1) \cdots \varphi(X_t + S_k);\ S_1 + X_t > 0,\ \ldots,\ S_k + X_t > 0 \mid X_t = x_t]. \end{aligned}$$

It follows that

$$\begin{aligned} &\mathrm{E}[\varphi(X_t)\varphi(X_{t+k})] \\ &\qquad = \kappa \sum_{n=0}^{k-1} p_n \, \mathrm{E}[\varphi(X_t)] \\ &\qquad\quad + \mathrm{E}[\varphi(X_t)\varphi(X_t + S_1) \cdots \varphi(X_t + S_k);\ S_1 + X_t > 0,\ \ldots,\ S_k + X_t > 0] \\ &\qquad = \kappa \sum_{n=0}^{k-1} p_n \, \mathrm{E}[\varphi(X_t)] + \kappa \sum_{n=k}^{\infty} p_n. \end{aligned}$$

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We deduce that

$$\operatorname{cov}(\varphi(X_t), \varphi(X_{t+k})) = \kappa^2 \sum_{n=k}^{\infty} p_n,$$

which completes the proof.

Proof of Proposition 8. First note that

$$\begin{aligned}
\mathrm{P}(A_1 > x) &= \sum_{n=0}^{\infty} \mathrm{E}[\mathbf{1}_{\{X_t > x\}};\ X_t > 0,\ \dots,\ X_{t-n+1} > 0,\ X_{t-n} = 0 \mid X_{t+1} = 0] \\
&= \frac{1}{\mathrm{P}(X = 0)} \sum_{n=0}^{\infty} \mathrm{E}[\mathbf{1}_{\{X_t > x\}};\ X_{t+1} = 0,\ X_t > 0,\ \dots,\ X_{t-n+1} > 0,\ X_{t-n} = 0] \\
&= \sum_{n=0}^{\infty} \mathrm{E}[\mathbf{1}_{\{X_t > x\}};\ X_{t+1} = 0,\ X_t > 0,\ \dots,\ X_{t-n+1} > 0 \mid X_{t-n} = 0] \\
&= \sum_{n=1}^{\infty} \mathrm{E}\Big[(1 - \varphi(S_n)) \prod_{k=0}^{n-1} \varphi(S_k);\ S_n > x\Big].
\end{aligned}$$

We now prove part (i). Since φ is a nondecreasing function,

$$(1 - \varphi_1)\kappa^{-1}\,\mathrm{P}(X > x) \le \mathrm{P}(A_1 > x) \le (1 - \varphi(x))\kappa^{-1}\,\mathrm{P}(X > x)$$

and the result follows.

To prove part (ii), as in the proof of Proposition 6(ii) let us define $v(x) = x^{\alpha}$ with $\frac{1}{2} < \alpha < 1$, $\underline{N}(x) := \lfloor (x - v(x))/\mu \rfloor$, $\overline{N}(x) := \lfloor (x + v(x))/\mu \rfloor$, $\underline{w}(\underline{N}(x)) := x - \mu \underline{N}(x)$, and $\overline{w}(\overline{N}(x)) := \mu \overline{N}(x) - x$. We have

$$\begin{aligned}
\mathrm{P}(A_1 > x) &= \sum_{n=1}^{\infty} \mathrm{E}\Big[(1 - \varphi(S_n)) \prod_{k=0}^{n-1} \varphi(S_k);\ S_n > x\Big] \\
&= \sum_{n=1}^{\underline{N}(x)} \mathrm{E}\Big[(1 - \varphi(S_n)) \prod_{k=0}^{n-1} \varphi(S_k);\ S_n > x\Big] \\
&\quad + \sum_{n=\underline{N}(x)+1}^{\lfloor x/\mu \rfloor} \mathrm{E}\Big[(1 - \varphi(S_n)) \prod_{k=0}^{n-1} \varphi(S_k);\ S_n > x\Big] \\
&\quad - \sum_{n=\lfloor x/\mu \rfloor+1}^{\overline{N}(x)-1} \mathrm{E}\Big[(1 - \varphi(S_n)) \prod_{k=0}^{n-1} \varphi(S_k);\ S_n \le x\Big] \\
&\quad - \sum_{n=\overline{N}(x)}^{\infty} \mathrm{E}\Big[(1 - \varphi(S_n)) \prod_{k=0}^{n-1} \varphi(S_k);\ S_n \le x\Big] + p_{\lfloor x/\mu \rfloor} \\
&=: T_1(x) + T_2(x) - T_3(x) - T_4(x) + p_{\lfloor x/\mu \rfloor}.
\end{aligned}$$

