

EXTREMALLY RICH C^* -CROSSED PRODUCTS AND THE CANCELLATION PROPERTY

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Abstract

A unital C^* -algebra A is called extremally rich if the set of quasi-invertible elements $A^{-1} \text{ex}(A)A^{-1}$ ($= A_q^{-1}$) is dense in A , where $\text{ex}(A)$ is the set of extreme points in the closed unit ball A_1 of A . In [7, 8] Brown and Pedersen introduced this notion and showed that A is extremally rich if and only if $\text{conv}(\text{ex}(A)) = A_1$. Any unital simple C^* -algebra with extremal richness is either purely infinite or has stable rank one ($\text{sr}(A) = 1$). In this note we investigate the extremal richness of C^* -crossed products of extremally rich C^* -algebras by finite groups. It is shown that if A is purely infinite simple and unital then $A \rtimes_\alpha G$ is extremally rich for any finite group G . But this is not true in general when G is an infinite discrete group. If A is simple with $\text{sr}(A) = 1$, and has the SP-property, then it is shown that any crossed product $A \rtimes_\alpha G$ by a finite abelian group G has cancellation. Moreover if this crossed product has real rank zero, it has stable rank one and hence is extremally rich.

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1. Introduction

In [7] Brown and Pedersen introduced the notion of extremal richness for C^* -algebras and proved many important results. In particular, it was pointed out [8] that a simple unital C^* -algebra with extremal richness is either purely infinite or has stable rank one. This result is closely related to the long-standing open problem for simple C^* -algebras: Is every infinite simple C^* -algebra purely infinite? (or is every finite simple C^* -algebra stably finite?) Still, there is no known example of a simple C^* -algebra which is neither stably finite nor purely infinite. Related to this problem it is important

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to determine the extremal richness of a simple C^* -crossed product of a simple unital C^* -algebra by a discrete group, and this reduces to the question if pure infiniteness (or the density of invertible elements) is preserved under taking crossed products by discrete groups.

In Section 3 we first define the (WS)-property for C^* -algebras which has been studied by several authors without having been given a specific name, and we discuss some basic results in connection with real rank and extremal richness. From observations by Blackadar [3], Brown and Pedersen [6], we see that this property is strictly weaker than the real rank zero property or extremal richness. It is proved that if A has the (WS)-property, then its real rank is at most one (Proposition 3.6). We also prove in Theorem 3.11 that for a simple unital C^* -algebra with stable rank greater than one, the (WS)-property implies the (SP)-property; hence we can say that for a unital simple C^* -algebra A , A is purely infinite if and only if every non-zero projection in A is infinite and A has the (WS)-property.

When A is a purely infinite simple unital C^* -algebra, the first author proved [13] that the simple crossed product $A \rtimes_{\alpha} G$ of A by an outer action α of a finite group G is also purely infinite, and in [14] both authors and Kodaka proved the pure infiniteness of crossed products by the integer group \mathbb{Z} , which was extended by Kishimoto and Kumjian [16] to any outer action by a discrete group G . Outerness of the action is essential for proving these results and actually it implies the simplicity of the crossed product [15].

In this paper we shall see that if G is a finite group, then any crossed product of a purely infinite simple unital C^* -algebra by G is extremally rich since it is a direct sum of purely infinite simple C^* -algebras, which follows from Theorem 4.2 and Rieffel's structural theorem on crossed products. In particular, if A is a purely infinite simple C^* -algebra and α is an action by a finite group so that the crossed product $A \rtimes_{\alpha} G$ is simple then $A \rtimes_{\alpha} G$ is purely infinite. Izumi kindly informed us that he also proved the same result using the C^* -index Theory. Even though outerness of action is essential in the proof of [16, Lemma 10] we can modify it to prove our Theorem 4.2 where we do not have to assume outerness any more, at least for finite groups.

The crossed product of a stably finite simple unital C^* -algebra A with a trace is always stably finite. But it is not easy to compute the stable rank of this kind of crossed product even when A is a UHF algebra. Note that Rieffel's theorem [24, Theorem 4.3, 7.1] and observation [2, Proposition 10.3.3] show that

$$\text{sr}(A \rtimes_{\alpha} \mathbb{Z}_n) \leq \text{sr}(A) + 1, \quad n = 1, 2, \dots,$$

where $\mathbb{Z}_1 = \mathbb{Z}$ and $\text{sr}(A)$ denotes the stable rank of A . Using this observation, Blackadar pointed out [4, Example 8.2.1] that there is a non-simple unital C^* -algebra A with $\text{sr}(A) = 1$ and an action α of \mathbb{Z}_2 such that the crossed product $A \rtimes_{\alpha} \mathbb{Z}_2$ has stable rank two, $\text{sr}(A \rtimes_{\alpha} \mathbb{Z}_2) = 2$. But it still seems to be conceivable that if A is

a simple unital C^* -algebra with stable rank one, then the stable rank of the simple crossed product by a finite group should also be one.

Recall that every C^* -algebra of stable rank one satisfies cancellation; it is not known if the converse is true for simple C^* -algebras. In Section 5, we prove cancellation for crossed products of simple unital C^* -algebras of stable rank one with the (SP)-property by finite abelian groups (Theorem 5.4). In particular, any C^* -crossed product $UHF \times_{\mathbb{Z}_n}$ ($n \geq 2$) has cancellation. Moreover, if these crossed products have the (WS)-property, then they have stable rank one. To see this we use Theorem 5.1 which is due to Rieffel [25] and Warfield, Jr.

