Appendix F

Summary of cosmological equations

Assuming an isotropic and homogeneous universe, the cosmological line element is the Friedman-Robertson-Walker line element, and can be written as:

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{F.1}$$

The function $a(t)$ is known as the scale factor. k is a parameter that is -1 for a hyperbolic or negatively curved universe, 0 for a flat universe, and $+1$ for a positively curved universe.

The equations of motion for a are derived from Einstein's equations assuming that the universe is filled with one or more fluids[1] with total energy density ρ and pressure p. Then:

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \tag{F.2}$$

$$\ddot{a} = -\frac{4\pi G}{3}(\rho + 3p)a \tag{F.3}$$

and energy-momentum conservation gives:

$$\frac{d}{da}(\rho a^3) = -3pa^2 \tag{F.4}$$

In addition, we need the equation of state for the fluid to connect p and ρ. Some examples of equations of state are: $p = -\rho$ (cosmological constant), $p = 0$ (dust), $p = \rho/3$ (radiation), and $p = -2\rho/3$ (slowly evolving wall network). Note that ρ may contain contributions from a large variety of forms of matter and then the corresponding energy densities and pressures must be added together.

[1] The fluid approximation means that the relaxation time of the various components – which in our case may be plasma, gas, stars, galaxies, walls, fundamental particles – is much shorter than the characteristic time for changes in the scale factor.

Assuming a single dominant component of energy density in a flat universe ($k = 0$), Eqs. (F.2) and (F.4) can be solved to obtain:

$$\begin{aligned} p &= -\rho \rightarrow a \propto \mathrm{e}^{Ht}, \quad \rho \propto a^0 \\ p &= 0 \rightarrow a \propto t^{2/3}, \quad \rho \propto \frac{1}{a^3} \\ p &= \frac{\rho}{3} \rightarrow a \propto t^{1/2}, \quad \rho \propto \frac{1}{a^4} \\ p &= -\frac{2}{3}\rho \rightarrow a \propto t^2, \quad \rho \propto \frac{1}{a} \end{aligned} \tag{F.5}$$

$$p = w\rho \rightarrow a \propto t^{2/3(w+1)}, \quad \rho \propto a^{-3(w+1)} \tag{F.6}$$

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