In a first step we give a lower bound for $p_{\lfloor x/\mu \rfloor}$. Then we explain how each term $T_i(x)$, $i = 1, \dots, 4$, can be neglected with respect to $p_{\lfloor x/\mu \rfloor}$. Let $0 < \eta < 3\tau - 2$.

Step 1: Lower bound for $p_{\lfloor x/\mu \rfloor}$. From Proposition 4,

$$p_n = C \exp\biggl(\sum_{i=1}^{n} \log \varphi(i\mu)\biggr)(1 + o(1)).$$

Thus, for large n,

$$p_n > \begin{cases} \exp(-n^{1-\tau+\eta}), & \frac{2}{3} < \tau < 1, \\ n^{-2c/\mu}, & \tau = 1. \end{cases}$$

Step 2: Analysis of $T_1(x)$. For $n \le \underline{N}(x)$, we have

$$\begin{aligned} \mathrm{E}\biggl[(1-\varphi(S_n))\prod_{k=0}^{n-1}\varphi(S_k);\ S_n > x\biggr] &\le \mathrm{E}\biggl[\prod_{k=0}^{n-1}\varphi(S_k);\ S_n > x\biggr] \\ &\le \mathrm{P}(S_n > x) \\ &= \mathrm{P}(S_n > \mu\underline{N}(x) + \underline{w}(\underline{N}(x))) \\ &\le 2(\mu K)^{-1} x \exp\biggl(-\frac{x^{2\alpha-1}}{(2\mu)^{2\alpha-2}2^2\sigma^2}\biggr). \end{aligned}$$

As in the proof of Proposition 6(ii), we can show that

$$T_1(x) \le 2(\mu K)^{-1} x \exp\biggl(-\frac{x^{2\alpha-1}}{(2\mu)^{2\alpha-2}2^2\sigma^2}\biggr),$$

since $x/2\mu < \underline{N}(x) < 2x/\mu$ for large x.

For $\frac{2}{3} < \tau < 1$, let us choose α such that $2\alpha - 1 > 1 - \tau + \eta$, that is, $1 - \tau/2 + \eta/2 < \alpha$. For $\tau = 1$, let us choose α such that $2\alpha - 1 > 0$. We then obtain

$$T_1(x) = o(p_{\lfloor x/\mu \rfloor}).$$

Step 3: Analysis of $T_4(x)$. For any $n \ge \overline{N}(x)$ and sufficiently large x, we have

$$\begin{aligned} \mathrm{E}\biggl[(1-\varphi(S_n))\prod_{k=0}^{n-1}\varphi(S_k);\ S_n \le x\biggr] &\le \mathrm{E}\biggl[\prod_{k=0}^{n-1}\varphi(S_k);\ S_n \le x\biggr] \\ &\le \mathrm{P}(S_n \le x) \\ &= \mathrm{P}(S_n - \mu n \le \mu(\overline{N}(x) - n) \\ &\qquad - (\overline{w}(\overline{N}(x)) - \overline{w}(n)) - \overline{w}(n)) \\ &\le \mathrm{P}(S_n - \mu n \le -\overline{w}(n)). \end{aligned}$$