2. Actions of finite groups on simple C^* -algebras

In this section we briefly review Rieffel’s results [23] for finite group actions on C^* -algebras, from which we can learn much about the structure of crossed products by finite groups. Let α be an action of a unital simple C^* -algebra A by a finite group G . Since A is simple, the set $N = \{g \in G \mid \alpha_g \text{ is inner on } A\}$ is a normal subgroup of G , and the crossed product $A \times_{\alpha} N$ can be viewed as a C^* -subalgebra of $A \times_{\alpha} G$ generated by A and unitaries $\{\delta_g \mid g \in N\}$, where $\{\delta_g \mid g \in G\}$ are the unitaries generating $A \times_{\alpha} G$ together with A . It is known from [23] that $A \times_{\alpha} N$ is a direct sum of matrix algebras over A . In fact, this crossed product is isomorphic to $A \otimes C^*\{u_g \delta_g \mid g \in N\}$, where $u_g \in A$ is the unitary in A implementing α_g . Then there exists an action β of G on $A \times_{\alpha} N$ defined by

$$\beta_t(f)(s) = \alpha_t(f(t^{-1}st)), \quad f \in A \times_{\alpha} N.$$

Since A is simple, the G -invariant (or, α -invariant) ideals of $A \times_{\alpha} N$ correspond exactly to the G -invariant (or, β -invariant) ideals of $C^*\{u_g \delta_g \mid g \in N\}$ ($= C$). Since C is a finite dimensional C^* -algebra, it is the direct sum of its G -simple ideals. Let C_1, \dots, C_k be the G -simple ideals of C , and for each i let $I_i = (A \times_{\alpha} G) (A \otimes C_i)$. Then I_1, \dots, I_k are mutually orthogonal closed two-sided ideals in $A \times_{\alpha} G$. Moreover, each I_i is simple.

THEOREM 2.1 [23, THEOREM 3.1]. *Let A be a simple unital C^* -algebra and let G be a finite group. Then $A \times_{\alpha} G$ is the finite direct sum of simple C^* -algebras $\{I_i \mid i = 1, \dots, k\}$.*

Let e denote the identity of G . We call the action α of G on A *outer* if every α_g ($g \neq e$) is an outer automorphism.

REMARK 2.2. If G is a finite group of prime order, then it follows from the above observation that $A \times_{\alpha} G$ is simple if and only if α is outer. More generally, if the

order of G is the product of distinct primes, we get the same conclusion ([21, Theorem 5]). But in the case of $G = \mathbb{Z}_4$ there is a non-outer action on a UHF algebra which provides a simple crossed product ([9, Theorem 4.2.5]) while the crossed product $A \times_\alpha G$ is always simple whenever α is an outer action by a discrete group G on a simple C^* -algebra A [28].

3. C^* -algebras with the (WS)-property

In this section we define the (WS)-property for C^* -algebras and discuss basic results. This property is strictly weaker than the properties of real rank zero, stable rank one and extremal richness. The notion seems to be helpful for characterizing simple C^* -algebras and for computing stable rank of crossed products of simple stably finite unital C^* -algebras, in particular simple unital AF-algebras, by finite groups, which we shall discuss in Section 5. Recall that an element x of a C^* -algebra A is *well-supported* if there is a projection $p \in A$ with $x = xp$ and x^*x is invertible in pAp [3, Definition 4.3.3].

DEFINITION 3.1. A C^* -algebra A has the (WS)-property if the set of all well-supported elements in A is dense.

Note that x is well-supported if and only if either x^*x is invertible or 0 is an isolated point of the spectrum $\text{sp}(x^*x)$.

Recall that the *stable rank*, $\text{sr}(A)$, of a unital C^* -algebra A is the least integer n such that the set $\{(x_1, \dots, x_n) \in A^n \mid \sum_{i=1}^n x_i^*x_i \text{ is invertible in } A\}$ is dense in A^n . Then $\text{sr}(A) = 1$ if and only if the set of invertible elements is dense in A . If A is non-unital, the stable rank of A is defined to be that of its unitization \tilde{A} ([24]).

Similarly, for a unital C^* -algebra A , the *real rank* $\text{RR}(A)$ of A is defined to be the least integer n such that the set $\{(x_0, x_1, \dots, x_n) \in A_{sa}^{n+1} \mid \sum_{i=0}^n x_i^2 \text{ is invertible in } A\}$ is dense in A_{sa}^{n+1} . If A is non-unital, we define the real rank of A by that of \tilde{A} [6].

A unital C^* -algebra A is called *extremally rich* [7] if the set of quasi-invertible elements ($= A_q^{-1}$) is dense in A , where $A_q^{-1} = A^{-1} \text{ex}(A)A^{-1}$ and $\text{ex}(A)$ is the set of extreme points in the closed unit ball of A . If A is non-unital, we call A *extremally rich* if \tilde{A} is extremally rich. Clearly every C^* -algebra with $\text{sr}(A) = 1$ is extremally rich.

The following is proved immediately from the characterization of a quasi-invertible element by Brown and Pedersen [7, Theorem 1.1].

PROPOSITION 3.2. *Let A be a unital C^* -algebra. If A is extremally rich, then A has the (WS)-property.*

PROOF. Since A is extremally rich, the set of quasi-invertible elements is dense in A . For each quasi-invertible element x , $|x|$ is invertible or 0 is an isolated point of the spectrum $\text{sp}(|x|)$ ([7, Theorem 1.1]), so x is well-supported.

The following is shown by Blackadar.

PROPOSITION 3.3 [3, Proposition 4.3.4]. *If $\text{RR}(A) = 0$, then A has the (WS)-property.*

But the converse of the proposition is not true in general as the following examples show.

