BULL. AUSTRAL. MATH. SOC. 06A55
VOL. 11 (1974), 157-158.

Abstracts of Australasian PhD theses

The problem of adjoining roots to ordered groups

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A long-standing open question in ordered group theory, first proposed by B.H. Neumann, is the following:-

Can an orderable group (O-group) be embedded in a divisible O-group?

This thesis, alas, contains no complete answer to this question. However, the sufficient conditions presented make something of a dint in the (to date) impregnable armour protecting this most vexing problem.

An *O-group* is a group, G, which can be linearly ordered in such a way that if $a \leq b$, then $ac \leq bc$ and $ca \leq cb$ (for a, b, c in G). A group, G, is *divisible* if for all g in G and integers $n > 0$, the equation $x^n = g$ has a (not necessarily unique) solution for x in G.

Chapter 1 contains an example of an O-group in which "long" commutators misbehave (in a certain sense). It is known (see Neumann [3]) that for elements a and b of an O-group, G, and integers $m \neq 0$:-

$$[a^m, b] = 1 \text{ implies } [a, b] = 1 ,$$

and

$$[a^m, b, a] = 1 \text{ implies } [a, b, a] = 1 .$$

Given an arbitrary, fixed subset T of the (strictly) positive integers, we construct an O-group, G, possessing elements g and h such that

Received 9 February 1974. Thesis submitted to the Australian National University, February 1974. Degree approved, June 1974. Supervisor: Professor B.H. Neumann.

$$[g^m, h, g, g, \ldots, g] \neq 1 \text{ if } m \in T ,$$
$$= 1 \text{ if } m \notin T ,$$

where the commutator has weight greater than, or equal to, 4 .

The embedding theorem (Theorem 3) of Chapter 2 generalizes theorems of Kopytov [2] and Conrad [1]. Theorem 3 says, in effect, that we can suitably ajoin roots to an element, a , of an O-group, G , if a lies in a normal, abelian subgroup of G . That such a weak condition is (apparently) the strongest known sufficient condition for adjoining roots to an element of an O-group, admirably demonstrates the difficulty of B.H. Neumann's question.

In Chapters 3 and 4 we turn our attention to the metabelian case. Even this case is proving obdurate, although, at the time of writing, is showing signs of weakening. Let G be a (non-abelian) metabelian O-group with A the isolated closure of the derived group of G . Then A is a normal, abelian proper subgroup of G , and the best we can say at this stage is that G can be embedded in a divisible, metabelian O-group if G/A is divisible. An example in Chapter 4 shows that if G always has a divisible, metabelian, orderable extension, then a minimal such extension need not be unique (up to isomorphism).

Note added in proof, 22 July 1974. A.I. Kokorin has informed the author that V.V. Bludov and N.Ya. Medvedev have shown that a metabelian O-group can be embedded in a divisible O-group.

References

[1] Paul Conrad, "Extensions of ordered groups", *Proc. Amer. Math. Soc.* 6 (1955), 516-528.

[2] В.М. Копытов [V.M. Kopytov], "О пополнении центра упорядоченной группы" [On the completion of the centre of an ordered group], *Ural. Gos. Univ. Mat. Zap.* 4 (1963), 20-24.

[3] B.H. Neumann, "On ordered groups", *Amer. J. Math.* 71 (1949), 1-18.