As in the proof of Proposition 6(ii), we can show that

$$T_4(x) \le \text{const.} \times \overline{N}(x)^{2(1-\alpha)} \exp\biggl(-\frac{\overline{w}^2(\overline{N}(x))}{2\sigma^2\overline{N}(x)}\biggr).$$

For $\frac{2}{3} < \tau < 1$, let us choose α such that $2\alpha - 1 > 1 - \tau + \eta$, that is, $1 - \tau/2 + \eta/2 < \alpha$. For $\tau = 1$, let us choose α such that $2\alpha - 1 > 0$. Then, as in step 2, we conclude that

$$T_4(x) = o(p_{\lfloor x/\mu \rfloor}).$$

Step 4: Analysis of $T_2(x)$ and $T_3(x)$. We have

$$\begin{aligned}|T_2(x) - T_3(x)| &\leq T_2(x) + T_3(x)\\ &\leq \sum_{n=\underline{N}(x)}^{\overline{N}(x)-1} (p_n - p_{n+1})\\ &\leq p_{\underline{N}(x)} - p_{\overline{N}(x)}.\end{aligned}$$

For $\alpha < \tau$,

$$\begin{aligned}p_{\underline{N}(x)} &= (1 + o(1))p_{\lfloor x/\mu \rfloor},\\ p_{\overline{N}(x)} &= (1 + o(1))p_{\lfloor x/\mu \rfloor},\end{aligned}$$

and we obtain

$$|T_2(x) - T_3(x)| = o(p_{\lfloor x/\mu \rfloor}).$$

Step 5. Finally, for $\frac{2}{3} < \tau < 1$ let us choose α such that $1 - \tau/2 + \eta/2 < \alpha < \tau$, which is possible since $0 < \eta < 3\tau - 2$, and for $\tau = 1$ let us choose α such that $\frac{1}{2} < \alpha < 1$. The result then follows.

Proof of Proposition 9. Recall that the distribution of X_0 is the stationary distribution. Since $\operatorname{var}(X_t) = \sigma_X^2$ is finite, we can discuss the asymptotic behavior of

$$\sum_{k=0}^{n-1} |\rho(k)|.$$

We have

$$\sum_{k=0}^{n-1} |\rho(k)| > \sum_{k=0}^{n-1} \frac{n-k}{n} |\rho(k)| > \frac{1}{2\sigma_X^2 n} \operatorname{var}\biggl(\sum_{t=0}^{n-1} X_t\biggr).$$

Let $N_n = \sup\{j : T_j \leq n - 1\}$; then

$$\begin{aligned}\frac{1}{n} \operatorname{var}\biggl(\sum_{t=0}^{n-1} X_t\biggr) &= \frac{1}{n} \operatorname{var}\biggl(\sum_{t=0}^{T_1-1} X_t + \sum_{j=1}^{N_n-1} \sum_{t=T_j}^{T_{j+1}-1} X_t + \sum_{t=T_{N_n}}^{n-1} X_t\biggr)\\ &\geq \frac{1}{n} \operatorname{var}\biggl(\sum_{j=1}^{N_n-1} \sum_{t=T_j}^{T_{j+1}-1} X_t\biggr),\end{aligned}$$

where the inequality has been deduced from the independence of the cycles $(X_t)_{T_j \leq t \leq T_{j+1}-1}$, $j \geq 0$. If

$$\sigma_C^2 = \operatorname{var}\biggl(\sum_{t=T_1}^{T_2-1} X_t - \frac{\mathrm{E}[\sum_{t=T_1}^{T_2-1} X_t]}{\mathrm{E}[T_2 - T_1]}(T_2 - T_1)\biggr) < \infty,$$

then

$$\lim_{n\to\infty}\frac{1}{n}\operatorname{var}\biggl(\sum_{j=1}^{N_n-1}\sum_{t=T_j}^{T_{j+1}-1}X_t\biggr)=\frac{1}{\mathrm{E}[T_2-T_1]}\sigma_C^2$$