EXAMPLE 3.4. The C^* -algebra $C[0, 1]$ of continuous functions on the interval $[0, 1]$ has stable rank one, so it has the (WS)-property, but its real rank is one since $\text{RR}(C[0, 1]) = \dim([0, 1]) = 1$ ([6]). Also Blackadar’s projectionless simple C^* -algebra ([1]) and the simple C^* -algebra A^3 from [5] have the (WS)-property and have real rank one.

EXAMPLE 3.5. The multiplier algebra of a non-unital finite matroid C^* -algebra has real rank zero ([6]), so it has the (WS)-property. But it is not extremally rich ([8]). More generally, the multiplier algebra of any simple AF algebra has real rank zero ([18]), but it is not extremally rich ([17]). Note that stable rank of these multiplier algebras are greater than one, because their corona algebras always contain proper isometries.

PROPOSITION 3.6. *Let A be a unital C^* -algebra with the (WS)-property. Then $\text{RR}(A) \leq 1$.*

PROOF. It follows immediately from the definition that for a unital C^* -algebra A , $\text{RR}(A) \leq 1$ if and only if for any $\varepsilon > 0$ and any element x in A there is an element $y \in A$ such that $\|x - y\| < \varepsilon$ and $y^*y + yy^*$ is invertible. Therefore we have only to show that any well-supported element x in A can be approximated by some element $y \in A$ such that $y^*y + yy^*$ is invertible.

From the definition, x can be written as $x = u|x|$, where $u^*u = p$ for a non-zero projection $p \in A$ and $|x|$ is invertible in pAp . For any $\varepsilon > 0$ set $y = x + \varepsilon(1 - q)$, where $q = uu^*$. Note that $(1 - q)x = 0$. Then

$$\begin{aligned} y^*y + yy^* &= (x^*x + \varepsilon^2(1 - q)) + (xx^* + \varepsilon(1 - q)x^* + \varepsilon x(1 - q) + \varepsilon^2(1 - q)) \\ &= x^*x + [\varepsilon^2(1 - q) + xx^* + \varepsilon(1 - q)x^* + \varepsilon x(1 - q) + \varepsilon^2(1 - q)] \\ &= x^*x + c. \end{aligned}$$

We show that c is invertible. Set $z = \varepsilon(1 - q)x^*$, $a = xx^*$. Then $a = qaq$ is invertible in qAq and $z = (1 - q)zq$. Since

$$\begin{aligned} (1 - za^{-1})c(1 - za^{-1})^* &= (1 - za^{-1})(\varepsilon^2(1 - q) + z + z^* + a + \varepsilon^2(1 - q))(1 - za^{-1})^* \\ &= \varepsilon^2(1 - q) - za^{-1}z^* + a + \varepsilon^2(1 - q) \end{aligned}$$

and $a = xx^*$ is invertible in qAq , it follows that $c = \varepsilon^2(1 - q) + z + z^* + a + \varepsilon^2(1 - q)$ is invertible if and only if $\varepsilon^2(1 - q) - za^{-1}z^* + \varepsilon^2(1 - q)$ is invertible in $(1 - q)A(1 - q)$. From

$$\begin{aligned} 0 &\leq (1 - za^{-1})(\varepsilon^2(1 - q) + z + z^* + a)(1 - za^{-1})^* \\ &= \varepsilon^2(1 - q) - za^{-1}z^* + a, \end{aligned}$$

we see that

$$\varepsilon^2(1 - q) - za^{-1}z^* + a + \varepsilon^2(1 - q) \quad (\geq \varepsilon^2(1 - q))$$

is invertible in $(1 - q)A(1 - q)$.

COROLLARY 3.7 [22, Theorem 10.4]. *Let A be a unital extremally rich C^* -algebra. Then $\text{RR}(A) \leq 1$.*

Recall that a unital C^* -algebra A has *cancellation of projections* if whenever p, q, r are projections in A with $p \perp r, q \perp r, p + r \sim q + r$, then $p \sim q$. A is said to have *cancellation* if for each $n \in \mathbb{N}$, $M_n(A)$ has cancellation of projections.

If $\text{RR}(A) = 0$, then A has cancellation if and only if $\text{sr}(A) = 1$ [3, Corollary 4.3]. More precisely, we can rewrite the fact as follows.

THEOREM 3.8 [3, Proposition 4.3.6]. *Let A be a unital C^* -algebra. Then the following are equivalent:*

- (1) A has stable rank one.
- (2) A has the (WS)-property and cancellation of projections.

We can also characterize the pure infiniteness of simple C^* -algebras in terms of the (WS)-property.

LEMMA 3.9. *Let x be a well-supported element, and a be an invertible element in a unital C^* -algebra A . Then xa is well-supported.*

PROOF. Since aa^* is invertible, there exists a $\delta > 0$ such that $aa^* \geq \delta$. Let $x = xp$ and x^*x be invertible in pAp . Then x can be written as $u|x|$ with $u^*u = p$. Let $q = uu^*$, then xx^* is invertible in qAq . Since $xaa^*x^* \geq \delta xx^*$ we see that xaa^*x^* is invertible in qAq , so it is invertible in A or 0 is an isolated point of its spectrum. Thus xa is well-supported in A .

The same argument as in [7, Theorem 3.5] proves the following lemma.

LEMMA 3.10. *Let A be a unital C^* -algebra with the (WS)-property. Then for a non-zero proper hereditary C^* -subalgebra B of A , \tilde{B} has the (WS)-property, where \tilde{B} denotes the unitization of B obtained by adding the unit of A to B . Furthermore if p is a non-zero projection in A then the hereditary C^* -subalgebra $B = pAp$ has the (WS)-property. If I is a closed two-sided ideal in A then A/I has the (WS)-property.*

The following result is an extension of [8].

THEOREM 3.11. *Let A be a unital C^* -algebra with the (WS)-property. If B is a non-zero hereditary C^* -subalgebra of A with $\text{sr}(B) > 1$ then B has a non-zero projection.*

PROOF. We may assume that $B = \overline{aAa}$ for a non-zero positive element $a \in A$.

Since $\text{sr}(B) = \text{sr}(\tilde{B}) > 1$ there is a non-zero element $x \in \tilde{B}$ and $\delta_0 > 0$ such that $\text{dist}(x, GL(\tilde{B})) > \delta_0$. Let $y \in \tilde{B}$ be a well-supported element such that $\|x - y\| < \delta_0/2$. Note that y is not invertible. Since y is well-supported, there is a partial isometry $u \in \tilde{B}$ such that $y = u|y|$, $u^*u = p$, and $|y|$ is invertible in $p\tilde{B}p$. Since u is not a unitary, either $u^*u \neq 1$ or $uu^* \neq 1$. Hence, either $1 - u^*u$ or $1 - uu^*$ is a non-zero projection contained in B .

Recall that a C^* -algebra A is called *purely infinite* if every non-zero hereditary C^* -subalgebra has an infinite projection.

COROLLARY 3.12. *Let A be a simple unital C^* -algebra. Then A is purely infinite if and only if any non-zero projection in A is infinite and A has the (WS)-property.*

PROOF. Since every purely infinite simple C^* -algebra A has real rank zero [29], A has the (WS)-property.

Conversely, A contains orthogonal isometries, so that $\text{sr}(A) = \infty$ by [24, Proposition 6.5]. It then follows from Theorem 3.11 that every hereditary C^* -subalgebra contains a non-zero projection which is infinite by the assumption.

We close this section examining some equivalent conditions to extremal richness of the C^* -algebra $C[0, 1] \otimes A$ of operator valued continuous functions on the interval $[0, 1]$.

LEMMA 3.13. *Let A be a unital C^* -algebra. Then*

$$\text{sr}(C[0, 1] \otimes A) = \text{sr}(C(S^1) \otimes A).$$

PROOF. Since $C[0, 1] \otimes A$ is isomorphic to a quotient C^* -algebra of $C(S^1) \otimes A$, $\text{sr}(C[0, 1] \otimes A) \leq \text{sr}(C(S^1) \otimes A)$.

Conversely, since $C(S^1) \otimes A \cong C_0(0, 1) \tilde{\vee} A$ and $C_0(0, 1) \otimes A$ is a closed two-sided ideal of $C[0, 1] \otimes A$, we get

$$\text{sr}(C(S^1) \otimes A) = \text{sr}(C_0(0, 1) \tilde{\vee} A) \leq \text{sr}(C[0, 1] \otimes A).$$

PROPOSITION 3.14. *Let A be a unital C^* -algebra. Then the following are equivalent:*

- (1) $C([0, 1]) \otimes A$ has the (WS)-property.
- (2) $C([0, 1]) \otimes A$ is extremally rich.
- (3) $C([0, 1]) \otimes A$ has stable rank one.

PROOF. The implication 3 implies 2 is obvious, and 2 implies 1 was shown in Proposition 3.2. We show that 1 implies 3. Suppose that $\text{sr}(C[0, 1] \otimes A) > 1$. Then for the ideal $I = C_0(0, 1) \otimes A$ of $C[0, 1] \otimes A$, \tilde{I} is isomorphic to $C(S^1) \otimes A$, so $\text{sr}(I) = \text{sr}(\tilde{I}) > 1$ by Lemma 3.13. Hence, by Theorem 3.11, $C_0(0, 1) \otimes A$ has a non-zero projection, which is not possible.

COROLLARY 3.15. *Let A be a unital C^* -algebra and let X be a finite CW-complex of dimension greater than two. Then $C(X) \otimes A$ is not extremally rich.*

PROOF. The C^* -algebra $C(X) \otimes A$ has $C[0, 1]^2 \otimes A$ as a quotient C^* -algebra. Since $\text{sr}(C[0, 1]^2 \otimes A) \geq 2$ [20], the tensor product is not extremally rich from Proposition 3.14, because it is isomorphic to $C[0, 1] \otimes (C[0, 1] \otimes A)$.

We have seen in Proposition 3.6 that the (WS)-property of a C^* -algebra implies $\text{RR}(A) \leq 1$. But the converse is not true as the following example shows.

EXAMPLE 3.16. Let A be a purely infinite simple unital C^* -algebra. Then $\text{sr}(C[0, 1] \otimes A) = \infty$ by [24]. So, $C[0, 1] \otimes A$ does not have the (WS)-property from the previous proposition. But $\text{RR}(C[0, 1] \otimes A) = 1$ [20].

4. Extremally rich infinite crossed products

When A is a prime C^* -algebra, the set of extremal points, $\text{ex}(A)$, in the closed unit ball of A is just the set of isometries and co-isometries, hence the set of quasi-invertible elements $A_q^{-1} (= A^{-1} \text{ex}(A)A^{-1})$ is the set of one-sided invertible elements in A . Rørdam [26] and Pedersen [22] proved that if A is a purely infinite simple C^* -algebra, then A is extremally rich. For a unital simple C^* -algebra A , A is extremally rich if and only if it is either purely infinite or $\text{sr}(A) = 1$ ([8]).

In the present and next section, we discuss the extremal richness of the C^* -crossed products of extremally rich simple unital C^* -algebras by finite groups.

Using the results on the (WS)-property obtained in Section 3, we first give a simple proof of the fact that a purely infinite simple unital C^* -algebra is extremally rich:

LEMMA 4.1 ([22], [26]). *A purely infinite unital simple C^* -algebra is extremally rich.*

PROOF. Since every quasi-invertible element in A is one-sided invertible, we show that every element in A can be approximated by one-sided invertible elements in A .