(see Theorem 3.2 of [1]). If $\sigma_C^2=\infty$ then it follows easily by monotonicity that

$$\lim_{n\to\infty}\frac{1}{n}\operatorname{var}\biggl(\sum_{j=1}^{N_n-1}\sum_{t=T_j}^{T_{j+1}-1}X_t\biggr)=\infty.$$

Therefore, we obtain

$$\lim_{n\to\infty}\sum_{k=0}^{n-1}|\rho(k)|\geq\frac{1}{2\sigma_X^2\,\mathrm{E}[T_2-T_1]}\operatorname{var}\biggl(\sum_{t=T_1}^{T_2-1}X_t-\frac{\mathrm{E}[\sum_{t=T_1}^{T_2-1}X_t]}{\mathrm{E}[T_2-T_1]}(T_2-T_1)\biggr).$$

Since

$$\mathrm{E}\biggl[\sum_{t=T_1}^{T_2-1}X_t\biggr]=\mathrm{E}[X]\,\mathrm{E}[T_2-T_1],$$

we have

$$\begin{aligned}\operatorname{var}\biggl(\sum_{t=T_1}^{T_2-1}X_t-\frac{\mathrm{E}[\sum_{t=T_1}^{T_2-1}X_t]}{\mathrm{E}[T_2-T_1]}(T_2-T_1)\biggr)&=\operatorname{var}\biggl(\sum_{t=0}^{D_1-1}X_{t+T_1}-\mathrm{E}[X]D_1\biggr)\\&\geq\operatorname{var}\biggl(\mathrm{E}\biggl[\sum_{t=0}^{D_1-1}X_{t+T_1}-\mathrm{E}[X]D_1\biggm| D_1\biggr]\biggr)\\&=\mathrm{E}\biggl[\mathrm{E}\biggl[\sum_{t=0}^{D_1-1}X_{t+T_1}-\mathrm{E}[X]D_1\biggm| D_1\biggr]\biggr]^2\\&=:\mathrm{E}[g(D_1)^2],\end{aligned}$$

where

$$\begin{aligned}g(n)&=\mathrm{E}\biggl[\sum_{t=0}^{n-1}(S_{t+T_1}-S_{T_1-1})-\mathrm{E}[X]n\biggm| D_1=n\biggr]\\&=\frac{\mathrm{E}[\sum_{i=1}^{n}S_i\,\mathbf{1}_{\{D_1=n\}}]}{\mathrm{P}(D_1=n)}-\mathrm{E}[X]n\\&=\frac{\mathrm{E}[(\sum_{i=1}^{n}S_i)(1-\varphi(S_n))\prod_{i=0}^{n-1}\varphi(S_i)]}{\mathrm{E}[(1-\varphi(S_n))\prod_{i=0}^{n-1}\varphi(S_i)]}-\mathrm{E}[X]n\\&=:h(n)-\mathrm{E}[X]n.\end{aligned}$$

We now look for an asymptotic lower bound for g. Let

$$a_n=n^{\tau-1/2-\nu}\quad\text{with }0<2\nu<\min\bigl(-2+3\tau,\tfrac{2}{3}\bigr).$$

We use the same notation as in the proof of Proposition 4:

$$H_{\mathrm{u},n} = \{i \le n\colon S_i \ge i\mu + a_i\sqrt{i}\}, \qquad C_{\mathrm{u},n} = \mathrm{card}(H_{\mathrm{u},n}), \qquad C_{\mathrm{u}} = C_{\mathrm{u},\infty},$$
$$H_{\mathrm{d},n} = \{i \le n\colon 0 < S_i \le i\mu - a_i\sqrt{i}\}, \qquad C_{\mathrm{d},n} = \mathrm{card}(H_{\mathrm{d},n}), \qquad C_{\mathrm{d}} = C_{\mathrm{d},\infty}.$$