Let x be an element in A . By Corollary 3.12, we may assume that x is well-supported. Then there exists a projection $p \in A$ such that $x = xp$ and x^*x is invertible in pAp . If $p = 1$, then x is left invertible. Suppose that $p \neq 1$. Put $u = x(x^*x)^{-1/2}$, where $(x^*x)^{-1}$ is an inverse of x^*x in pAp . Then $x = u(x^*x)^{1/2}$ and $u^*u = p$. Put $q = uu^*$. If $q = 1$, then xx^* is invertible in A , so x is right invertible. If $q \neq 1$, since A is purely infinite simple, there is a partial isometry $v \in A$ such that $v^*v = 1 - p$ and $vv^* \leq 1 - q$ ([11]). Put $y = x + \varepsilon v$. Then, $\|x - y\| < \varepsilon$ and $y^*y = x^*x + \varepsilon^2(1 - p)$, so y is left invertible.

Recall that a C^* -algebra A has the (SP)-property if every non-zero hereditary C^* -subalgebra of A has a non-zero projection. From [6], we see that any C^* -algebra with $RR(A) = 0$ has this property. But the converse is not true in general ([22]).

The following extends the result in [16, Lemma 10].

THEOREM 4.2. *Let A be a simple unital C^* -algebra with the (SP)-property and let α be an action by a discrete group G . Suppose that the normal subgroup $N = \{g \in G \mid \alpha_g \text{ is inner on } A\}$ of G is finite. Then any non-zero hereditary C^* -subalgebra of the reduced crossed product $A \times_{\alpha_r} G$ has a non-zero projection which is equivalent to a projection in $A \times_{\alpha} N$.*

PROOF. Let $a \in A \times_{\alpha_r} G$ be a non-zero positive element. We may assume that $\|a\| = 1$. We show that the hereditary C^* -subalgebra $\overline{a(A \times_{\alpha_r} G)a}$ has a non-zero projection which is equivalent to some projection in $A \times_{\alpha} N$.

Let $\varepsilon > 0$ be any sufficiently small number such that $(192/\|a_0^2\| + 15)\varepsilon < 1$, where a_0 is the image of a under the canonical faithful conditional expectation from $A \times_{ar} G$ to A .

Let $\{\delta_g \mid g \in G\}$ be the unitaries in $A \times_{ar} G$ implementing α . Since the $*$ -algebra generated by $\{b\delta_g \mid b \in A, g \in G\}$ is dense in $A \times_{ar} G$, we can approximate $a^{1/2}$ by elements of the form $b = \sum_{i=1}^n c_i \delta_{g_i}$ so that b^*b approximates a to within ε . Write

$$b^*b = b_0 + \sum_{g_i \in N \setminus \{e\}} b_i \delta_{g_i} + \sum_{g_l \notin N} b_l \delta_{g_l}$$

with $b_0 \geq 0$ in A and $B = \sum_{g_i \in N \setminus \{e\}} b_i \delta_{g_i} = B^* \in A \times_{\alpha} N$. Then, since $\|a_0 - b_0\| \leq \|a - b^*b\| < \varepsilon$, we have $0 < \|a_0\| - \varepsilon \leq \|b_0\|$, and $\|b^*b\| < \|a\| + \varepsilon = 1 + \varepsilon$.

By [15, Lemma 3.2] there exists a positive element $x \in A$ with $\|x\| = 1$ such that

$$\|xb_0x\| > (1 - \varepsilon)\|b_0\| \quad \text{and} \quad \|xb_l \delta_{g_l}\| < \varepsilon/n^2, \quad g_l \notin N.$$

Then

$$\begin{aligned} \|(xax)^2 - (xb^*bx)^2\| &\leq \|xax\| \|xax - xb^*bx\| + \|xax - xb^*bx\| \|xb^*bx\| \\ &< \varepsilon + \varepsilon \|xb^*bx\| \leq \varepsilon + \varepsilon \|b^*b\| < 3\varepsilon, \end{aligned}$$

and

$$\|xb^*bx\| \geq \|xb_0x\| > (1 - \varepsilon)\|b_0\|.$$

On the other hand, $\|xBx + x(\sum_{g_l \notin N} b_l \delta_{g_l})x\| = \|x(b^*b)x - xb_0x\| \leq \|xb^*bx\| + \|b_0\| \leq 2 + 2\varepsilon$, hence $\|xBx\| \leq \|x(\sum_{g_l \notin N} b_l \delta_{g_l})x\| + 2 + 2\varepsilon \leq 2 + 3\varepsilon$.

Since $\|(xb^*bx)\|^2 \leq (\|x(b_0 + B)x\| + \varepsilon)^2$ and $\|x(b_0 + B)x\| \leq \|xb_0x\| + 2 + 3\varepsilon \leq 3 + 4\varepsilon$, we see that

$$(1 - \varepsilon)\|b_0\| \leq \|xb^*bx\| \leq \|x(b_0 + B)x\| + \varepsilon < 3 + 5\varepsilon.$$

Let $d = (x(b_0 + B)x)^2 \in A \times_{ar} N$. Note that $0 < \{\|a_0\| - 4\varepsilon\}^2/4 \leq \|d\|$. Then $0 < \|a_0\|^2/64 < \|d\|$ because $(1 - \varepsilon)(\|a_0\| - \varepsilon) - \varepsilon \leq \|d\|^{1/2}$. Since $\varepsilon < 1/2$, the left hand side in the previous inequality is larger than

$$\frac{1}{2}(\|a_0\| - \varepsilon) - \varepsilon = \frac{1}{2}\|a_0\| - \frac{3}{2}\varepsilon \geq \frac{1}{2}\|a_0\| - \frac{3}{4}\|a_0\| = \frac{1}{8}\|a_0\|.$$

Consider the continuous functions f and g defined by

$$f(t) = \max(0, t - (1 - \varepsilon)\|d\|), \quad g(t) = \min(t, (1 - \varepsilon)\|d\|).$$

Note that $fg = (1 - \varepsilon)\|d\|f$.