Step 1: Lower bound for the numerator of h. We have

$$\begin{aligned}
&\Big(\sum_{i=1}^{n} S_i\Big)(1-\varphi(S_n))\prod_{i=0}^{n-1}\varphi(S_i)\\
&\quad\ge \Bigg[\sum_{i=1}^{n-C_{\mathrm{d},n}}(i\mu - a_i\sqrt{i})\Bigg]\underline{\varphi}^{C_{\mathrm{d},n}}\prod_{i=0}^{n-1}\varphi(i\mu - a_i\sqrt{i})(1-\varphi(n\mu + a_n\sqrt{n})\,\mathbf{1}_{\{S_n<n\mu+a_n\sqrt{n}\}})\\
&\qquad - \Bigg[\sum_{i=1}^{n-C_{\mathrm{d},n}}(i\mu - a_i\sqrt{i})\Bigg]\underline{\varphi}^{C_{\mathrm{d},n}}\prod_{i=0}^{n-1}\varphi(i\mu - a_i\sqrt{i})\,\mathbf{1}_{\{S_n\ge n\mu+a_n\sqrt{n}\}}\\
&\quad\ge \Bigg[\sum_{i=1}^{n-C_{\mathrm{d},n}}(i\mu - a_i\sqrt{i})\Bigg]\underline{\varphi}^{C_{\mathrm{d},n}}\prod_{i=0}^{n-1}\varphi(i\mu - a_i\sqrt{i})(1-\varphi(n\mu + a_n\sqrt{n}))\\
&\qquad - 2\Bigg[\sum_{i=1}^{n}(i\mu - a_i\sqrt{i})\Bigg]\prod_{i=0}^{n-1}\varphi(i\mu - a_i\sqrt{i})\,\mathbf{1}_{\{S_n\ge n\mu+a_n\sqrt{n}\}}\\
&\quad\ge \Bigg[\sum_{i=1}^{n}(i\mu - a_i\sqrt{i})\Bigg]\underline{\varphi}^{C_{\mathrm{d}}}\prod_{i=0}^{n-1}\varphi(i\mu - a_i\sqrt{i})(1-\varphi(n\mu + a_n\sqrt{n}))\\
&\qquad - C_{\mathrm{d}}(n\mu - a_n\sqrt{n})\prod_{i=0}^{n-1}\varphi(i\mu - a_i\sqrt{i})(1-\varphi(n\mu + a_n\sqrt{n}))\\
&\qquad - 2\Bigg[\sum_{i=1}^{n}(i\mu - a_i\sqrt{i})\Bigg]\prod_{i=0}^{n-1}\varphi(i\mu - a_i\sqrt{i})\,\mathbf{1}_{\{S_n\ge n\mu+a_n\sqrt{n}\}}.
\end{aligned}$$

Thus, by using the same arguments as for Proposition 4,

$$\begin{aligned}
&\mathrm{E}\Bigg[\Big(\sum_{i=1}^{n} S_i\Big)(1-\varphi(S_n))\prod_{i=0}^{n-1}\varphi(S_i)\Bigg]\\
&\quad\ge \mathrm{E}[\underline{\varphi}^{C_{\mathrm{d}}}]\Bigg[\sum_{i=1}^{n}(i\mu - a_i\sqrt{i})\Bigg]\prod_{i=0}^{n-1}\varphi(i\mu - a_i\sqrt{i})(1-\varphi(n\mu + a_n\sqrt{n}))\\
&\qquad - \mathrm{E}[C_{\mathrm{d}}]n\mu\prod_{i=0}^{n-1}\varphi(i\mu - a_i\sqrt{i})(1-\varphi(n\mu + a_n\sqrt{n}))\\
&\qquad - 2\Bigg[\sum_{i=1}^{n}(i\mu - a_i\sqrt{i})\Bigg]\prod_{i=0}^{n-1}\varphi(i\mu - a_i\sqrt{i})\,\mathrm{P}(S_n \ge n\mu + a_n\sqrt{n})\\
&\quad= \mathrm{E}[\underline{\varphi}^{C_{\mathrm{d}}}]\Bigg[\sum_{i=1}^{n}(i\mu - a_i\sqrt{i})\Bigg]\prod_{i=0}^{n-1}\varphi(i\mu - a_i\sqrt{i})(1-\varphi(n\mu + a_n\sqrt{n}))(1+o(1)).
\end{aligned}$$

Step 2: Upper bound for the denominator of h. We have

$$
\begin{aligned}
(1-\varphi(S_n))\prod_{i=0}^{n-1}\varphi(S_i) &\le (1-\varphi(n\mu - a_n\sqrt{n})\,\mathbf{1}_{\{S_n\ge n\mu - a_n\sqrt{n}\}})\prod_{i=C_{\mathrm{u},n}}^{n-1}\varphi(i\mu + a_i\sqrt{i}) \\
&\quad + \prod_{i=C_{\mathrm{u},n}}^{n-1}\varphi(i\mu + a_i\sqrt{i})\,\mathbf{1}_{\{S_n< n\mu - a_n\sqrt{n}\}} \\
&\le \biggl(\prod_{i=1}^{C_\mathrm{u}}\varphi(i\mu + a_i\sqrt{i})\biggr)^{-1}\prod_{i=1}^{n-1}\varphi(i\mu + a_i\sqrt{i})(1-\varphi(n\mu - a_n\sqrt{n})) \\
&\quad + \biggl(\prod_{i=1}^{C_\mathrm{u}}\varphi(i\mu + a_i\sqrt{i})\biggr)^{-1}\prod_{i=1}^{n-1}\varphi(i\mu + a_i\sqrt{i})\times 2\,\mathbf{1}_{\{S_n< n\mu - a_n\sqrt{n}\}}.
\end{aligned}
$$

Thus, by using the same arguments as for Proposition 4,

$$
\begin{aligned}
&\mathrm{E}\biggl[(1-\varphi(S_n))\prod_{i=0}^{n-1}\varphi(S_i)\biggr] \\
&\qquad\le \mathrm{E}\biggl[\biggl(\prod_{i=1}^{C_\mathrm{u}}\varphi(i\mu + a_i\sqrt{i})\biggr)^{-1}\biggr]\prod_{i=1}^{n-1}\varphi(i\mu + a_i\sqrt{i})(1-\varphi(n\mu - a_n\sqrt{n}))(1+o(1)).
\end{aligned}
$$

Step 3: Asymptotic lower bound for g. We deduce from step 1 and step 2 that

$$
\begin{aligned}
h(n) &\ge \biggl(\sum_{i=1}^{n}(i\mu - a_i\sqrt{i})\biggr)\frac{(1+o(1))\,\mathrm{E}[\underline{\varphi}^{C_\mathrm{d}}]}{\mathrm{E}[(\prod_{i=1}^{C_\mathrm{u}}\varphi(i\mu - a_i\sqrt{i}))^{-1}]} \\
&\quad\times\frac{\prod_{i=0}^{n-1}\varphi(i\mu - a_i\sqrt{i})(1-\varphi(n\mu + a_n\sqrt{n}))}{\prod_{i=1}^{n-1}\varphi(i\mu + a_i\sqrt{i})(1-\varphi(n\mu - a_n\sqrt{n}))}.
\end{aligned}
$$

By using the same arguments as in step 3 of the proof of Proposition 4, we deduce that $h(n)\ge O(n^2)$ and that there exists a constant B_g such that

$$g(n)\ge B_g n^2, \qquad n\ge 1.$$

The result then follows.

Proof of Corollary 1. Since $c>3\mu$, $|\rho(k)|<\infty$ for each k. Moreover, if $c<4\mu$ then, by Propositions 4 and 6 and Example 1(ii), $\mathrm{E}[D_1^4]=\infty$, and, by Proposition 9, $\sum_{k=0}^{\infty}|\rho(k)|=\infty$.

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