Since α_g is inner for $g \in N$, the crossed product $A \times_\alpha N$ is isomorphic to a direct sum of matrix algebras over A , and thus it has the (SP)-property. Let p be a non-zero projection in $\overline{f(d)(A \times_\alpha N)f(d)}$. Then there exists a non-zero element $y \in \overline{f(d)(A \times_\alpha N)f(d)}$ such that $p = yf(d)y^*$. Let

$$z_0 = (1 - \varepsilon)^{-1/2}\|d\|^{-1/2}yf(d)^{1/2}.$$

Then $\|z_0\| = (1 - \varepsilon)^{-1/2}\|d\|^{-1/2}$ and $z_0g(d)z_0^* = p$. Since $g(d) \leq d$, we have $p = z_0g(d)z_0^* \leq z_0dz_0^*$. Thus there exists an element $z \in A \times_\alpha N$ such that

$$zdz^* = p, \quad \|z\| \leq \|z_0\| = (1 - \varepsilon)^{-1/2}\|d\|^{-1/2} \leq 8(1 - \varepsilon)^{-1/2}/\|a_0\|.$$

Since

$$\begin{aligned} & \|z(x(b^*b)x)^2z^* - p\| \\ &= \|z\{(x(b_0 + B)x)^2 + x(b_0 + B)x^2Cx + xCx^2(b_0 + B)x + xCx^2Cx\}z^* - p\| \\ &\leq \|zdz^* - p\| + 2\|x(b_0 + B_0)x\|\varepsilon + \varepsilon^2 < 2(3 + 4\varepsilon)\varepsilon + \varepsilon^2 < 15\varepsilon, \end{aligned}$$

where $C = \sum_{g_l \notin N} b_l \delta_{g_l}$, we have

$$\begin{aligned} \|z((xax)^2)z^* - p\| &\leq \|z((xax)^2 - (xb^*bx)^2)z^*\| + \|z(xb^*bx)^2z^* - p\| \\ &\leq 3\varepsilon\|z\|^2 + 15\varepsilon \\ &\leq 3\frac{\varepsilon}{1 - \varepsilon} \times \frac{64}{\|a_0\|^2} + 15\varepsilon \\ &\leq \left(\frac{192}{\|a_0\|^2} + 15\right)\varepsilon < 1. \end{aligned}$$

Therefore, there exists an element $z_1 \in A \times_{\alpha r} G$ such that $z_1(xax)^2z_1^* = p$. Thus $(xax)z_1^*z_1(xax)$ is a projection in the hereditary C^* -subalgebra $\overline{(xax)(A \times_{\alpha r} G)(xax)}$. Since $\overline{(xax)(A \times_{\alpha r} G)(xax)}$ is isomorphic to the hereditary subalgebra $\overline{(a^{1/2}x)(x^*a^{1/2})(A \times_{\alpha r} G)(a^{1/2}x)(x^*a^{1/2})}$ [10, 1.4], the hereditary C^* -subalgebra $\overline{a(A \times_{\alpha r} G)a}$ has a projection which is equivalent to p .

COROLLARY 4.3 [16, Remark 8]. *Let A be a simple unital C^* -algebra with the (SP)-property and let G be a discrete group. If either the action α of G on A is outer or G is finite, then the reduced crossed product $A \times_{\alpha r} G$ has the (SP)-property.*

COROLLARY 4.4. *Let A be a purely infinite simple unital C^* -algebra and let α be an action by a discrete group G such that the normal subgroup N is finite as in the above theorem. Then $A \times_{\alpha r} G$ is purely infinite. In particular, if either α is outer or G is finite then $A \times_{\alpha r} G$ is purely infinite.*

PROOF. Note that $A \times_{\alpha} N$ is the direct sum of purely infinite simple C^* -algebras (see Section 2). So, any non-zero projection in $A \times_{\alpha} N$ is infinite. Therefore, any non-zero hereditary C^* -algebra of the crossed product has a non-zero infinite projection.

THEOREM 4.5. *Let A be a purely infinite simple unital C^* -algebra and let G be a finite group. Then $A \times_{\alpha} G$ is isomorphic to a direct sum of purely infinite simple C^* -algebras, so that it is extremally rich.*

PROOF. By Theorem 2.1, the crossed product $A \times_{\alpha} G$ is the direct sum of simple C^* -algebras. Since this crossed product is purely infinite by Corollary 4.4, each simple direct summand is purely infinite and so extremally rich by Lemma 4.1.

The following is an extension of [13, Theorem 1].

COROLLARY 4.6. *Let A be a purely infinite simple unital C^* -algebra, and let G be a finite group. Then $A \times_{\alpha} G$ is purely infinite simple if and only if it is simple.*

REMARK 4.7. (1) Izumi kindly informed that he also obtained the same result as in Theorem 4.5 by the application of C^* -index theory.

(2) Let A be a purely infinite simple unital C^* -algebra and let α be an action by an infinite discrete group. Suppose that the reduced crossed product $A \times_{\alpha r} G$ is simple. Since $A \times_{\alpha r} G$ contains A as a C^* -subalgebra, $A \times_{\alpha r} G$ has infinite projections. However, it is not clear whether $A \times_{\alpha r} G$ is purely infinite or not. If the action α is outer then the crossed product is purely infinite as was mentioned before in Corollary 4.4.

The following proposition shows that extremal richness may not be preserved under taking the crossed products by infinite discrete groups.

PROPOSITION 4.8. *Let A be a purely infinite simple unital C^* -algebra and α be an outer automorphism on A . Suppose $\alpha^n = \text{id}$ for some $n \geq 2$. Then $A \times_{\alpha} \mathbb{Z}$ is not extremally rich.*

PROOF. Since $\alpha^n = \text{id}$, $A \times_{\alpha} \mathbb{Z}$ can be realized as a mapping torus of $A \times_{\alpha} \mathbb{Z}_n$, that is, $A \times_{\alpha} \mathbb{Z}$ is isomorphic to the C^* -algebra $\{f : [0, 1] \rightarrow A \times_{\alpha} \mathbb{Z}_n \mid f(1) = \hat{\alpha}(f(0))\}$ ([2, Proposition 10.3.3]). Then we have the following C^* -exact sequence:

$$0 \rightarrow C_0(0, 1) \otimes (A \times_{\alpha} \mathbb{Z}_n) \rightarrow A \times_{\alpha} \mathbb{Z} \rightarrow A \times_{\alpha} \mathbb{Z}_n \rightarrow 0.$$

Since α is outer, $A \times_{\alpha} \mathbb{Z}_n$ is purely infinite simple by Corollary 4.4 (cf. [13]). It then follows from Proposition 3.14 that the unitization of $C_0(0, 1) \otimes (A \times_{\alpha} \mathbb{Z}_n)$ is not extremally rich. Hence, $A \times_{\alpha} \mathbb{Z}$ is not extremally rich [7, Theorem 3.5].

5. Cancellation and extremal richness of finite crossed products

In this section we find a condition under which the crossed product $A \times_\alpha G$ has cancellation when G is a finite abelian group.

We begin with recalling the following theorem due to Rieffel and Warfield, Jr. [25].

THEOREM 5.1 [3, Theorem 4.2.2]. *Let A be a simple unital C^* -algebra. Suppose A contains a sequence (p_k) of projections such that*

- (1) *For each k there is a projection r_k such that $2p_{k+1} \oplus r_k$ is equivalent to a subprojection of $p_k \oplus r_k$.*
- (2) *There is a constant K such that $\text{sr}(p_k A p_k) \leq K$ for all k .*

Then A has cancellation.

PROPOSITION 5.2. *Let A be a simple unital C^* -algebra with $\text{sr}(A) = 1$ and the (SP)-property. If α is an automorphism of A with $\alpha^n = \text{id}$, then $A \times_\alpha \mathbb{Z}_n$ has cancellation.*

PROOF. We first assume that the crossed product $A \times_\alpha \mathbb{Z}_n$ is simple.

Since the fixed point algebra A^α can be identified with a hereditary C^* -subalgebra of $A \times_\alpha \mathbb{Z}_n$ ([27]), A^α has the (SP)-property by Corollary 4.3. Thus there is a sequence of projections $\{p_k\} \in A^\alpha$ such that $2[p_{k+1}] \leq [p_k]$ by [19, Lemma 2.2], where $[p]$ denotes the equivalence class of p . Since $p_k \in A^\alpha$, $p_k(A \times_\alpha \mathbb{Z}_n)p_k$ is isomorphic to $(p_k A p_k) \times_\alpha \mathbb{Z}_n$ for each $k \in \mathbb{N}$. Note that each $p_k A p_k$ has stable rank one since $p_k A p_k$ is stably isomorphic to A . From [2, Proposition 10.3.3] and [24, Theorem 7.1] $\text{sr}(p_k A p_k \times_\alpha \mathbb{Z}_n) \leq 2$. Therefore, the assertion follows from Theorem 5.1 ($K = 2, r_k = 0$).

Now assume that $A \times_\alpha \mathbb{Z}_n$ is not simple. Then from Theorem 2.1 $A \times_\alpha \mathbb{Z}_n$ is a direct sum $I_1 \oplus \dots \oplus I_m$ of simple C^* -algebras. Since the fixed point algebra A^α is isomorphic to a hereditary subalgebra of the crossed product it has the (SP)-property by Corollary 4.3. Note that A^α is also a direct sum $\bigoplus_{j=1}^k B_j$ ($k \leq m$) of simple C^* -subalgebras B_j ([23, Corollary 3.5]), and each simple direct summand B_j also has the (SP)-property. Thus we can find a sequence $\{p_k^{(j)}\}$ of projections in B_j such that $2[p_{k+1}^{(j)}] \leq [p_k^{(j)}]$ by [19, Lemma 2.2]. Hence A^α contains full projections $\{q_k\}$ with $2[q_{k+1}] \leq [q_k]$ where $q_k = \sum p_k^{(j)}$. Since A is a subalgebra of $A \times_\alpha \mathbb{Z}_n$ and $A^\alpha \subset A$, we can write $q_k = e_k^{(1)} \oplus \dots \oplus e_k^{(m)}$ for some projections $e_k^{(j)} \in I_j$. Note that each projection $e_k^{(j)}$ is non-zero since q_k is full in A^α which contains the unit of the crossed product. It then follows that $2[e_{k+1}^{(j)}] \leq [e_k^{(j)}]$ for each $j = 1, \dots, m$ and $k = 1, 2, 3, \dots$. Since $q_k \in A^\alpha$ we have $\text{sr}(q_k(A \times_\alpha \mathbb{Z}_n)q_k) = \text{sr}(q_k A q_k \times_\alpha \mathbb{Z}_n) \leq 2$. But $q_k(A \times_\alpha \mathbb{Z}_n)q_k = \bigoplus_{j=1}^m e_k^{(j)}(A \times_\alpha \mathbb{Z}_n)e_k^{(j)}$, and hence $\text{sr}(e_k^{(j)}(A \times_\alpha \mathbb{Z}_n)e_k^{(j)}) = \text{sr}(e_k^{(j)} I_j e_k^{(j)}) \leq 2$. Thus each

simple direct summand I_j has cancellation by Theorem 5.1 and therefore the crossed product $A \times_\alpha \mathbb{Z}_n = \bigoplus_{j=1}^m I_j$ has cancellation.

For a discrete semidirect product group $G = H \times K$ with the group operation given by $(h_1, k_1)(h_2, k_2) = (h_1k_1h_2k_1^{-1}, k_1k_2)$, if α is an action of G on a C^* -algebra A then there is an isomorphism

$$\phi : A \times_{\alpha r} G \rightarrow (A \times_{(\alpha|_H)r} H) \times_{\beta r} K,$$

where $\alpha|_H$ denotes the restriction of α to H and $\beta : K \rightarrow \text{Aut}(A \times_{(\alpha|_H)r} H)$ is the action defined by $\beta(k)(au_{(h,1)}) = \alpha_{(1,k)}(a)u_{(khk^{-1},1)}$, $a \in A$ (here $u_{(h,k)}$ is the unitary implementing $\alpha_{(h,k)}$, that is, $\alpha_{(h,k)}(a) = u_{(h,k)}au_{(h,k)}^*$). Then ϕ is defined by $\phi(au_{(h,1)}) = au_{(h,1)}v_k$, where v_k is the unitary such that $\beta(k)(x) = v_kxv_k^*$, $x \in A \times_{(\alpha|_H)r} H$, $k \in K$.

LEMMA 5.3. *Let A be a unital C^* -algebra with finite stable rank and let $G = \mathbb{Z}_{n_1} \times \dots \times \mathbb{Z}_{n_k}$ be a finite abelian group and α be an action of A by G . Then*

$$\text{sr}(A \times_\alpha G) \leq \text{sr}(A) + k.$$

PROOF. This follows from [2, Proposition 10.3.3], [24, Theorem 7.1], and the above argument.

THEOREM 5.4. *Let A be a simple unital C^* -algebra with stable rank one and the (SP)-property, and let G be a semidirect product of finite abelian groups and α be an action of A by G . Then $A \times_\alpha G$ has cancellation.*

PROOF. It suffices to prove the assertion in case of $G = \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}$. Note that the fixed point algebra A^α is a hereditary subalgebra of the crossed product $A \times_\alpha (\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2})$ which is a direct sum of simple C^* -algebras. Hence we can choose a sequence of projections $\{q_k\}$ in A^α such that $2[q_{k+1}] \leq [q_k]$ as in the proof of Proposition 5.2. Then, from Lemma 5.3 we see that $\text{sr}(q_k((A \times_{\alpha|_{\mathbb{Z}_{n_1}}} \mathbb{Z}_{n_1}) \times_{\beta} \mathbb{Z}_{n_2})q_k) = \text{sr}((q_k A q_k \times_{\alpha|_{\mathbb{Z}_{n_1}}} \mathbb{Z}_{n_1}) \times_{\beta} \mathbb{Z}_{n_2}) \leq \text{sr}(q_k A q_k \times_{\alpha|_{\mathbb{Z}_{n_1}}} \mathbb{Z}_{n_1}) + 1 \leq \text{sr}(q_k A q_k) + 2 = 3$ for any $k \in \mathbb{N}$. Therefore, $A \times_\alpha G$ has cancellation from the same argument as in Proposition 5.2.

COROLLARY 5.5. *Under the same assumptions as in the previous theorem, if $A \times_\alpha G$ has the (WS)-property, then it has stable rank one.*

PROOF. It follows from Theorem 3.8 and Theorem 5.4.

COROLLARY 5.6. *Under the same assumptions, if $A \times_\alpha G$ has real rank zero, then it has stable rank one.*

REMARK 5.7. Even when A is a UHF-algebra, there is an outer action α on A with $\alpha^2 = \text{id}$ such that $RR(A \times_{\alpha} \mathbb{Z}_2) \neq 0$ ([12]). But $\text{sr}(A \times_{\alpha} \mathbb{Z}_2) = 1$. It would be very important and interesting to find the stable rank of crossed products satisfying the assumptions in Theorem 5.4.

A unital C^* -algebra A with $\text{sr}(A) = 1$ always has cancellation, but it is not known whether the converse is true for simple C^* -algebras while there is a (non-commutative) non-simple C^* -algebra with cancellation whose stable rank is two. Indeed, $C[0, 1] \otimes B$ (B is a Bunce-Deddens algebra) has cancellation and its stable rank is 2. In fact that can be proved by the following proposition.

PROPOSITION 5.8. *For a unital C^* -algebra A , $C[0, 1] \otimes A$ has cancellation if and only if A has.*

PROOF. We have only to show that if two projections p, q in $C[0, 1] \otimes A$ are equivalent, then they are unitarily equivalent (see [3, Proposition 6.4.1]).

Note that there exist unitaries u, v in $C[0, 1] \otimes A$ (see [28]) such that $u(1 \otimes p(0))u^* = p$ and $v(1 \otimes q(0))v^* = q$. Since A has cancellation there is a unitary w in A such that $wp(0)w^* = q(0)$ because $p(0)$ and $q(0)$ are equivalent in A . Then

$$(v(1 \otimes w)u^*)p(v(1 \otimes w)u^*)^* = q.$$

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